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The Euler Class Group of a Noetherian Ring

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Abstract. For a commutative Noetherian ring A of finite Krull dimension containing the field of rational numbers, an Abelian group called the Euler class group is defined. An element of this group is attached to a projective A-module of rank $=$ dim$A$ and it is shown that the vanishing of this element is necessary and sufficient for P to split off a free summand of rank 1. As one of the applications of this result, it is shown that for any n-dimensional real affine domain, a projective module of rank n (with trivial determinant), all of whose generic sections have n generated vanishing ideals, necessarily splits off a free direct summand of rank 1.

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1. Introduction

Let A be a commutative Noetherian ring of (Krull) dimension n. A classical theorem of Serre ([Se]) asserts that if P is a projective A-module of rank $> n$, then P splits off a free summand of rank 1 (i.e. P has a unimodular element). It is well known that this result is not in general true if rank $P = n$. In ([Mu], Theorem 3.8), Murthy proved that if P is a projective module of rank n over the coordinate ring of a smooth n-dimensional affine variety X over an algebraically closed field, then a necessary and sufficient condition for P to split off a free summand of rank 1 is the vanishing of its 'top Chern class' $C_n(P)$ in the Chow group $CH_0(X)$ of zero cycles modulo rational equivalence (see also [MK-Mu] and [MK 2] for earlier results in this direction). However, this result of Murthy is not true for smooth varieties over non-algebraically closed fields, as is evidenced by the example of the tangent bundle of the real 2-sphere.

To tackle this question for smooth varieties over arbitrary base fields, Nori defined the notion of the 'Euler class group' of a smooth affine variety $X = \text{Spec}\, A$, attached to any projective A-module P of rank $=$ dim A, an element in this group, called the 'Euler class' of P, and asked whether the vanishing of the Euler class of P would ensure that P splits off a free summand of rank 1.

In ([B-RS 1]), we settled this question of Nori in the affirmative for projective modules of trivial determinant. In fact, this paper also contained an explicit description of the Euler class group of a smooth affine variety which was crucial for our

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solution. This approach appeared amenable for a plausible generalisation to arbitrary Noetherian rings (where there is no guarantee that smooth maximal ideals exist). It is natural to ask whether such a general definition of the Euler class group does exist, and with this definition whether one can prove results similar to those in ([B-RS 1]) for arbitrary noetherian rings.

This programme is indeed accomplished in the present paper for noetherian rings A which contain the field of rational numbers.

Let A be a Noetherian ring of dimension $n \geqslant 2$. The Euler class group $E(A)$ (with respect to the trivial line bundle over A) is defined roughly as follows: (for details see Section 4)

First, one takes the free Abelian group on pairs (J, ω_J), where $J \subset A$ is an ideal of height n and ω_J a set of n generators of J/J^2. The group $E(A)$ is a quotient of this group by the subgroup generated by (J, ω_J), where $J = (a_1, \cdots, a_n)$ and ω_J is the induced set of generators of J/J^2.

The underlying reason why this group detects the obstruction for a projective A-module P of rank n (with trivial determinant) to split off a free summand of rank 1 is the following:

By a result of Eisenbud and Evans ([E-E], Remark following Theorem A), most linear maps $\alpha: P \to A$ have the property that height $(J = \alpha(P)) = n$. In such a situation, a result of Mohan Kumar ([MK 2], Theorem 1, second implication), asserts that a necessary condition for P to split off a free summand is that J is generated by n elements. In the other direction, the proof of ([RS 1], Theorem 5) essentially shows that, if J is generated by n elements, which are lifts of a certain set of generators of J/J^2 (arising out of α and a generator of $\wedge^n(P)$), then P splits off a free direct summand of rank 1.

In our set up, we have, apart from the Euler class group $E(A)$, a certain canonical quotient $E_0(A)$ of this group called the 'weak Euler class group' which roughly corresponds to the Chow group in the geometric situation. If $n = \dim A$ is even, interestingly, the kernel of the canonical map $E(A) \twoheadrightarrow E_0(A)$ is a homomorphic image of the orbit space $Um_{n+1}(A)/SL_{n+1}(A)$ with the group structure introduced by Van der Kallen [VK 1]. If $n = 2$, in fact, $Um_3(A)/SL_3(A)$ is precisely the kernel (see (7.3) and (7.6)). This implies, in particular, that if $[v]$ and $[w] \in Um_{n+1}(A)/SL_{n+1}(A)$, and if the projective modules corresponding to any two of $[v]$, $[w]$ and $[v].[w]$ split off free direct summands of rank 1, then so does the projective module corresponding to the third (see (7.7)). An interesting consequence of (7.3) is that if $X = \operatorname{Spec} A$ is a smooth affine surface over the field $\mathbf{R}$ of real numbers such that the canonical module K_A is trivial, then $Um_3(A)/SL_3(A)$ is a free Abelian group of rank t, where t is the number of compact connected components of the topological space $X(\mathbf{R})$ consisting of the set of real points of X (see (7.8)).

If A is an affine domain of dimension n over an algebraically closed field and P is a projective A-module of rank n, then a result of Mohan Kumar ([MK 2], Theorem 1) asserts that if P has a generic section ideal which is generated by n elements, then P splits off a free summand of rank 1 and hence all its generic section ideals are gen-

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erated by n elements. But this is not necessarily true if the base field is not algebraically closed. For example, all the *reduced* generic section ideals (and there are plenty) of the tangent bundle of the real 2-sphere are complete intersections (see [B-RS 2], (5.6,(i))). There are however *non-reduced* generic section ideals of the tangent bundle which are not complete intersections (see for example [B-RS 1], (5.2)). This phenomenon is explained by the result (5.9) of this paper, which asserts that for any n-dimensional real affine domain, a projective module of rank n (with trivial determinant), all of whose generic section ideals are generated by n elements, necessarily splits off a free direct summand of rank 1.

The layout of this paper is as follows: In Section 4, we first define the notion of the *Euler class group* $E(A, L)$ with respect to a line bundle L over A. We attach to the pair (P, χ), where P is a projective A-module of rank n with $\chi: \wedge^n(P) \xrightarrow{\sim} L$ an isomorphism, an element of $E(A, L)$ called the *Euler class* of (P, χ). Among other results, we show that P splits off a free direct summand of rank 1 if and only if the Euler class of (P, χ) is zero (see (4.4)). The main result of Section 5 is (5.9), mentioned earlier. In Section 6, we define the notion of the *weak Euler class group* $E_0(A, L)$ as a certain quotient of $E(A, L)$. We show that, even though the group $E(A, L)$ may vary with the line bundle L, the group $E_0(A, L)$ is *independent* of L (see (6.8)). In Section 7, we establish a connection between $E(A), E_0(A)$ and the group $Um_{n+1}(A)/SL_{n+1}(A)$ defined by Van der Kallen, if dim $A = n$ is even (see (7.3) and (7.6)). In Section 3, we prove some addition and subtraction principles which are crucial for the proofs of the results of Section 4. In Section 2, we quote some results which are used in the later sections.

2. Some Preliminary Results

In this section we prove some preliminary results which will be used later.

All rings considered in this paper are commutative and Noetherian. All modules considered, are assumed to be finitely generated.

LEMMA 2.1. *Let A be a Noetherian ring. Let L be a projective A-module of rank* 1. *Let θ be an element of* $\mathrm{Hom}_A(L, A)$ *and l an element of L. Then, the composite map $L \xrightarrow{\theta} A \xrightarrow{l} L$ is scalar multiplication by $\theta(l)$.*

LEMMA 2.2. *Let A be a Noetherian ring with dim $A = n$ and let P, P_1 be projective A-modules of rank n. Let $J \subset A$ be an ideal of height n, let $\alpha: P \twoheadrightarrow J/J^2$ and $\beta: P_1 \twoheadrightarrow J/J^2$ be surjections. Let $\Psi: P \to P_1$ be a homomorphism such that $\beta\Psi = \alpha$. Then, $\Psi \otimes A/J : P/JP \to P_1/JP_1$ is an isomorphism.*

Proof. Let K be the radical of J. It is enough to prove that $\Psi \otimes A/K : P/KP \to P_1/KP_1$ is an isomorphism. Since height $(J) = n = \dim A$ and J/J^2 is a surjective image of P, it follows that J/KJ is a free A/K-module of rank n. Hence, $\alpha \otimes A/K: P/KP \to J/KJ$ and $\beta \otimes A/K : P_1/KP_1 \to J/KJ$ are isomorphisms. Now the the result follows from the fact that $\beta\Psi = \alpha$. $\square$

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LEMMA 2.3. *Let A be a Noetherian ring and L a projective A-module of rank* 1. *Let J be a proper ideal of A and α, β be surjections from $L \oplus A$ to J. Let Ψ' be an automorphism of $L/JL \oplus A/J$ such that $\overline{\beta}\Psi' = \overline{\alpha}$, where $\overline{\beta}$ and $\overline{\alpha}$ denote surjections from $L/JL \oplus A/J$ to J/J^2 induced by β and α, respectively. Suppose that* $\det(\Psi') = 1$ *Then, there exists an automorphism Δ of $L \oplus A$ such that* (i) $\beta\Delta = \alpha$ *and* (ii) *det* $(\Delta) = 1$.

Proof. First we show that there exists an endomorphism Ψ of $L \oplus A$ such that Ψ is a lift of Ψ' and $\beta\Psi = \alpha$.

Let $\widetilde{\Psi}$ be a lift of Ψ'. Then $(\beta\widetilde{\Psi} - \alpha)(L \oplus A) \subset J^2$. Since $\beta(J(L \oplus A)) = J^2$, there exists a homomorphism $\eta: L \oplus A \to J(L \oplus A)$ such that $\beta\widetilde{\Psi} - \alpha = \beta\eta$. Since $\mathrm{Hom}(P, JP) = J\mathrm{Hom}(P, P)$ for any finitely generated projective A-module P, setting $\Psi = \widetilde{\Psi} - \eta$ we see that Ψ is a lift of Ψ' and $\beta\Psi = \alpha$.

Let $\beta = (\phi, a)$ and $\alpha = (\psi, b)$ where $\phi, \psi \in \mathrm{Hom}\ (L, A)$. Then, we can write the equality $\beta\Psi = \alpha$ in the following matrix form:

$$\begin{pmatrix} c & \theta \\ l & d \end{pmatrix}\begin{pmatrix} \phi \\ a \end{pmatrix} = \begin{pmatrix} \psi \\ b \end{pmatrix},$$

where $\theta \in \mathrm{Hom}(L, A)$, $l \in L = \mathrm{Hom}\ (A, L)$. Moreover c, d denote homothety of the modules L and A respectively.

By (2.1), det $(\Psi) = cd - \theta(l)$. Since Ψ is a lift of Ψ' and $\det(\Psi') = 1$, we see that $cd - \theta(l) = 1 - f, f \in J$. As $\alpha = (\psi, b)$ is a surjection, it follows that there exist $l' \in L$ and $e \in A$ such that $f = eb - \psi(l')$. Since $\psi = c\phi + a\theta$ and $b = \phi(l) + ad$, it is easy to see by (2.1) and by computing determinants, that the endomorphism Δ of $L \oplus A$ given by

$$\begin{pmatrix} c + ea & \theta - e\phi \\ l + al' & d - \phi(l') \end{pmatrix}$$

is an automorphism of determinant 1 with $\beta\Delta = \alpha$. $\square$

The following lemma is proved in ([MK-2], Lemma 1) in the case where A is reduced.

LEMMA 2.4. *Let A be a Noetherian ring of dimension n and $J \subset A$ be an ideal of height n. Let P, P_1 be two projective A-modules of rank n and let $\alpha : P \twoheadrightarrow J$, $\beta : P_1 \twoheadrightarrow J$ be surjections. Then, there exists an injective homomorphism $\Psi : P \hookrightarrow P_1$ such that $\beta\Psi = \alpha$.*

Proof. Since P is projective and β is surjective, there exists a homomorphism $\Phi : P \to P_1$ such that $\beta\Phi = \alpha$. Moreover, by (2.2), given any such homomorphism Φ, we have $\Phi \otimes A/J: P/JP \to P_1/JP_1$ is an isomorphism. Hence, there exists $a \in A$ which is a unit modulo J such that Φ_a is an isomorphism. If a is a non-zero divisor, then Φ is injective and hence we are through.

If Φ is not injective, then we show below that there exists a homomorphism $\Theta : P \to \ker(\beta)$ such that $\Psi = \Phi + \Theta$ is an injective homomorphism from P to

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P_1. Note that by construction $\beta\Psi = \alpha$. Let K, N denote the kernels of α and β respectively.

