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The intersection of a continuum of open dense sets

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It is proved that every metrizable topological space without isolated points is the union of a continuum or fewer nowhere dense subsets.

Introduction

In this paper a theorem is proved which shows that for the class of complete metric spaces the Baire Category Theorem may not be extended to uncountable families of open dense subsets if the Continuum Hypothesis is assumed. A cardinal invariant is defined and used to prove that certain topological spaces cannot be expressed as a product having a non trivial metrizable component. Finally it is deduced that each metrizable space without isolated points having the property that nowhere dense subsets have cardinality less than $2^{\aleph_0}$ is itself of cardinality at most $2^{\aleph_0}$.

THEOREM 1. *Let* (X, d) *be a metric space having no isolated points. Then there exist a family* $\left(B_\sigma : \sigma \in \Sigma\right)$ *of open dense subsets of* X *satisfying* $\bigcap_{\sigma\in\Sigma} B_\sigma = \emptyset$ *and* $|\Sigma| \leq c$.

Proof. The proof will be divided into four cases: where X is separable, where X is locally separable, where X contains no non-empty open separable subsets, and lastly where X is an arbitrary metric space without isolated points.

(a) Let X be separable; then $|X| \leq c$. Let $B_p = X - \{p\}$ for

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each p in X. Then each B_p is open and dense and $\bigcap_{p \in X} B_p = \emptyset$.

(b) Let (X, d) be locally separable; then by [1, p. 200, Problem C] there exists a partition $(P_\lambda : \lambda \in \Lambda)$ of X with each P_λ open and separable. Let Σ be a set with $|\Sigma| = c$ and, for each λ in Λ, let $\theta_\lambda : \Sigma \to P_\lambda$ be a surjective function. For each σ in Σ let $B_\sigma = \bigcup_{\lambda \in \Lambda} (P_\lambda - \theta_\lambda(\sigma))$. Then B_σ is open and dense and $\bigcap_{\sigma \in \Sigma} B_\sigma = \emptyset$.

(c) Suppose that X contains no non-empty separable subsets. For any non-empty open subset $P \subset X$ there is a family $(G_n)_{n \in \mathrm{N}}$ of non-empty open subsets of P with $G_n \cap G_m = \emptyset$ for $n \neq m$. Define a function $f : X \to \mathrm{R}$ by setting $f(x) = 0$ if $x \notin \bigcup_{n=1}^{\infty} G_n$ and $f(x) = n$ if $x \in G_n$. Then f is lower semi-continuous on X, unbounded on P and 0 on the complement of P.

Let $\mathcal{B} = (\mathcal{B}_n : n \in \mathrm{N}) = ((B_{n,\lambda} : \lambda \in I_n) : n \in \mathrm{N})$ be a σ-discrete base for the topology of X. Let $f_{n,\lambda}$ be a lower semi-continuous function which is unbounded on $B_{n,\lambda}$ and zero on its complement. Let, for each n in N,

$$f_n = \sum_{\lambda \in I_n} f_{n,\lambda} .$$

Then the sequence of functions $f_n : X \to [0, \infty)$ satisfies the following: for each non-empty open set P there exists an n in N such that f_n is unbounded on P.

Let Σ be the set of all functions $\sigma : \mathrm{N} \to \mathrm{N}$ and let

$$F_\sigma = \{x \in X : f_n(x) \leq \sigma(n), \ \forall n \in \mathrm{N}\} .$$

Then, as each f_n is lower semi-continuous, F_σ is closed. We will show that F_σ has an empty interior. If $P \neq \emptyset$ is open and $P \subset F_\sigma$ then there exists f_n unbounded on P. But then f_n is unbounded on F_σ, which is false. We will now show that $\bigcap_{\sigma\in\Sigma} B_\sigma = \emptyset$. If $x \in B_\sigma$ for all $\sigma \in \Sigma$, then $f_n(x) > \sigma(n)$ for all $\sigma \in \Sigma$. By the definition of Σ this is clearly impossible.

(d) Let X be a metric space. Let the subset $P = \{\cup G : G \subset X$ is open and separable$\}$ be non-empty and $Y = X - P$ be closed and non-empty. If P or Y is empty the proof is complete by cases (b) and (c) above. Otherwise decompose P into the union of disjoint open separable sets $P = \cup\left(V_\theta : \theta \in \Xi\right)$. Let Γ be a set with $|\Gamma| = c$ and, for each $\theta \in \Xi$, let $\alpha_\theta : \Gamma \to V_\theta$ be a surjective function. Let $P_\gamma = P - \{\alpha_\theta(\gamma) : \theta \in \Xi\}$ for each γ in Γ .

Let $B = \left(\left(G_{n,\delta} : \delta \in \Delta_n\right) : n \in \mathbb{N}\right)$ be a σ-discrete base for the topology of X .