From the above discussion, it follows that if Φ is not injective, then there exists at least one associated prime ideal of A which is comaximal with J. Let $\mathfrak{q}_i$, $1 \leqslant i \leqslant t$ be the associated prime ideals which are comaximal with J. Let $\mathfrak{m}_i$ be a maximal ideal containing $\mathfrak{q}_i$ and $J' = \cap \mathfrak{m}_i$.

Let bar denote reduction modulo J'. Since $J + J' = A$, we have the following commutative diagram of split short exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \overline{K} & \to & \overline{P} & \stackrel{\overline{\alpha}}{\to} & J/J'J = A/J' & \to & 0 \\
 & & \downarrow & & \overline{\Phi}\downarrow & & \| & & \\
0 & \to & \overline{N} & \to & \overline{P_1} & \stackrel{\overline{\beta}}{\to} & J/J'J = A/J' & \to & 0
\end{array}$$

Let $b \in J$ be such that $1 - b \in J'$ and $p \in P$ be such that $\alpha(p) = b$. Let $\Phi(p) = q$. Then, from the above diagram, it follows that $\overline{P} = \overline{K} \oplus \overline{A}\overline{p}$ and $\overline{P_1} = \overline{N} \oplus \overline{A}\overline{q}$. Therefore, as $\overline{K}, \overline{N}$ are free A/J'-modules of rank $n-1$ and $\Phi(K) \subset N$, it is easy to see that there exists $\theta \in \mathrm{Hom}_{A/J'}(\overline{P}, \overline{N})$ such that $\overline{\Phi} + \theta : \overline{P} \to \overline{P_1}$ is an isomorphism.

Let $\Theta \in \mathrm{Hom}_A(P, N)$ be a lift of θ and $\Psi = \Phi + \Theta$. Then $\overline{\Psi} : \overline{P} \to \overline{P_1}$ is an isomorphism. Moreover, as $\beta\Psi = \alpha$, by (2.2), $\Psi \otimes A/J : P \otimes A/J \to P_1 \otimes A/J$ is also an isomorphism. Now, since no associated prime ideal of A is comaximal with $J \cap J'$, it follows that Ψ is injective. $\square$

LEMMA 2.5. *Let A be a Noetherian ring of dimension 2 and $J \subset A$ an ideal of height 2 such that $J = (f, g) + J^2$. Let L be a projective A-module of rank 1. Then, there exists a projective A-module P of rank 2 having determinant isomorphic to L and a surjection from P to J.*

Proof. Let $Q = L \oplus A$ and $S = 1 + J$. Then, since Q_S is a free A_S-module of rank 2 and $J_S = (f, g)$, J_S is a surjective image of Q_S. Hence, there exists an element $b \in S$ such that J_b is a surjective image of Q_b. Let $a \in J$ be such that $b = 1 + a$ and $\alpha : Q_b \twoheadrightarrow J_b$ a surjection.

Since $a \in J$, $J_a = A_a$. Hence, there exists a surjection $\beta : Q_a (= L_a \oplus A_a) \twoheadrightarrow J_a$ such that $\beta(0, 1) = 1$.

Thus, we obtain two surjections α_a, β_b from Q_{ab} to $J_{ab} = A_{ab}$ such that $\ker(\alpha_a) \xrightarrow{\sim} L_{ab} \xrightarrow{\sim} \ker(\beta_b)$. Therefore, we get an automorphism Δ of Q_{ab} such that $\det(\Delta) = 1$ and $\beta_b \Delta = \alpha_a$.

Now, patching Q_a and Q_b via Δ, we get a projective A-module P of rank 2 and a surjection from P to J. Since $\det(\Delta) = 1$, it follows that $\det(P) \xrightarrow{\sim} L$. $\square$

LEMMA 2.6. *Let A be a ring and P a projective A-module of rank n. Let α be any element of P^*. Let $p_0, p_1, \cdots, p_n$ be $n+1$ elements of P.*

Let $\omega_i \in \wedge^n(P)$ be defined as follows: $\omega_0 = \alpha(p_0)(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$ and $\omega_i = \alpha(p_i)(p_0 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_n)$, $1 \leqslant i \leqslant n$. Then $\sum_{i=0}^{n} (-1)^i \omega_i = 0$.

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Proof. Let e denote the element $(1,0) \in A \oplus P$. The map $x \to e \wedge x$ is an isomorphism from $\wedge^n(P)$ to $\wedge^{n+1}(A \oplus P)$.

Let ω denote the element $\sum_{i=0}^{n}(-1)^i \omega_i$. Now consider the map $\gamma : P \to A \oplus P$ defined by $\gamma(p) = (\alpha(p), p)$. We obtain an induced map $\wedge^{n+1}\gamma : \wedge^{n+1}P \to \wedge^{n+1}(A \oplus P)$.

The image of the element $p_0 \wedge \cdots \wedge p_n$ of $\wedge^{n+1}(P)$ under $\wedge^{n+1}\gamma$ is $e \wedge \omega$. The lemma now follows from the fact that $\wedge^{n+1}(P)$ is zero (P being of rank n). □

LEMMA 2.7. *Let A be a Noetherian ring and P a projective A-module of rank n. Suppose that we are given the following short exact sequence*

$$0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\to} A \to 0.$$

Let $(a_0, p_0) \in A \oplus P$ be such that $a_0 b - \alpha(p_0) = 1$. Let $q_i = (a_i, p_i) \in P_1, 1 \leqslant i \leqslant n$. Then,

(i) *The map $\delta\colon \wedge^n(P_1) \to \wedge^n(P)$ given by $\delta(q_1 \wedge \cdots \wedge q_n) = a_0(p_1 \wedge p_2 \wedge \cdots \wedge p_n) + \sum_{i=1}^{n}(-1)^i a_i(p_0 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_n)$ is an isomorphism.*

(ii) $\delta(bq_1 \wedge \cdots \wedge q_n) = p_1 \wedge \cdots \wedge p_n$.

Proof. Let $e = (1,0), f = (a_0, p_0)$. Then $A \oplus P = Af \oplus P_1$ and $f \wedge q_1 \wedge \cdots \wedge q_n = e \wedge \omega$ in $\wedge^{n+1}(A \oplus P)$, where $\omega = a_0(p_1 \wedge p_2 \wedge \cdots \wedge p_n) + \sum_{i=1}^{n}(-1)^i a_i(p_0 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_n)$. Therefore (i) follows.

Since $q_i = (a_i, p_i) \in P_1$, we have $ba_i = \alpha(p_i)$. Moreover $ba_0 = 1 + \alpha(p_0)$. Therefore (ii) follows from (2.6). □

The proof of the following Lemma follows easily from (2.6) and (2.7).

LEMMA 2.8. *Let A be a Noetherian ring and P a projective A-module of rank n. Suppose that we are given the following short exact sequence*

$$0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\to} A \to 0.$$

Then,

(i) *The map $\beta : P_1 \to A$ given by $\beta(q) = c$, where $q = (c, p)$, has the property that $\beta(P_1) = \alpha(P)$.*

(ii) *The map $\Phi : P \to P_1$ given by $\Phi(p) = (\alpha(p), bp)$ has the property that $\beta\Phi = \alpha$ and $\delta \wedge^n (\Phi)$ (where δ is as in (2.7)) is scalar multiplication by b^{n-1}.*

The following Lemma is easy to prove and hence we omit the proof.

LEMMA 2.9. *Let A be a Noetherian ring and P a finitely generated projective A-module. Let $P[T]$ denote the projective $A[T]$-module $P \otimes_A A[T]$. Let $\alpha(T)\colon P[T] \twoheadrightarrow A[T]$ and $\beta(T)\colon P[T] \twoheadrightarrow A[T]$ be two surjections such that*

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$\alpha(0)=\beta(0)$. *Suppose further that the projective* $A[T]$*-modules* $\ker\alpha(T)$ *and* $\ker\beta(T)$ *are extended from* A. *Then there exists an automorphism* $\sigma(T)$ *of* $P[T]$ *with* $\sigma(0)=id$ *such that* $\beta(T)\sigma(T)=\alpha(T)$.

The following Lemma follows from the well known Quillen Splitting Lemma ([Q], Lemma 1) and its proof is essentially contained in ([Q], Theorem 1).

LEMMA 2.10. *Let* A *be a Noetherian ring and* P *a finitely generated projective* A*-module. Let* $a,b\in A$ *be such that* $Aa+Ab=A$. *Let* $\sigma(T)$ *be an* $A_{ab}[T]$*-automorphism of* $P_{ab}[T]$ *such that* $\sigma(0)=id$. *Then* $\sigma(T)=\tau(T)_a\theta(T)_b$, *where* $\tau(T)$ *is an* $A_b[T]$*-automorphism of* $P_b[T]$ *such that* $\tau(T)=id$ *modulo the ideal* (aT) *and* $\theta(T)$ *is an* $A_a[T]$*-automorphism of* $P_a[T]$ *such that* $\theta(T)=id$ *modulo the ideal* (bT).

LEMMA 2.11. *Let* A *be a Noetherian ring and let* J *be a proper ideal of* A. *Let* $J_1\subset J$ *and* $J_2\subset J^2$ *be two ideals of* A *such that* $J_1+J_2=J$. *Then* $J=J_1+(e)$ *for some* $e\in J_2$ *and* $J_1=J\cap J'$, *where* $J_2+J'=A$.

Proof. Since J/J_1 is an idempotent ideal of a Noetherian ring A/J_1 and J_2 maps surjectively onto J/J_1, there exists an element $e\in J_2$ such that $J_1+(e)=J$ and $e(1-e)\in J_1$. Therefore the result follows by taking $J'=J_1+(1-e)$. □

We conclude this section by quoting a theorem of Eisenbud and Evans ([E-E]) as stated in ([P], p. 1420) and deducing some consequences which will be used later.

THEOREM 2.12. *Let* A *be a Noetherian ring and* M *be a finitely generated* A*-module. Let* S *be a subset of Spec* A *and* $d:S\to\mathbf{N}$ *be a generalized dimension function. Assume that* $\mu_Q(M)\geqslant 1+d(Q)$ *for all* $Q\in S$. *Let* $(m,a)\in M\oplus A$ *be basic at all prime ideals* $Q\in S$. *Then there exists an element* $m'\in M$ *such that* $m+am'$ *is basic at all primes* $Q\in S$.

As a consequence of (2.12), we have the following result.

COROLLARY 2.13. *Let* A *be a Notherian ring and* P *be a projective* A*-module of rank* n. *Let* $(\alpha,a)\in(P^*\oplus A)$. *Then there exists an element* $\beta\in P^*$ *such that* $\mathrm{ht}(I_a)\geqslant n$, *where* $I=(\alpha+a\beta)(P)$. *In particular if the ideal* $(\alpha(P),a)$ *has height* $\geqslant n$ *then* $\mathrm{ht}\, I\geqslant n$. *Further, if* $(\alpha(P),a)$ *is an ideal of height* $\geqslant n$ *and* I *is a proper ideal of* A, *then* $\mathrm{ht}\, I=n$.

Proof. Let S denote the subset of Spec A consisting of all prime ideals Q of A with the property: $a\notin Q$ and height of $Q\leqslant n-1$. Then by ([P], Example 1), there exists a generalized dimension function $d:S\to\mathbf{N}$ such that $d(Q)\leqslant n-1$ for all $Q\in S$. As $a\notin Q$ for all $Q\in S$, the element (α,a) of $P^*\oplus A$ is unimodular and hence basic at every member of S. Therefore by (2.12), there exists an element $\beta\in P^*$ such that $\alpha+a\beta$ is basic and hence (as P^* is projective) is unimodular at all prime ideals $Q\in S$.

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Let $I = (\alpha + a\beta)(P)$. As $\alpha + a\beta$ is unimodular at all prime ideals $Q \in S$, we have $I_Q = A_Q$ for every $Q \in S$. Hence $\mathrm{ht}(I_a) \geqslant n$. Since I is a surjective image of P, I is locally generated by n elements and hence if I is a proper ideal, then ht $I \leqslant n$. Therefore the rest of the conclusions follow. □

As an application of (2.11) and (2.13) we have

COROLLARY 2.14. *Let A be a Noetherian ring of dimension $n \geqslant 2$ and let P be a projective A-module of rank n. Let $J \subset A$ be an ideal of height n and let $\overline{\alpha} : P/JP \twoheadrightarrow J/J^2$ be a surjection. Then there exists an ideal $J' \subset A$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:*

(i) $J + J' = A$.
(ii) $\beta \otimes A/J = \overline{\alpha}$.
(iii) height $(J') \geqslant n$
(iv) Further, given finitely many ideals $J_1, J_2, \cdots, J_r$ of height $\geqslant 1$, J' can be chosen with the additional property that J' is comaximal with $J_1, J_2, \cdots, J_r$.

Proof. Let $K = J^2 \cap J_1 \cdots \cap J_r$ Then, by the assumption, ht $K \geqslant 1$. Therefore there exists an element $a \in K$ such that ht $Aa \geqslant 1$ and hence dim $A/Aa \leqslant n-1$.

CLAIM: The surjection $\overline{\alpha}$ can be lifted to a surjection from P/aP to J/Aa.

Proof of the claim. First note that $a \in J^2$ and hence $(J/Aa)^2 = J^2/Aa$. Let δ be a lift of $\overline{\alpha}$ in $\mathrm{Hom}_{A/Aa}(P/aP, J/Aa)$. Then $\delta(P/aP) + J^2/Aa = J/Aa$ and hence, by (2.11), there exists $c' \in J^2/Aa$ such that $\delta(P/aP) + (c') = J/Aa$. Now applying (2.13) to the element (δ, c') of $(P/aP)^* \oplus A/Aa$, we see that there exists $\gamma \in (P/aP)^*$ such that height of the ideal $N_{c'} \geqslant n$, where $N = (\delta + c'\gamma)(P/aP)$. Since dim$A/Aa \leqslant n-1$, this implies that $c'^r \in N$ for some positive integer r. Therefore, as $N + (c') = J/Aa$ and $c' \in (J/Aa)^2$, we have $N = J/Aa$. Thus, as $\delta + c'\gamma$ is also a lift of $\overline{\alpha}$, the claim is proved.

Let $\theta \in \mathrm{Hom}_A(P, J)$ be a lift of $\delta + c'\gamma$. Then, as $J/Aa = (\delta + c'\gamma)(P/aP)$, we have $\theta(P) + Aa = J$. Again applying (2.13) to the element (θ, a) of $P^* \oplus A$, we see that there exists $\psi \in P^*$ such that ht $J_1 = n$ where $J_1 = (\theta + a\psi)(P)$.

Since $J_1 + Aa = J$ and $a \in J^2$, by (2.11), $J_1 = J \cap J'$ and $Aa + J' = A$. Now, setting $\beta = \theta + a\psi$, the proof of the corollary is complete. □

3. Addition and Subtraction Principles

LEMMA 3.0. *Let A be a Noetherian ring of dimension n and P a projective A-module of rank n. Let $\lambda : P \twoheadrightarrow J_0$ and $\mu : P \twoheadrightarrow J_1$ be surjections, where $J_0, J_1 \subset A$ are ideals of height n. Then, there exists an ideal I of $A[T]$ of height n and a surjection*

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$\alpha(T) : P[T] \twoheadrightarrow I$ such that $I(0) = J_0, \alpha(0) = \lambda$ and $I(1) = J_1, \alpha(1) = \mu$, where for $a \in A$, $I(a) = \{F(a) : F(T) \in I\}$.

Proof. Let $\lambda(T) = \lambda \otimes A[T]$ and $\mu(T) = \mu \otimes A[T]$. Let $\alpha(T) = T\mu(T) + (1 - T)\lambda(T)$. Then $\alpha(0) = \lambda, \alpha(1) = \mu$. Further $\alpha(T)(P[T]) + (T(1 - T)) = (J_0A[T], T) \cap (J_1A[T], T - 1)$. Therefore replacing $\alpha(T)$ by $\alpha(T) + T(1 - T)\beta(T)$ for a suitable $\beta(T) \in P[T]^*$, we may assume, by (2.13), that $\alpha(P[T]) = I$ has height n. This proves the lemma. □

PROPOSITION 3.1. *Let A be a Noetherian ring of dimension $n \geqslant 2$ such that $(n-1)!$ is invertible in A. Let P and L be projective A-modules of rank n and 1 respectively such that the determinant of P is isomorphic to L. Let $P' = L \oplus A^{n-1}$ and let $\chi : \wedge^n(P') \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Suppose that $\alpha(T) : P[T] \twoheadrightarrow I$ is a surjection, where $I \subset A[T]$ is an ideal of height n. Then, there exists a homomorphism $\phi : P' \to P$, an ideal $K \subset A$ of height $\geqslant n$ which is comaximal with $I \cap A$ and a surjection $\rho(T) : P'[T] \twoheadrightarrow I \cap KA[T]$ such that:*

(i) $\wedge^n(\phi) = u\chi$ *where* $u = 1$ *modulo* $I \cap A$.
(ii) $(\alpha(0)\phi)(P') = I(0) \cap K$.
(iii) $\alpha(T).\phi(T) \otimes A[T]/I = \rho(T) \otimes A[T]/I$.
(iv) $\rho(0) \otimes A/K = \rho(1) \otimes A/K$.

Proof. We first show the existence of ϕ satisfying (i) and (ii).

Let $N = (I \cap A)^2$. Since height $(I) = n$, height $I \cap A \geqslant n - 1$ and hence $\dim A/N \leqslant 1$. Now since P, P' have determinant L, there exists an isomorphism $P'/NP' \xrightarrow{\sim} P/NP$. Now, using the fact that P'/NP' has a unimodular element, we can alter the given isomorphism by an automorphism of P'/NP' to obtain an isomorphism $\overline{\delta}$ such that $\wedge^n(\overline{\delta}) = \overline{\chi}$ where bar denotes reduction modulo N. Let δ be a lift of $\overline{\delta}$. Note that $\delta_{1+N} : P'_{1+N} \to P_{1+N}$ is an isomorphism.

Let $J = I(0)$, where $I(0) = \{F(0) | F(T) \in I\}$ and $\beta = \alpha(0) : P \twoheadrightarrow J$. The equality $\delta(P') + NP = P$, shows that $(\beta\delta)(P') + NJ = J$. Since $NJ \subset J^2$, by (2.11) there exists $c \in NJ$ such that $(\beta\delta)(P') + (c) = J$. Therefore, applying (2.13) to $(\beta\delta, c)$, we see that there exists $\gamma \in P'^*$ such that the ideal $(\beta\delta + c\gamma)(P')$ has height n. Since $(\beta\delta + c\gamma)(P') + (c) = J$ and $c \in J^2$, by (2.11), $(\beta\delta + c\gamma)(P') = J \cap K$, where K is either $= A$ or an ideal of height n which is comaximal with (c), hence with N and J.

Since $c \in NJ$, $c = \sum a_i d_i$, where $a_i \in N$ and $d_i \in J$. Any element of P'^* of the form $d\gamma$ (where $d \in J$) has its image contained in J. Now, as P' is projective, $d_i \in J$ and $\beta(P) = J$, it follows that there exists $v_i : P' \to P$ such that $\beta v_i = d_i\gamma$. Let $v = \sum a_i v_i$. Then, $c\gamma = \beta v$ where $v = 0$ modulo N. Let $\phi = (\delta + v)$. Then ϕ is also a lift of $\overline{\delta}$ and hence $\wedge^n\phi = u\chi$, where $u = 1$ modulo N. Moreover ϕ has the property that $\beta\phi(P') = J \cap K$. This proves (i) and (ii).

Since $K + N = A$, we have $I + KA[T] = A[T]$. Let $I' = I \cap KA[T]$. Then $I'(0) = J \cap K$ and $I'/I'^2 = I/I^2 \oplus KA[T]/K^2A[T]$.

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Let $B = A_{1+N}$. Note that $\phi_{1+N} : P' \otimes B \to P \otimes B$ is an isomorphism and $I'_{1+N} = I_{1+N}$. Therefore, the map $(\alpha(T)\phi(T))_{1+N} : P'[T] \otimes_{A[T]} B[T] \to I'_{1+N}$ is surjective. We choose $a \in N$ such that $1 + a \in K$ and $(\alpha(T)\phi(T))_{1+a}(P'_{1+a}[T]) = I'_{1+a}$. Since $a \in N \subset I$, $I'_a = KA_a[T]$. Therefore, we get a surjection $(\beta\phi) \otimes A_a[T] : P'_a[T] \twoheadrightarrow I'_a$. The elements $\beta(T)\phi(T)_{a(1+aA)}$ and $\alpha(T)\phi(T)_{a(1+aA)}$ are unimodular elements of $P'_{a(1+aA)}[T]^*$ and as $\alpha(0) = \beta$, they are equal modulo (T). Since $\dim A_{a(1+aA)} = n - 1$, rank $P' = n$ and $(n-1)!$ is invertible in A, by ([Ra], Corollary 2.5), the kernels of the surjections $\beta(T)\phi(T)_{a(1+aA)}$ and $\alpha(T)\phi(T)_{a(1+aA)}$ are locally free projective modules and hence by Quillen's local global principle ([Q], Theorem 1) these kernels are projective modules which are extended from $A_{a(1+aA)}$. Hence, by (2.9), there exists an automorphism $\sigma(T)$ of $P'_{a(1+aA)}[T]$ such that $\sigma(0) = \mathrm{id}$ and $(\alpha(T)\phi(T))_{a(1+aA)}\sigma(T) = (\beta\phi) \otimes A_{a(1+aA)}[T]$. Therefore, there exists an element $b \in A$ of the form $1 + ca$ such that b is a multiple of $1 + a$ and $\sigma(T)$ is an automorphism of $P'_{ab}[T]$ with $\sigma(0) = \mathrm{id}$. Hence, by (2.10), we see that $\sigma(T) = \tau(T)_a.\theta(T)_b$, where $\tau(T)$ is an $A_b[T]$-automorphism of $P'_b[T]$ such that $\tau(T) = \mathrm{id}$ modulo the ideal (aT) and $\theta(T)$ is an $A_a[T]$-automorphism of $P'_a[T]$ such that $\theta(T) = \mathrm{id}$ modulo the ideal (bT).

The surjections $(\alpha(T)\phi(T))_b.\tau(T) : (P'_b[T]) \twoheadrightarrow I'_b$ and $(\beta\phi) \otimes A_a(T).(\theta(T))^{-1} : P'_a[T] \twoheadrightarrow I'_a$ patch to yield a surjection $\rho(T) : P'[T] \twoheadrightarrow I'$.

Since $\theta(T) = \mathrm{id}$ modulo the ideal (bT), it follows from the construction of $\rho(T)$ that $\rho(0) \otimes A/K = \rho(1) \otimes A/K$. Further, using the fact that $\tau(T) = \mathrm{id}$ modulo the ideal (aT), we see that $\alpha(T).\phi(T) \otimes A[T]/I = \rho(T) \otimes A[T]/I$.

This proves (iii) and (iv) and hence the proposition. $\square$

THEOREM 3.2. (Addition Principle) *Let A be a Noetherian ring of dimension $n \geqslant 2$. Let J_1 and J_2 be two comaximal ideals of height n and $J_3 = J_1 \cap J_2$. Let Q be a projective A-module of rank $n - 1$ and $P = Q \oplus A$. Let $\theta_1 : P \twoheadrightarrow J_1$ and $\theta_2 : P \twoheadrightarrow J_2$ be surjections. Then, there exists a surjection $\theta : P \twoheadrightarrow J_3$ such that: $\theta \otimes A/J_1 = \theta_1 \otimes A/J_1$ and $\theta \otimes A/J_2 = \theta_2 \otimes A/J_2$.*

Proof. We regard θ_i as elements of $P^* = Q^* \oplus A$ and write $\theta_i = (\beta_i, a_i)$, $i = 1, 2$.

Let bar denote reduction modulo J_2. Since $\dim A/J_2 = 0$, the projective module $\overline{Q^*}$ has a unimodular element, say β'. Moreover the unimodular element $(\overline{\beta_1}, \overline{a_1})$ of $\overline{Q^*} \oplus \overline{A}$ can be taken to $(\beta', 0)$ by an elementary automorphism σ' of $\overline{Q^*} \oplus \overline{A}$. By ([B-R],Proposition 4.1), σ' can be lifted to an automorphism σ of $Q^* \oplus A$ which has determinant 1. Hence, we may replace (β_1, a_1) by $\sigma(\beta_1, a_1)$ and assume that $a_1 \in J_2$ and $\beta_1(Q)$ is comaximal with J_2. Now, since $\theta_1(P)$ has height n, by (2.13), there exists $\alpha_1 \in Q^*$ such that $(\beta_1 + a_1\alpha_1)(Q)$ has height $n - 1$. Note that, since $a_1 \in J_2$ and $\beta_1(Q)$ is comaximal with J_2, $(\beta_1 + a_1\alpha_1)(Q) + J_2 = A$. Since height $((\beta_1 + a_1\alpha_1)(Q)) = n - 1$, it follows that $\dim A/(\beta_1 + a_1\alpha_1)(Q) \leqslant 1$. Since the element (β_1, a_1) can be taken to $(\beta_1 + a_1\alpha_1, a_1)$ by a transvection of $Q^* \oplus A$, we can replace (β_1, a_1) by $(\beta_1 + a_1\alpha_1, a_1)$ and assume that (1) $\beta_1(Q) + J_2 = A$ and (2) $\dim A/\beta_1(Q) \leqslant 1$.