If $G_{n,\delta}$ contains an infinite disjoint family of non-empty open subsets define $f_{n,\delta} : X \to [0, \infty)$ as in case (c) above. If not let $f_{n,\delta} \equiv 0$ on X . Then let

$$f_n = \sum_{\delta\in\Delta_n} f_{n,\delta} .$$

Let $\Sigma = \{\sigma : \sigma : \mathbb{N} \to \mathbb{N}\}$ and let

$$A_{i,\sigma} = \{x \in X : f_i(x) > \sigma(i)\} .$$

Finally let

$$B_{\sigma,\gamma} = \bigcup_{i\in\mathbb{N}} A_{i,\sigma} \cup P_\gamma .$$

We will show that

(1) $B_{\sigma,\gamma}$ is open,

(2) $B_{\sigma,\gamma}$ is dense, and

(3) $\bigcap_{\sigma,\gamma} B_{\sigma,\gamma} = \emptyset$.

This will complete the proof as $|\Sigma \times \Gamma| \le c$.

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(1) Immediate.

(2) Let $G \subset X$ be open and non-empty and suppose that, firstly, for all non-empty open $V \subset G$, V is not separable.

Let $G_{n,\delta} \subset G$. Then f_n is unbounded on G and so $\{x : f_n(x) > \sigma(n)\} \cap G \neq \emptyset$. Thus $B_{\sigma,\gamma} \cap G \neq \emptyset$. Secondly, suppose that there exists a non-empty open separable subset $V \subset G$. Then $V \subset P$ and there exists $\theta \in \Xi$ with $V_\theta \cap V \neq \emptyset$. Therefore $V \cap (V_\theta - \alpha_\theta(\gamma)) \neq \emptyset$, for all γ, as X has no isolated points. Thus $V \cap P_\gamma \neq \emptyset$ and so $G \cap B_{\sigma,\gamma} \neq \emptyset$. Therefore $B_{\sigma,\gamma}$ is dense in X.

(3) Let $x \in \bigcap_{\sigma,\gamma} B_{\sigma,\gamma}$. If $x \in \bigcup_i A_{i,\sigma}$ for all σ in Σ, then for all σ there exists an i in $\mathbb{N}$ such that $\sigma(i) < f_i(x)$. This is impossible, and we may assume that there exists a σ with $x \notin \bigcup_i A_{i,\sigma}$. Necessarily $x \in \bigcap_{\gamma \in \Gamma} P_\gamma$. As $P_\gamma \subset P$, $x \in V_\theta$ for some θ and so $x = \alpha_\theta(\gamma_0)$ for some γ_0 in Γ. But then $x \notin P_{\gamma_0}$, a contradiction. Thus $\bigcap_{\sigma,\gamma} B_{\sigma,\gamma} = \emptyset$.

This completes the proof.

Clearly the result is false if X has an isolated point.

Applications

First the relevant definitions are described.

DEFINITION 1. Let κ be a cardinal. A space is κ-Baire if the intersection of fewer than κ dense open sets is dense; Tall [5, p. 317].

DEFINITION 2. Let X be a topological space without isolated points. Let $\alpha(X)$ be the minimum cardinality of a cover of X by nowhere dense subsets.

Note that $\alpha(X) \leq |X|$ and that if X is κ-Baire then $\kappa \leq \alpha(X)$. If X is metrizable with no isolated points then, by Theorem 1, $\alpha(X) \leq 2^{\aleph_0}$.

The minimum number of nowhere dense subsets needed to cover $\mathbb{R}$ has been considered recently by several authors using a variety of models for set theory: for example, [2], [3], [4], and [7]. In this paper only ZFC + AC has been assumed unless stated otherwise.

DEFINITION 3. A topological space is generalized Lusin if it is uncountable and every nowhere dense subset has cardinality less than $2^{\aleph_0}$; Tall [6].

The following theorems are easy applications of Theorem 1 and these definitions.

THEOREM 2. *If* X *is metrizable, has no isolated points, and is* κ*-Baire, then* $\kappa \leq 2^{\aleph_0}$.

THEOREM 3. *If* X *is metrizable, has no isolated points, and is generalized Lusin, then* $|X| \leq 2^{\aleph_0}$.

THEOREM 4. *If* X *is not a singleton and has no isolated points and* Y *is any topological space then* $\alpha(X \times Y) \leq \alpha(X)$.

THEOREM 5. *Let* X *have no isolated points, be* κ*-Baire with* $\kappa > 2^{\aleph_0}$ *, and be expressible as a product* $X = Y \times Z$ *. If* Y *is metrizable then* Y *has an isolated point.*

THEOREM 6. *Let* X *be connected and* κ*-Baire with* $\kappa > 2^{\aleph_0}$ *. If* X *is expressible as a product* $X = Y \times Z$ *and* Y *is metrizable then* Y *is a singleton.*

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