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Let $K = \beta_1(Q)$ and let $S = 1 + K$. Then, since $K + J_2 = A$, (β_2, a_2) is a unimodular element of ${Q^*}_S \oplus A_S$. Moreover, since K_S is in the Jacobson radical of A_S and $\dim A/K \leqslant 1$, (β_2, a_2) can be taken to the element $(0, 1)$ by an automorphism of ${Q^*}_S \oplus A_S$ of determinant 1. In fact if $n \geqslant 3$, then by ([Ba 1], Section 3, p. 178), this automorphism can be chosen to be a product of transvections. Therefore, there exists $s \in S$ and an automorphism Γ of ${Q^*}_s \oplus A_s$ of determinant 1 such that $\Gamma(\beta_2, a_2) = (0, 1)$. Since $S \cap J_2 \neq \emptyset$, without loss of generality we may assume that $s \in J_2$. Therefore $(J_3)_s = (J_1)_s$. Hence, we can regard $(\beta_1, a_1)_s$ as a surjection from $Q_s \oplus A_s (= P_s)$ to $(J_3)_s$.

Let $s = 1 + t, t \in K$. Then $(J_3)_t = (J_2)_t$. Hence, we can regard $(\beta_2, a_2)_t$ as a surjection from $Q_t \oplus A_t (= P_t)$ to $(J_3)_t$. Note that, as $t \in K = \beta_1(Q)$, β_1 becomes a unimodular element of ${Q^*}_t$ and hence (β_1, a_1) can be taken to the element $(0, 1)$ by an automorphism Δ of ${Q^*}_t \oplus A_t$ which is a product of transvections.

Thus, we obtain two surjections

$$(\beta_i, a_i)_{st} : Q_{st} \oplus A_{st} \twoheadrightarrow (J_3)_{st} = A_{st}, i = 1, 2.$$

Note that as elements of ${Q^*}_{st} \oplus A_{st}$ we have $\Phi {\Gamma_t}^{-1}(\beta_1, a_1) = (\beta_2, a_2)$, where $\Phi = (\Gamma^{-1})_t \Delta_s \Gamma_t$. Now since Δ_s is a product of transvections, so also is Φ. Hence Φ is isotopic to the identity automorphism of ${Q^*}_{st} \oplus A_{st}$. Hence, by ([Q]), $\Phi = (\Phi_2)_s (\Phi_1)_t$, where Φ_2 is an automorphism of ${Q^*}_t \oplus A_t$ of determinant 1 and Φ_1 is an automorphism of ${Q^*}_s \oplus A_s$ of determinant 1.

Let $\Phi_1 \Gamma^{-1}(\beta_1, a_1) = \psi_1$ and ${\Phi_2}^{-1}(\beta_2, a_2) = \psi_2$. Then ψ_1 and ψ_2 are surjections from $Q_s \oplus A_s (= P_s)$ to $(J_3)_s$ and $Q_t \oplus A_t (= P_t)$ to $(J_3)_t$ respectively, which patch up to give a surjection ψ from $Q \oplus A (= P)$ to J_3. By construction, $\psi \otimes A/J_i$ and $\theta_i \otimes A/J_i$ differ by an element of $SL(P/J_iP), i = 1, 2$. As $SL(P/J_iP) = E(P/J_iP)$, using ([B-R],Proposition 4.1), we alter ψ by an element of $SL(P)$ to obtain a surjection $\theta : P \twoheadrightarrow J_3$ such that: $\theta \otimes A/J_1 = \theta_1 \otimes A/J_1$ and $\theta \otimes A/J_2 = \theta_2 \otimes A/J_2$. $\square$

THEOREM 3.3. (Subtraction Principle) *Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let P and Q be projective A-modules of rank n and $n - 1$, respectively such that $\wedge^n(P) \xrightarrow{\sim} \wedge^{n-1}(Q)$. Let $\chi : \wedge^n(P) \xrightarrow{\sim} \wedge^n(Q \oplus A)$ be an isomorphism. Let $J \subset A$ be an ideal of height $\geqslant n$ and J' be an ideal of height n which is comaximal with J. Let $\alpha : P \twoheadrightarrow J \cap J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be surjections. Let bar denote reduction modulo J' and $\overline{\alpha} : \overline{P} \twoheadrightarrow J'/J'^2$, $\overline{\beta} : \overline{Q \oplus A} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively. Suppose that there exists an isomorphism $\delta : \overline{P} \xrightarrow{\sim} \overline{Q \oplus A}$ such that (i) $\overline{\beta}\delta = \overline{\alpha}$, (ii) $\wedge^n(\delta) = \overline{\chi}$. Then, there exists a surjection $\theta : P \twoheadrightarrow J$ such that $\theta \otimes A/J = \alpha \otimes A/J$.*

Proof. We note that to prove the result, we can always replace β by $\beta\sigma$ where σ is an automorphism of $Q \oplus A$ of determinant 1.

Let ν be the restriction of β to Q. Let $\beta(0, 1) = a$. Then, as an element of $Q^* \oplus A$, (ν, a) is unimodular modulo J^2. Let tilde denote reduction modulo J^2. Since $\dim A/J^2 = 0$, the projective module $\widetilde{Q^*}$ has a unimodular element say ν'. Moreover the unimodular element $(\widetilde{\nu}, \widetilde{a})$ of $\widetilde{Q^*} \oplus \widetilde{A}$ can be taken to $(\nu', 0)$ by an elementary

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automorphism $\widetilde{\sigma}$ of $\widetilde{Q^*} \oplus \widetilde{A}$. By ([B-R],Proposition 4.1), $\widetilde{\sigma}$ can be lifted to an automorphism σ^* of $Q^* \oplus A$ which has determinant 1. The element σ^* induces an automorphism σ of $Q \oplus A$ of determinant 1. Hence, we may replace β by $\beta\sigma$, and assume that $\beta = (v, a)$ has the property that $a \in J^2$ and $v(Q)$ is comaximal with J^2. Now, by (2.13), there exists $\tau \in Q^*$ such that the ideal $(v + a\tau)(Q)$ has height $n - 1$. Note that, since $a \in J^2$ and $v(Q)$ is comaximal with J^2, $(v + a\tau)(Q) + J^2 = A$. As the element (v, a) can be taken to $(v + a\tau, a)$ by a transvection of $Q^* \oplus A$, (which has determinant 1), we can as before assume by altering β that

(1) $v(Q) + J^2 = A$.
(2) $ht\ (v(Q)) = n - 1$.

Further, using (1), we may replace a by $v(q) + a$ for a suitable $q \in Q$ and assume that $a = 1$ modulo J^2. Note that by (2), dim $A/(v(Q)) \leqslant 1$.

We set

$$R = A[Y], K_1 = (v(Q)A[Y], Y + a), K_2 = JA[Y], K_3 = K_1 \cap K_2.$$

We note that $K_1(0) = J', K_3(0) = J \cap J'$.

We claim that there exists a surjection

$$\eta(Y) : P[Y] \twoheadrightarrow K_3$$

such that $\eta(0) = \alpha$.

We first show that the theorem follows from the claim. Specialising η at $Y = 1 - a$, we obtain a surjection

$$\theta : P \twoheadrightarrow J.$$

Since $a = 1$ modulo J^2, we have

$$\theta \otimes A/J = \eta(1 - a) \otimes A/J = \eta(0) \otimes A/J = \alpha \otimes A/J.$$

Hence the theorem follows.

We first prove the claim when $n \geqslant 3$.

Note that $A[Y]/K_1 \xrightarrow{\sim} A/(v(Q))$. Therefore dim $A[Y]/K_1 \leqslant 1$. Since the projective modules P and $Q \oplus A$ have the same determinant, it follows that there exists an isomorphism $\kappa(Y) : P[Y]/K_1P[Y] \xrightarrow{\sim} Q[Y]/K_1Q[Y] \oplus A[Y]/K_1$. We choose $\kappa(Y)$ such that $\wedge^n\kappa(Y) = \chi \otimes A[Y]/K_1$. We can choose an isomorphism with the above property, by choosing any isomorphism and altering it by a suitable automorphism of $Q[Y]/K_1Q[Y] \oplus A[Y]/K_1$. Since $\wedge^n(\delta) = \chi \otimes A/J'$, it follows that $\kappa(0)$ and δ differ by an element of $SL(Q/J'Q \oplus A/J')$. Therefore, by ([B-R], Proposition 4.1), we may alter $\kappa(Y)$ by an element of $SL(Q[Y]/K_1Q[Y] \oplus A[Y]/K_1)$ and assume that $\kappa(0) = \delta$. We have a surjection $(v \otimes A[Y], Y + a) : Q[Y] \oplus A[Y] \twoheadrightarrow K_1$. Tensoring this surjection with $A[Y]/K_1$ we obtain a surjection $\varepsilon(Y) : Q[Y]/K_1Q[Y] \oplus A[Y]/K_1 \twoheadrightarrow K_1/K_1^2$. Thus, we obtain a surjection $\pi(Y) = \varepsilon(Y)\kappa(Y) :$

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$P[Y]/K_1P[Y] \twoheadrightarrow K_1/K_1^2$. Since $\overline{\beta}\delta = \overline{\alpha}$, $\varepsilon(0) = \overline{\beta}$ and $\kappa(0) = \delta$, we have $\pi(0) = \alpha \otimes A/J'$. Therefore, by ([B-RS 1], Prop 3.7, [M-RS], Theorem 2.3), we obtain

$$\eta(Y) : P[Y] \twoheadrightarrow K_3$$

such that $\eta(0) = \alpha$.

Now we consider the case when $n = 2$.

With the notation as above, we show that there is a surjection $\eta(Y) : P[Y] \twoheadrightarrow K_3$ $\eta(0) = \alpha$.

Let $N = \nu(Q)$. Let $S = 1 + N$. We claim that there exists a surjection $\vartheta(Y) : P_{1+N}[Y] \twoheadrightarrow (K_3)_{1+N}$ such that $\vartheta(0) = \alpha_{1+N}$.

Note that since $N + J = A$, $(K_3)_{1+N} = (K_1)_{1+N}$. We have seen above that $\dim A/N \leqslant 1$, where $N = \nu(Q)$. It follows that $P_{1+N} \tilde{\to} Q_{1+N} \oplus A_{1+N}$. We choose an isomorphism $\xi : P_{1+N} \tilde{\to} Q_{1+N} \oplus A_{1+N}$ such that $\wedge^n(\xi) = \chi \otimes A_{1+N}$. This induces an isomorphism $\xi(Y) : P_{1+N}[Y] \tilde{\to} Q_{1+N}[Y] \oplus A_{1+N}[Y]$. We have a surjection $\pi(Y) = (\nu \otimes A_{1+N}[Y], Y + a) : Q_{1+N}[Y] \oplus A_{1+N}[Y] \twoheadrightarrow (K_3)_{1+N}$. Composing with $\xi(Y)$, we obtain a surjection

$$\vartheta(Y) = \pi(Y)\xi(Y) : P_{1+N}[Y] \twoheadrightarrow (K_3)_{1+N} = (K_1)_{1+N}.$$

Since $K_3(0) = J \cap J'$ and $J + N = A$, we have surjections $\vartheta(0) : P_{1+N} \twoheadrightarrow (J')_{1+N}$ and $\alpha_{1+N} : P_{1+N} \twoheadrightarrow (J')_{1+N}$.

Since $J'_{1+N}/J'^2_{1+N} = J'/J'^2$ and $\vartheta(0) = \beta_{1+N}\xi$, the above surjections give rise to surjections

$$\overline{\beta\xi} = \overline{\vartheta(0)} : P/J'P \tilde{\to} Q/J'Q \oplus A/J' \twoheadrightarrow J'/J'^2.$$

and

$$\overline{\beta}\delta = \overline{\alpha} : P/J'P \tilde{\to} Q/J'Q \oplus A/J' \twoheadrightarrow J'/J'^2.$$

Since $\wedge^n(\delta) = \chi \otimes A/J'$ and $\wedge^n(\xi) = \chi \otimes A_{1+N}$, it follows that $\vartheta(0) \otimes A/J'$ and $\alpha_{1+N} \otimes A/J'$ differ by an element of $SL(P/J'P)$. Since $P_{1+N} \tilde{\to} Q_{1+N} \oplus A_{1+N}$, by (2.3), $\vartheta(0)$ and α_{1+N} differ by an automorphism of P_{1+N} of determinant 1. Since $\mathrm{Aut}\, P_{1+N} \subset \mathrm{Aut}\, P_{1+N}[Y]$, we may alter $\vartheta(Y)$ by an automorphism of P_{1+N} and assume that $\vartheta(0) = \alpha_{1+N}$. Since $N + J = A$, we can choose an element $s \in J$ of the type $1 + t, t \in N$ such that there is a surjection $\vartheta(Y) : P_s[Y] \twoheadrightarrow (K_3)_s$ and $\vartheta(0) = \alpha_s$.

We claim that $\vartheta(Y)$ and $\alpha \otimes A_t[Y]$ can be modified suitably to yield a surjection

$$\eta(Y) : P[Y] \twoheadrightarrow K_3$$

such that $\eta(0) = \alpha$.

As $s = 1 + t \in J$ and $t \in N$, we have $(K_3)_{st} = A_{st}[Y]$. Therefore $(\vartheta(Y))_t$ and $\alpha \otimes A_{st}[Y]$ are surjections from $P_{st}[Y]$ to $A_{st}[Y]$. Hence, as rank $P = 2$, the kernels of $\vartheta(Y)_t$ and $\alpha \otimes A_{st}[Y]$ are isomorphic to the extended projective $A_{st}[Y]$-module

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$\wedge^2(P_{st})[Y]$. Hence, as $\vartheta(0) = \alpha_s$, by (2.9), there exists an automorphism $\Psi(Y)$ of $P_{st}[Y]$ such that $\Psi(0) = \mathrm{id}$ and $\vartheta(Y)_t\Psi(Y) = \alpha \otimes A_{st}[Y]$.

Therefore, by (2.10), we see that $\Psi(Y) = \Theta(Y)_t.\Phi(Y)_s$, where $\Phi(Y)$ is an $A_t[Y]$-automorphism of $P_t[Y]$ such that $\Phi(Y) = \mathrm{id}$ modulo Y and $\Theta(Y)$ is an $A_s[Y]$-automorphism of $P_s[Y]$ such that $\Theta(Y) = \mathrm{id}$ modulo Y.

The surjections $\vartheta(Y)\Theta(Y)$ and $\alpha \otimes A_t[Y].\Phi(Y)^{-1}$ patch to yield a surjection

$$\eta(Y) : P[Y] \twoheadrightarrow K_3$$

such that $\eta(0) = \alpha$. Thus the claim is proved and hence the proof of the theorem is complete. $\square$

Taking $J = A$ in the above theorem we obtain the following

COROLLARY 3.4. *Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let P and Q be projective A-modules of rank n and $n-1$ respectively such that $\wedge^n(P) \xrightarrow{\sim} \wedge^{n-1}(Q)$. Let $\chi : \wedge^n(P) \xrightarrow{\sim} \wedge^n(Q \oplus A)$ be an isomorphism. Let $J' \subset A$ be an ideal of height n. Let $\alpha : P \twoheadrightarrow J'$ and $\beta : Q \oplus A \twoheadrightarrow J'$ be two surjections. Let bar denote reduction modulo J' and $\overline{\alpha} : \overline{P} \twoheadrightarrow J'/J'^2$, $\overline{\beta} : \overline{Q \oplus A} \twoheadrightarrow J'/J'^2$ be surjections induced from α and β respectively. Suppose that there exists an isomorphism $\delta : \overline{P} \xrightarrow{\sim} \overline{Q \oplus A}$, such that (i) $\overline{\beta}\delta = \overline{\alpha}$, (ii) $\wedge^n(\delta) = \overline{\chi}$. Then P has a unimodular element.*

4. The Euler Class Group of a Noetherian Ring

For the rest of this paper, we assume that all rings considered contain the field $\mathbf{Q}$ of rational numbers. We make this assumption as we need to apply (3.1) to show that the 'Euler class' of a projective module is well defined.

Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let L be a rank 1 projective A-module. We define the *Euler Class group* of A *with respect to* L (denoted by $E(A, L)$) as follows:

Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let α and β be two surjections from $L/JL \oplus (A/J)^{n-1}$ to J/J^2. We say that α and β are *related* if there exists an automorphism σ of $L/JL \oplus (A/J)^{n-1}$ of determinant 1 such that $\alpha\sigma = \beta$. It is easily seen that this is an equivalence relation on the set of surjections from $L/JL \oplus (A/J)^{n-1}$ to J/J^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ a *local L-orientation of J*.

Note that since $\dim A/J = 0$, $SL_{A/J}(L/JL \oplus (A/J)^{n-1}) = E_{A/J}(L/JL \oplus (A/J)^{n-1})$ and therefore, by ([B-R], Proposition 4.1), the canonical map from $SL_A(L \oplus A^{n-1})$ to $SL_{A/J}(L/JL \oplus (A/J)^{n-1})$ is surjective. Hence if a surjection α from $L/JL \oplus (A/J)^{n-1}$ to J/J^2 can be lifted to a surjection $\theta : L \oplus A^{n-1} \twoheadrightarrow J$ then so can any β equivalent to α.

A local L-orientation $[\alpha]$ of J is called a *global L-orientation of J* if the surjection $\alpha : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ can be lifted to a surjection $\theta : L \oplus A^{n-1} \twoheadrightarrow J$.

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Hence we shall also, from now on, identify a surjection α with the equivalence class $[\alpha]$ to which α belongs.

Let $\mathcal{M} \subset A$ be a maximal ideal of height n and $\mathcal{N}$ be an $\mathcal{M}$-primary ideal such that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements. Let $\omega_{\mathcal{N}}$ be a local L-orientation of $\mathcal{N}$. Let G be the free Abelian group on the set of pairs $(\mathcal{N}, \omega_{\mathcal{N}})$, where $\mathcal{N}$ is a $\mathcal{M}$ primary ideal and $\omega_{\mathcal{N}}$ is a local L-orientation of $\mathcal{N}$.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary ($\mathcal{M}_i \subset A$ are distinct maximal ideals of height n). Assume that J/J^2 is generated by n elements. Let ω_J be a local L-orientation of J. Then, ω_J gives rise, in a natural way, to a local L-orientation $\omega_{\mathcal{N}_i}$ of $\mathcal{N}_i$. We associate to the pair (J, ω_J), the element $\sum(\mathcal{N}_i, \omega_{\mathcal{N}_i})$ of G. By abuse of notation, we denote the element $\sum(\mathcal{N}_i, \omega_{\mathcal{N}_i})$ by (J, ω_J).

Let H be the subgroup of G generated by set of pairs (J, ω_J), where J is an ideal of height n and ω_J is a global L-orientation of J.

We define $E(A, L) = G/H$. Thus $E(A, L)$ can be thought of as the quotient of the group of local L-orientations by the subgroup generated by global L-orientations. If $L = A$, we denote the group $E(A, L)$ by $E(A)$.

Now we discuss (3.1) in the context of the Euler class group $E(A, L)$. Let $I \subset A[T]$ be an ideal of height n which is a surjective image of a projective $A[T]$-module $P[T]$, where P is a projective A-module of rank n having determinant L. Further assume that $I(0)$ and $I(1)$ are ideals of height n. Now using the surjection from $P[T]$ to I and ϕ we get a 'local $L[T]$-orientation' $\omega(T)$ of I, which in its turn gives rise to local L-orientations $\omega(0)$ and $\omega(1)$ of $I(0)$ and $I(1)$ respectively. The gist of (3.1) is that there exists an ideal $K \subset A$ of height n and a local orientation ω_K of K such that

$$(I(0), \omega(0)) + (K, \omega_K) = 0 = (I(1), \omega(1)) + (K, \omega_K)$$

in $E(A, L)$. Therefore $(I(0), \omega(0)) = (I(1), \omega(1))$ in $E(A, L)$.

Let P be a projective A-module of rank n with determinant L. Let χ be an isomorphism from $\wedge^n(L \oplus A^{n-1})$ to $\wedge^n(P)$. We call χ an *L-orientation of P*. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(A, L)$ as follows:

Let $\lambda : P \twoheadrightarrow J_0$ be a surjection, where $J_0 \subset A$ is an ideal of height n. Let bar denote reduction modulo J_0. We obtain an induced surjection $\overline{\lambda} : P/J_0P \twoheadrightarrow J_0/{J_0}^2$. We choose an isomorphism $\overline{\gamma} : L/J_0L \oplus (A/J_0)^{n-1} \xrightarrow{\sim} P/J_0P$, such that $\wedge^n(\overline{\gamma}) = \overline{\chi}$. Let ω_{J_0} be the local L-orientation of J_0 given by $\overline{\lambda}\overline{\gamma} : L/J_0L \oplus (A/J_0)^{n-1} \twoheadrightarrow J_0/{J_0}^2$. Let $e(P, \chi)$ be the image in $E(A, L)$ of the element (J_0, ω_{J_0}) of G. (We say that (J_0, ω_{J_0}) is *obtained* from the pair (λ, χ)). We show that the assignment sending the pair (P, χ) to the element $e(P, \chi)$ of $E(A, L)$ is well defined.

Let $\mu : P \twoheadrightarrow J_1$ be another surjection, where $J_1 \subset A$ is an ideal of height n. Then, by (3.0), there exists a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ (where $I \subset A[T]$ is an ideal of height n) with $\alpha(0) = \lambda$ and $\alpha(1) = \mu$. It now follows from the above discussion, that $e(P, \chi)$ is a well defined element of $E(A, L)$.

We define the *Euler Class* of (P, χ) to be $e(P, \chi)$.

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Remark 4.0. If the ring A is Cohen–Macaulay and J is an ideal of height n such that J/J^2 is generated by n elements, then J/J^2 is a free A/J-module of rank n and hence a local L-orientation ω_J of J gives rise to a unique isomorphism $L/JL \xrightarrow{\sim} \wedge^n(J/J^2)$. Conversely, an isomorphism $L/JL \xrightarrow{\sim} \wedge^n(J/J^2)$ gives rise to a local L-orientation of J.

PROPOSITION 4.1. *Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let $J, J_1, J_2 \subset A$ be ideals of height n such that J is comaximal with J_1 and J_2. Assume further that there exist surjections*

$$\alpha : L \oplus A^{n-1} \twoheadrightarrow J \cap J_1, \beta : L \oplus A^{n-1} \twoheadrightarrow J \cap J_2$$

with $\alpha \otimes A/J = \beta \otimes A/J$. Suppose that there exists an ideal J_3 of height n such that: (i) J_3 is comaximal with J, J_1 and J_2, and (ii) there exists a surjection $\gamma : L \oplus A^{n-1} \twoheadrightarrow J_3 \cap J_1$ with $\alpha \otimes A/J_1 = \gamma \otimes A/J_1$. Then, there exists a surjection $\delta : L \oplus A^{n-1} \twoheadrightarrow J_3 \cap J_2$ with $\delta \otimes A/J_3 = \gamma \otimes A/J_3$ and $\delta \otimes A/J_2 = \beta \otimes A/J_2$.

Proof. By (2.14), there exists an ideal J_4 of height $\geqslant n$ such that J_4 is comaximal with J, J_1, J_2, J_3; and $\eta : L \oplus A^{n-1} \twoheadrightarrow J \cap J_4$ with $\alpha \otimes A/J = \eta \otimes A/J$. Thus, we have the following equations:

$$\alpha \otimes A/J = \beta \otimes A/J \tag{1}$$
$$\alpha \otimes A/J_1 = \gamma \otimes A/J_1 \tag{2}$$
$$\alpha \otimes A/J = \eta \otimes A/J \tag{3}$$

Now, applying (3.2) with $Q = L \oplus A^{n-2}$, we obtain a surjection $\mu : L \oplus A^{n-1} \twoheadrightarrow (J_3 \cap J_1) \cap (J \cap J_4)$ such that

$$\mu \otimes A/J_3 \cap J_1 = \gamma \otimes A/J_3 \cap J_1 \tag{4}$$
$$\mu \otimes A/J \cap J_4 = \eta \otimes A/J \cap J_4 \tag{5}$$

Therefore, using equations (3,5) and (2,4), we see that $\mu \otimes A/J \cap J_1 = \alpha \otimes A/J \cap J_1$. Since there exists a surjection $\mu : L \oplus A^{n-1} \twoheadrightarrow (J_3 \cap J_1) \cap (J \cap J_4) = (J \cap J_1) \cap (J_3 \cap J_4)$, applying (3.3) (with $Q = L \oplus A^{n-2}$ and $P = L \oplus A^{n-1}$), we see that there exists a surjection $\nu : L \oplus A^{n-1} \twoheadrightarrow J_3 \cap J_4$ such that:

$$\mu \otimes A/J_3 \cap J_4 = \nu \otimes A/J_3 \cap J_4 \tag{6}$$

Now, by (3.2), there exists a surjection $\lambda : L \oplus A^{n-1} \twoheadrightarrow (J \cap J_2) \cap (J_3 \cap J_4)$ such that:

$$\lambda \otimes A/J \cap J_2 = \beta \otimes A/J \cap J_2 \tag{7}$$
$$\lambda \otimes A/J_3 \cap J_4 = \nu \otimes A/J_3 \cap J_4 \tag{8}$$

Therefore, using equations (1,3,7) and (5,6,8), we see that $\lambda \otimes A/J \cap J_4 = \eta \otimes A/J \cap J_4$. Since there exist surjections $\lambda : L \oplus A^{n-1} \twoheadrightarrow (J \cap J_4) \cap (J_3 \cap J_2)$ and $\eta : L \oplus A^{n-1} \twoheadrightarrow J \cap J_4$, applying (3.3), we get a surjection $\delta : L \oplus A^{n-1} \twoheadrightarrow$

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$J_3 \cap J_2$ such that $\delta \otimes A/J_3 \cap J_2 = \lambda \otimes A/J_3 \cap J_2$. Now applying (4,6,8) and (7), the proposition follows. $\square$

THEOREM 4.2. *Let A be a Noetherian ring of dimension $n \geqslant 2$. Let L be a rank 1 projective A-module. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements, and let $\omega_J : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ be a local L-orientation of J. Suppose that the image of (J, ω_J) is zero in the Euler Class group $E(A, L)$ of A. Then, ω_J is a global L-orientation of J.*

Proof. Since (J, ω_J) is zero in $E(A, L)$, there exists a family of (not necessarily distinct) pairs $\{(J_t, \omega_t) | 1 \leqslant t \leqslant r + s\}$ such that (1) J_t are ideals of height n (2) there exist surjections $\alpha_t : L \oplus A^{n-1} \twoheadrightarrow J_t$ such that $\omega_t = \alpha_t \otimes A/J_t$ and (3) the following equality

$$(J, \omega_J) + \sum_{l=r+1}^{r+s} (J_l, \omega_l) = \sum_{t=1}^{r} (J_t, \omega_t) \qquad (*)$$

holds in the free Abelian group G.

We first consider the case when $J_1, J_2, \cdots, J_r$ are pairwise comaximal. In this case $J, J_{r+1}, \cdots, J_{r+s}$ are also pairwise comaximal. Let $J'' = \cap_{t=1}^{r} J_t$ and $J' = \cap_{l=r+1}^{r+s} J_l$. From the equality $(*)$ in the group G and the addition principle (3.2), we have:

(i) $J \cap J' = J''$.
(ii) There exists a surjection $\alpha' : L \oplus A^{n-1} \twoheadrightarrow J'$ such that if $\omega_{J'} = \alpha' \otimes A/J'$ then $(J', \omega_{J'}) = \sum_{l=r+1}^{r+s} (J_l, \omega_l)$ in G.
(iii) There exists a surjection $\alpha'' : L \oplus A^{n-1} \twoheadrightarrow J''$ such that if $\omega_{J''} = \alpha'' \otimes A/J''$ then $(J'', \omega_{J''}) = \sum_{t=1}^{r} (J_t, \omega_t)$ in G.

Thus, we may assume by $(*)$ that $\alpha'' \otimes A/J' = \alpha' \otimes A/J'$. Therefore, applying (3.3), we obtain a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$, such that $\alpha \otimes A/J = \alpha'' \otimes A/J$. It is easy to see that $\omega_J = \alpha \otimes A/J$. Therefore, the theorem is proved in this case.

We now consider the case when $J_1, J_2, \cdots, J_r$ are not pairwise comaximal. Given an equality of the type $(*)$, we associate a non-negative integer $n(*)$ in the following manner: For a maximal ideal $\mathcal{M}$ of A, we associate a number $n(\mathcal{M})$ as follows: $n(\mathcal{M}) + 1$ is the cardinality of the set $\{t | \mathcal{M} \in V(J_t), 1 \leqslant t \leqslant r\}$. Let $n(*) = \sum n(\mathcal{M})$, where the summation is over all those maximal ideals $\mathcal{M}$ of A such that $n(\mathcal{M}) \geqslant 0$. We note that $n(*) = 0$ if and only if $J_1, J_2, \cdots, J_r$ are pairwise comaximal.

We now consider the case when $J_1, J_2, \cdots, J_r$ are not pairwise comaximal (i.e. $n(*)$ is positive). Let $\mathcal{M}$ be a maximal ideal such that $n(\mathcal{M})$ is positive. This implies that there exists an $\mathcal{M}$-primary ideal $\mathcal{N}$, a surjection $\omega_{\mathcal{N}} : L/\mathcal{N}L \oplus A/\mathcal{N}^{n-1} \twoheadrightarrow \mathcal{N}/\mathcal{N}^2$ and integers l, t such that

(i) $r + 1 \leqslant l \leqslant r + s, 1 \leqslant t \leqslant r$.

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(ii) $\mathcal{N}$ is a primary component of J_l and J_t.
(iii) The surjections $\alpha_l : L \oplus A^{n-1} \twoheadrightarrow J_l$, $\alpha_t : L \oplus A^{n-1} \twoheadrightarrow J_t$ have the property that $\alpha_l \otimes A/\mathcal{N} = \omega_{\mathcal{N}}$ and $\alpha_t \otimes A/\mathcal{N} = \omega_{\mathcal{N}}$.

We can assume, without loss of generality, that $l = r+1$ and $t = 1$. Let $\mathcal{N} \cap K_1 = J_1, \mathcal{N} \cap K_2 = J_{r+1}$ where $\mathcal{N} + K_1 = A = \mathcal{N} + K_2$.

By (2.14), we can find an ideal K_3 of height $\geqslant n$, such that:

(1) K_3 is comaximal with $J, J_j, 1 \leqslant j \leqslant r+s$.
(2) There exists a surjection $\beta_1 : L \oplus A^{n-1} \twoheadrightarrow K_3 \cap K_1$
(3) $\alpha_1 \otimes A/K_1 = \beta_1 \otimes A/K_1$

Therefore, applying (4.1), we see that there exists a surjection $\beta_{r+1} : L \oplus A^{n-1} \twoheadrightarrow K_3 \cap K_2$ such that $\alpha_{r+1} \otimes A/K_2 = \beta_{r+1} \otimes A/K_2$, and $\beta_1 \otimes A/K_3 = \beta_{r+1} \otimes A/K_3$. Hence, the following equality

$$(J, \omega_J) + (\widetilde{J_{r+1}}, \omega_{\widetilde{J_{r+1}}}) + \sum_{l=r+2}^{r+s} (J_l, \omega_l) = (\widetilde{J}_1, \omega_{\widetilde{J}_1}) + \sum_{t=2}^{r} (J_t, \omega_t) \tag{$**$}$$

holds in the free Abelian group G, where $\widetilde{J_{r+1}} = K_3 \cap K_2, \omega_{\widetilde{J_{r+1}}} = \beta_{r+1} \otimes A/K_3 \cap K_2$ and $\widetilde{J}_1 = K_3 \cap K_1, \omega_{\widetilde{J}_1} = \beta_1 \otimes A/K_3 \cap K_1$. It is easy to see that $n(**) \leqslant (n(*) - 1)$. Therefore, by induction, the proof is complete. $\square$

COROLLARY 4.3. *Let A be a Noetherian ring of dimension $n \geqslant 2$. Let P be a projective A-module of rank n with determinant L and χ be an L-orientation of P. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let ω_J be be a local L-orientation of J. Suppose that $e(P, \chi) = (J, \omega_J)$ in $E(A, L)$. Then, there exists a surjection $\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is obtained from (α, χ).*

Proof. We can regard ω_J as a surjection $L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$. We choose an isomorphism $\lambda : P/JP \xrightarrow{\sim} L/JL \oplus (A/J)^{n-1}$ such that $\wedge^n(\lambda) = \overline{\chi}$ (where bar denotes reduction modulo J). We consider the surjection $\overline{\alpha} = \omega_J \lambda : P/JP \twoheadrightarrow J/J^2$. By (2.14), there exists an ideal $J' \subset A$ and a surjection $\beta : P \twoheadrightarrow J \cap J'$ such that:

(i) $J + J' = A$.
(ii) $\beta \otimes A/J = \overline{\alpha}$.
(iii) height $(J') \geqslant n$.

If $J' = A$, the surjection β satisfies the required property. Otherwise, we have $e(P, \chi) = (J, \omega_J) + (J', \omega_{J'})$ in $E(A, L)$. Hence, by the assumption of the theorem, $(J', \omega_{J'}) = 0$ in $E(A, L)$. Therefore, by (4.2), there exists a surjection $\gamma : L \oplus A^{n-1} \twoheadrightarrow J'$ such that $\omega_{J'} = \gamma \otimes A/J'$. Now applying (3.3), with $Q = L \oplus A^{n-2}$, we get a surjection $\alpha : P \twoheadrightarrow J$ such that (J, ω_J) is obtained from the pair (α, χ). $\square$

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COROLLARY 4.4. *Let A be a Noetherian ring of dimension $n \geqslant 2$. Let P be a projective A-module of rank n with determinant L and χ be an L-orientation of P. Then $e(P, \chi) = 0$ if and only if P has a unimodular element. In particular if the determinant of P is trivial and P has a unimodular element then P maps onto any ideal of height n generated by n elements.*

Proof. Let $\alpha : P \twoheadrightarrow J'$ be a surjection where J' is an ideal of height n. Let $e(P, \chi) = (J', \omega_{J'})$ in $E(A, L)$, where $(J', \omega_{J'})$ is obtained from the pair (α, χ).

First assume that $e(P, \chi) = 0$. Then, by (4.2), there exists a surjection $\beta : L \oplus A^{n-1} \twoheadrightarrow J'$ such that and $\omega_{J'} = \beta \otimes A/J'$.

Now applying (3.4) with $Q = L \oplus A^{n-2}$, we see that P has a unimodular element.

Now we assume that $P = Q \oplus A$. Then $\alpha = (\theta, a)$ as an element of $P^* = Q^* \oplus A$. By performing an elementary automorphism of P, we may assume by (2.13), that height $(\theta(Q)) = n - 1$. Let $K = \theta(Q)$.

Note that the since determinant of Q is isomorphic to L, without loss of generality we may assume that χ is induced by an isomorphism $\chi' : \wedge^{n-1}(L \oplus A^{n-2}) \xrightarrow{\sim} \wedge^{n-1}(Q)$.

Let $Q_1 = L \oplus A^{n-2}$. Then, since $\dim A/K \leqslant 1$, there exists an isomorphism $\gamma' : Q_1/KQ_1 \xrightarrow{\sim} Q/KQ$ such that $\wedge^{n-1}\gamma' = \chi'$ modulo K. The surjection $(\theta \otimes A/K)\gamma' : Q_1/KQ_1 \twoheadrightarrow K/K^2$ can be lifted to a map $\delta : Q_1 \to K$ such that $\delta(Q_1) + K^2 = K$. Let $\delta(Q_1) = K'$. Then, since $K' + K^2 = K$, by (2.11) we have $K' + (e) = K$ with $e \in K^2$ and $e^2 - e \in K'$. Therefore, by ([MK 1], Lemma 1), $J' = K' + (b)$, where $b = e + (1 - e)a$.

Now consider the surjection $(\delta, b) : L \oplus A^{n-1} \twoheadrightarrow J'$. As $e \in K^2$, it is easy to see that $\omega_{J'}$ is obtained by tensoring the surjection (δ, b) with A/J'. Hence, by definition, $e(P, \chi) = 0$ in $E(A, L)$.

The last assertion of the corollary follows from (4.3). $\square$

Remark 4.5. Let A and P be as in (4.4). Further assume that $P = Q \oplus A$. One can also show that $e(P, \chi) = 0$ in the following manner. We only give an indication and leave the details to the reader. Let $\beta : Q \oplus A \twoheadrightarrow J$ be any surjection, where J has height n and $\gamma : Q \oplus A \twoheadrightarrow A$ be the projection map. As in (3.0), we can obtain a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ with $\alpha(0) = \beta$ and $\alpha(1) = \gamma$, where $I \subset A[T]$ is an ideal of height n. Now using (3.1), and the discussion in Section 4 preceding the definition of $e(P, \chi)$, it is easy to show that $e(P, \chi) = 0$. We have however preferred the approach of (4.4) as it is more elementary.

Let A be a Noetherian ring of dimension $n \geqslant 2$ and L a projective A-module of rank 1. Let N denote the nil radical of A and let $\overline{A} = A/N, \overline{L} = L/NL$. Let $J \subset A$ be an ideal of height n with primary decomposition $J = \cap \mathcal{N}_i$. Then, $\overline{J} = (J + N)/N \subset \overline{A}$ is an ideal of height n with primary decomposition $\overline{J} = \cap \overline{\mathcal{N}_i}$. Moreover, any surjection $\omega_J : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ induces a surjection $\overline{\omega}_{\overline{J}}$ from $\overline{L}/\overline{J}\overline{L} \oplus (\overline{A}/\overline{J})^{n-1}$ to $J + N/J^2 + N$. From this discussion, it follows that the assignment sending (J, ω_J) to $(\overline{J}, \overline{\omega}_{\overline{J}})$ gives rise to a group homomorphism $\Phi : E(A, L) \to E(\overline{A}, \overline{L})$.

As a consequence of (4.2), we have the following:

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COROLLARY 4.6. *The homomorphism* $\Phi : E(A, L) \to E(\overline{A}, \overline{L})$ *is an isomorphism.*

Proof. We only give the salient points of the proof.

Let $J \supset N$ be an ideal of height n and $\alpha : L \oplus A^{n-1} \to A$ be a linear map such that $K + J^2 + N = J$, where $K = \alpha(L \oplus A^{n-1})$. Then, there exists $e \in J^2$ such that

(1) $K + N + Ae = J$ and (2) $e^2 - e \in K + N$. Since N is nilpotent and idempotent elements can be lifted modulo a nilpotent ideal, it follows that there exists $f \in A$ such that $f - e \in K + N$ and $f^2 - f \in K$. Let $J_1 = K + Af$. Then $K + {J_1}^2 = J_1$ and $J_1 + N = K + N + Ae = J$. This shows that Φ is surjective.

Now suppose that J is an ideal of height n and ω_J is a local orientation of J such that the image of $(J, \omega_J) = 0$ in $E(\overline{A}, \overline{L})$.

This means that we are given surjections $\alpha : L \oplus A^{n-1} \twoheadrightarrow J/J^2$ (corresponding to ω_J) and $\beta : L \oplus A^{n-1} \twoheadrightarrow J + N/N = J/J \cap N$ such that they induce the same surjective map from $L \oplus A^{n-1}$ to $J/(J^2 + J \cap N)$. Since $J/J^2 \cap N$ is the fibre product of J/J^2 and $J/J \cap N$ over $J/(J^2 + J \cap N)$, α, β patch to yield a map $\delta : L \oplus A^{n-1} \to J/J^2 \cap N$. Let $\theta : L \oplus A^{n-1} \to J$ be a lift of δ. Then θ is a lift of α and β. Hence we have (1) $\theta(L \oplus A^{n-1}) + J^2 = J$, (2) $\theta(L \oplus A^{n-1}) + (J \cap N) = J$. Since N is nilpotent, we see, by (2), that $\theta(L \oplus A^{n-1})$ and J have the same radical. Therefore, by (1), $\theta(L \oplus A^{n-1}) = J$. Since θ is a lift of α, we see that $(J, \omega_J) = 0$ in $E(A, L)$. Hence Φ is injective. $\square$

Remark 4.7. Let A be a smooth affine domain of dimension $n \geqslant 2$ over a field of characteristic zero and let L be a projective A-module of rank 1. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Then, (see (4.0)), a local L-orientation ω_J of J induces in a canonical way an isomorphism from L/JL to $\wedge^n(J/J^2)$ and conversely. By abuse of notation, let ω_J also denote the corresponding isomorphism from L/JL to $\wedge^n(J/J^2)$. Therefore, in this case, we can give the following description of $E(A, L)$.

Let S be the set of pairs (m, ω_m), where m is a maximal ideal of A and ω_m is a local L-orientation of m. Let G be the free Abelian group generated by S. Let $J = \cap m_i$ be the intersection of finitely many maximal ideals and ω_J be a local L-orientation of J. Then ω_J gives rise, in a natural way, to local L-orientations ω_{m_i} of m_i.

We associate to the pair (J, ω_J), the element $\sum(m_i, \omega_{m_i})$ of G. By abuse of notation, we denote the element $\sum(m_i, \omega_{m_i})$ by (J, ω_J).

Let H be the subgroup of G generated by the set of pairs (J, ω_J), where J is the intersection of finitely many maximal ideals and ω_J is a global L-orientation of J.

Now suppose that $J \subset A$ is an ideal height n such that J/J^2 is generated by n elements and ω_J is a surjection from $L/JL \oplus (A/J)^{n-1}$ to J/J^2. Then, by Swan's Bertini theorem ([Sw],(1.3) and (1.4)), there exists an ideal $J' \subset A$ and a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J \cap J'$ such that (1) $J + J' = A$, (2) J' is a finite intersection of maximal ideals or $J' = A$ and (3) $\alpha \otimes A/J = \omega_J$.

In view of this, using (4.2), it follows easily that the canonical map from G/H to $E(A, L)$ is an isomorphism.

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We conclude this section by giving an example to show that the groups $E(A, L)$ may vary with L.

EXAMPLE 4.8. Let $X = \mathrm{Spec}\ A$ be an affine open subvariety of the projective 2-space $\mathbf{P}^2(\mathbf{R})$ which is the complement of $V(X^2 + Y^2 + Z^2)$. Then $E(A) = \mathbf{Z}/2$ (see ([B-RS 2], Corollary 6.3) for the proof). However, if L is the canonical module of A over $\mathbf{R}$, then $E(A, L) = \mathbf{Z}$. This can be shown using the methods of ([B-RS 2], (4.12) and (4.13)). But we do not go into the details since the proofs are rather involved.

5. Some Results on $E(A, L)$

Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let $J \subset A$ be an ideal of height n and $\omega_J : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ be a local L-orientation of J. Let $\overline{b} \in A/J$ be a unit. Composing ω_J with an automorphism of $L/JL \oplus (A/J)^{n-1}$ with determinant $\overline{b}$, we obtain another local L-orientation of J which we denote by $\overline{b}\omega_J$.

Remark 5.0. Let A and J as above and let $\omega_J, \widetilde{\omega}_J$ be two local L-orientations of J. Then, it is easy to see from (2.2), that $\widetilde{\omega}_J = \overline{b}\omega_J$ for some unit $\overline{b} \in A/J$.

The following lemma can be deduced from (2.7) and (2.8). The statement is a little complicated. Briefly it says that if there is an L-oriented projective module P of rank n whose Euler class is $= (J, \omega_J)$ and a is a unit modulo J, then there exists an L-oriented projective module P_1 of rank n such that P_1 is stably isomorphic to P and its Euler class $= (J, \overline{a^{n-1}}\omega_J)$.

LEMMA 5.1. *Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let P be a projective A-module of rank n with determinant L. Let χ be an L-orientation of P. Let $\alpha : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ is an ideal of height n and let (J, ω_J) be obtained from (α, χ). Let $a, b \in A$ be such that $ab = 1$ modulo J and let P_1 be the kernel of the surjection $A \oplus P \overset{(b,-\alpha)}{\twoheadrightarrow} A$. Let $\beta : P_1 \twoheadrightarrow J$ be as in (2.8) and χ_1 be the L-orientation of P_1 given by $\delta^{-1}\chi : \wedge^n(L \oplus A^{n-1}) \tilde{\to} \wedge^n(P_1)$ (where δ is as in (2.7)). Then $(J, \overline{a^{n-1}}\omega_J)$ is obtained from (β, χ_1).*

LEMMA 5.2. *Let A be a ring and $J \subset A$ be an ideal which is generated by two elements a_1, a_2. Let $a \in A$ be a unit modulo J and $b \in A$ be such that $ab = 1$ modulo J. Suppose that the unimodular row $(b, a_2, -a_1)$ is completable to a matrix in $SL_3(A)$. Then, there exists a matrix $\tau \in M_2(A)$ with $\det(\tau) = a$ modulo J such that $[a_1, a_2]\tau^t = [b_1, b_2]$, where b_1, b_2 generate J.*

Proof. We choose a completion $\sigma \in SL_3(A)$ of the unimodular row $(b, a_2, -a_1)$. Suppose that the second and the third rows of σ are $(d, \lambda_{11}, \lambda_{12})$ and $(e, \lambda_{21}, \lambda_{22})$. Let $\gamma : A^3 \twoheadrightarrow J$ be defined by setting $\gamma(1, 0, 0) = 0, \gamma(0, 1, 0) = a_1$ and $\gamma(0, 0, 1) = a_2$. The vectors $(b, a_2, -a_1)$, $(d, \lambda_{11}, \lambda_{12})$ and $(e, \lambda_{21}, \lambda_{22})$ generate A^3, since they are the rows of an invertible matrix. Hence their images under γ generate J.

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Hence $J = (\lambda_{11}a_1 + \lambda_{12}a_2, \lambda_{21}a_1 + \lambda_{22}a_2)$. Let τ be the 2×2 matrix whose entries are λ_{ij}. Then, since $\sigma \in SL_3(A)$ and $a_1, a_2 \in J$, it follows that det $\tau = a$ modulo J. It follows that the elements $b_1 = \lambda_{11}a_1 + \lambda_{12}a_2$ and $b_2 = \lambda_{21}a_1 + \lambda_{22}a_2$ satisfy the required properties. This proves the lemma. $\square$

LEMMA 5.3. *Let A be a Noetherian ring of dimension $n \geqslant 2$, $J \subset A$ an ideal of height n and $\omega_J : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ a surjection. Suppose that ω_J can be lifted to a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J$. Let $a \in A$ be a unit modulo J. Let θ be an automorphism of $L/JL \oplus (A/J)^{n-1}$ with determinant $\overline{a^2}$. Then, the surjection $\omega_J\theta : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ can be lifted to a surjection $\gamma : L \oplus A^{n-1} \twoheadrightarrow J$.*

Proof. We first prove lemma for $n \geqslant 3$. Let $P_2 = L \oplus A^{n-3}$. Then, we can regard α as a surjection from $P_2 \oplus A^2$ to J. Let $J_2 = \alpha(P_2)$ and tilde denote reduction modulo J_2. Let $\widetilde{\alpha}(0, 1, 0) = \widetilde{a_1}$ and $\widetilde{\alpha}(0, 0, 1) = \widetilde{a_2}$. By (5.2), there exists a matrix $\widetilde{\tau}$ in $M_2(\widetilde{A})$ such that $[\widetilde{a_1}, \widetilde{a_2}]\widetilde{\tau} = [\widetilde{b_1}, \widetilde{b_2}]$, where $\widetilde{J} = (\widetilde{b_1}, \widetilde{b_2})$ and det $(\widetilde{\tau}) = \widetilde{a}^2$ modulo J. We define a surjection $\gamma' : P_2 \oplus A^2 \twoheadrightarrow J$ by setting $\gamma'|P_2 = \alpha$, $\gamma'((0, 1, 0)) = b_1$ and $\gamma'((0, 0, 1)) = b_2$. From the construction of γ', it is easy to see that there exists an automorphism θ' of $L/JL \oplus (A/J)^{n-1}$ with determinant $\overline{a^2}$ such that the surjection $\omega_J\theta' = \gamma' \otimes A/J$. Since the map $SL(L \oplus A^{n-1}) \to SL(L/JL \oplus (A/J)^{n-1})$ is surjective ([B-R]), Proposition 4.1) and det $(\theta') =$ det $(\theta) = \overline{a^2}$, it follows that the surjection $\omega_J\theta : L/JL \oplus (A/J)^{n-1} \twoheadrightarrow J/J^2$ can be lifted to a surjection $\gamma : L \oplus A^{n-1} \twoheadrightarrow J$.

We now consider the case if $n = 2$. Let $\chi = id : \wedge^2(L \oplus A) \xrightarrow{\sim} \wedge^2(L \oplus A)$. Let $b \in A$ be such that $ba = 1$ modulo J. We consider the exact sequence

$$0 \to P_1 \to A \oplus L \oplus A \overset{(b^2, -\alpha)}{\to} A \to 0.$$

By ([Mu-Sw],Theorem 4), $P_1 \xrightarrow{\sim} L \oplus A$.

Further, by (5.1), we see that there exists a surjection $\beta : P_1 (= L \oplus A) \twoheadrightarrow J$ and $\chi_1 : \wedge^2(L \oplus A) \xrightarrow{\sim} \wedge^2(L \oplus A)$ such that $(J, \overline{a^2}\omega_J)$ is obtained from (β, χ_1). Note that $\chi = u\chi_1$, for $u \in A^*$. Now, using the fact that the map det: $\mathrm{Aut}(L \oplus A) \to A^*$ is surjective, it follows that there exists a surjection $\eta : L \oplus A \twoheadrightarrow J$, such that $(J, \overline{a^2}\omega_J)$ is obtained from (η, χ) (where $\chi = id$ as above). Now, using the fact that the map $SL(L \oplus A) \to SL(L/JL \oplus A/J)$ is surjective ([B-R], Proposition 4.1), it follows that there exists a surjection $\gamma : L \oplus A \twoheadrightarrow J$ such that $\gamma \otimes A/J = \omega_J\theta$. This proves the lemma. $\square$

We deduce from the above lemma the following result (see also ([B-RS 2], Lemma 3.4)).

LEMMA 5.4. *Let A be a Noetherian ring of dimension $n \geqslant 2$, $J \subset A$ an ideal of height n and ω_J be a local L-orientation of J. Let $\overline{a} \in A/J$ be a unit. Then $(J, \omega_J) = (J, \overline{a^2}\omega_J)$ in $E(A, L)$.*

Proof. If $(J, \omega_J) = 0$ in $E(A, L)$, then the result follows from (5.3). We assume therefore that $(J, \omega_J) \neq 0$ in $E(A, L)$. Then, by (2.14), there exists an ideal J_1 of height

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n which is comaximal with J and a surjection $\alpha : L \oplus A^{n-1} \twoheadrightarrow J \cap J_1$ such that $\alpha \otimes A/J = \omega_J$. Let $\alpha \otimes A/J_1 = \omega_{J_1}$. Let $b \in A$ be such that $b = a^2$ modulo J and $b = 1$ modulo J_1. Applying (5.3), we see that there exists a surjection $\gamma : L \oplus A^{n-1} \twoheadrightarrow J \cap J_1$ such that $\gamma \otimes A/J = \overline{a^2}\omega_J$ and $\gamma \otimes A/J_1 = \omega_{J_1}$. From the surjection α we get $(J, \omega_J) + (J_1, \omega_{J_1}) = 0$ in $E(A, L)$. From the surjection γ we get $(J, \overline{a^2}\omega_J) + (J_1, \omega_{J_1}) = 0$ in $E(A, L)$. Thus, $(J, \omega_J) = (J, \overline{a^2}\omega_J)$ in $E(A, L)$. This completes the proof of the proposition. $\square$

LEMMA 5.5. *Let A be a Noetherian domain of dimension $n \geqslant 2$. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let ω_J be a local orientation of J. Suppose that $(J, \omega_J) \neq 0$ in $E(A)$. Then, there exists an ideal J_1 of height n which is comaximal with J and a local orientation ω_{J_1} of J_1 such that $(J, \omega_J) + (J_1, \omega_{J_1}) = 0$ in $E(A)$. Further, given any non-zero element $f \in A$, J_1 can be chosen with the additional property that it is comaximal with (f).*

Proof. Let $\alpha : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection corresponding to ω_J. Then, by (2.14), there exists an ideal J_1 of height $\geqslant n$ which is comaximal with fJ and a surjection $\beta : A^n \twoheadrightarrow J \cap J_1$ such that $\beta \otimes A/J = \alpha$. Since $(J, \omega_J) \neq 0$ in $E(A)$, J_1 is a proper ideal of height n. Let ω_{J_1} be the local orientation of J_1 induced by β. Then $(J, \omega_J) + (J_1, \omega_{J_1}) = 0$ in $E(A)$. $\square$

LEMMA 5.6. *Let A be an affine domain over a field k of dimension $n \geqslant 2$ and f be a non-zero element of A. Let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Suppose that $(J, \omega_J) \neq 0$ in $E(A)$, but the image of $(J, \omega_J) = 0$ in $E(A_f)$. Then, there exists an ideal J_2 of height n such that $(J_2)_f = A_f$ and $(J, \omega_J) = (J_2, \omega_{J_2})$ in $E(A)$.*

Proof. Since $(J, \omega_J) \neq 0$ in $E(A)$ and $(J, \omega_J) = 0$ in $E(A_f)$, we see that the element $f \in A$ is not a unit. By (5.5), we can choose an ideal J_1 of height n which is comaximal with J and (f) such that $(J, \omega_J) + (J_1, \omega_{J_1}) = 0$ in $E(A)$. Since the image of $(J, \omega_J) = 0$ in $E(A_f)$, it follows that the image of $(J_1, \omega_{J_1}) = 0$ in $E(A_f)$.

Since the image of $(J_1, \omega_{J_1}) = 0$ in $E(A_f)$, by (4.2), we have, $(J_1)_f = (b_1, \cdots, b_n)$ and $\omega_{J_1} \otimes A_f$ is induced by the set of generators $b_1, \cdots, b_n$ of $(J_1)_f$ modulo ${(J_1)^2}_f$. Let f^k be so chosen such that $f^{2k} b_i \in J_1$, $1 \leqslant i \leqslant n$. Since f is a unit modulo J_1, by (5.4), we have $(J_1, \omega_{J_1}) = (J_1, \overline{f^{2kn}}\omega_{J_1})$ in $E(A)$. Therefore, without loss of generality, we can assume that $b_i \in J_1$. Moreover, adapting the proof of ([B], Proposition 3.1), it is easy to see that there exists an element $\sigma \in E_n(A_f)$ such that if $[b_1, \cdots, b_n]\sigma = [c_1, \cdots, c_n]$, then $c_i \in J_1$ and $c_1, \cdots, c_n$ generates an ideal of height n in A. Thus, $(c_1, \cdots, c_n) = J_1 \cap J_2$, where J_2 is an ideal of height n such that $(J_2)_f = A_f$. Since $(f) + J_1 = A$ and $(J_2)_f = A_f$, it follows that $J_1 + J_2 = A$. Further, by the construction of $c_1, \cdots, c_n$, it follows that ω_{J_1} is given by the set of generators $c_1, \cdots, c_n$ of J_1 modulo ${J_1}^2$. Therefore $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2}) = 0$ in $E(A)$, (where ω_{J_2} is given by the set of generators $c_1, \cdots, c_n$ of J_2 modulo ${J_2}^2$). Since $(J, \omega_J) + (J_1, \omega_{J_1}) = 0 = (J_2, \omega_{J_2}) + (J_1, \omega_{J_1})$ in $E(A)$, it follows that $(J, \omega_J) = (J_2, \omega_{J_2})$ in $E(A)$. This proves the lemma $\square$

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The following lemma is an immediate consequence of (4.3), (4.4) and (5.6).

LEMMA 5.7. *Let A be an affine domain over a field k of dimension $n \geqslant 2$ and P a projective A-module of rank n having trivial determinant. Let $f \in A$ be a non-zero element. Suppose that the projective A_f-module P_f has a unimodular element. Then there exists a surjection $\alpha : P \twoheadrightarrow J$ where $J \subset A$ is an ideal of height n such that $J_f = A_f$.*

Let A be a Noetherian ring of dimension n and P a projective module of rank n. Let $\alpha : P \twoheadrightarrow J$, be a surjection. We say that α is a *generic* surjection, if J has height n.

LEMMA 5.8. *Let A be an affine domain over a field k of dimension $n \geqslant 2$ and P a projective A-module of rank n with trivial determinant. Let $f \in A$ be a non-zero element. Assume that every generic surjection ideal of P is generated by n elements. Then, every generic surjection ideal of P_f is generated by n elements.*

Proof. Let $\beta : P_f \twoheadrightarrow \widetilde{J}$ be a generic surjection and $J' = \widetilde{J} \cap A$. Then, $J' \subset A$ is an ideal of height n which is comaximal with (f) such that $J'_f = \widetilde{J}$. Let χ be a generator of $\wedge^n(P)$ and let $(J'_f, \omega_{J'_f})$ be obtained from (β, χ_f). Using (5.4), we may replace $\omega_{J'_f}$ by $\overline{f^m}\omega_{J'_f}$ for some large suitably chosen even integer m and assume that $\omega_{J'_f}$ is given by a set of generators of J'/J'^2 which induce $\omega_{J'}$. The element $e(P, \chi) - (J', \omega_{J'})$ of $E(A)$ is zero in $E(A_f)$. It is easy to see that $e(P, \chi) - (J', \omega_{J'}) = (J_2, \omega_{J_2})$ in $E(A)$, where $J_2 \subset A$ is an ideal of height n. Moreover, by (5.6), we can assume that $(J_2)_f = A_f$. Since $(J_2)_f = A_f$ and J' is comaximal with (f), it follows that $J' + J_2 = A$. Since $e(P, \chi) = (J', \omega_{J'}) + (J_2, \omega_{J_2})$ in $E(A)$, it follows from (4.3), that there is a surjection $\gamma : P \twoheadrightarrow J' \cap J_2$. By the hypothesis of the lemma, $J' \cap J_2$ is generated by n elements. Hence $J'_f = (J' \cap J_2)_f$ is generated by n elements. This proves the lemma. $\square$

THEOREM 5.9. *Let A be an affine domain over* $\mathbf{R}$ *of dimension $n \geqslant 2$ and P a projective A-module of rank n and trivial determinant. Assume that for every generic surjection $\alpha : P \twoheadrightarrow J$, J is generated by n elements. Then P has a unimodular element.*

Proof. To any generic surjection $\alpha : P \twoheadrightarrow J$, we associate an integer $N(P, \alpha)$, which is equal to the number of real maximal ideals containing J. Let $t(P) = \min N(P, \alpha)$, where α varies over all generic surjections of P.

Case 1. Suppose that $t(P) = 0$. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection with $N(P, \alpha) = 0$. This means that J is contained only in complex maximal ideals. By assumption, J is generated by n elements. These n elements give rise to $\widetilde{\omega}_J$ such that the element $(J, \widetilde{\omega}_J) = 0$ in $E(A)$. Let χ be a generator of $\wedge^n(P)$ and $e(P, \chi) = (J, \omega_J)$ in $E(A)$. Then, by (2.2), $(J, \omega_J) = (J, \overline{u}\widetilde{\omega}_J)$ in $E(A)$, where $\overline{u} \in A/J$ is a unit. Since J is contained only in complex maximal ideals, $\overline{u}$ is a square. It follows now from (5.4), that $e(P, \chi) = (J, \omega_J) = (J, \overline{u}\widetilde{\omega}_J) = (J, \widetilde{\omega}_J) = 0$ in $E(A)$. Therefore, by (4.4), P has a unimodular element.

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Case 2. Suppose that $t(P) = 1$. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection with $N(P, \alpha) = 1$. This means that J is contained only in one real maximal ideal. By assumption J is generated by n elements. Hence, there exists $\widetilde{\omega}_J$ such that the element $(J, \widetilde{\omega}_J) = 0 = (J, -\widetilde{\omega}_J)$ in $E(A)$. Let χ be a generator of $\wedge^n(P)$ and $e(P, \chi) = (J, \omega_J)$ in $E(A)$. Let $(J, \omega_J) = (J, \overline{u}\widetilde{\omega}_J)$ in $E(A)$. Then, since J is contained only in one real maximal ideal, it follows as in Case 1 that either $\overline{u} \in A/J$ is a square or $-\overline{u}$ is a square. Therefore, it follows that either $(J, \omega_J) = (J, \widetilde{\omega}_J)$ or $(J, \omega_J) = (J, -\widetilde{\omega}_J)$ in $E(A)$. In any case, $(J, \omega_J) = 0$ in $E(A)$ and hence, by (4.4), P has a unimodular element.

Now we complete the proof by showing that under the assumption of the theorem $t(P) \leqslant 1$. Let $\alpha : P \twoheadrightarrow J$ be a generic surjection. If $N(P, \alpha) \leqslant 1$ there is nothing to prove. Now suppose $N(P, \alpha) = r \geqslant 2$. Let $m_1, \cdots, m_r$ be the real maximal ideals containing J. Let $f \in A$ be chosen so that f belongs to only the real maximal ideals $m_2, \cdots, m_r$. Then $N(P_f, \alpha_f) = 1$ and hence $t(P_f) \leqslant 1$. Since for every generic surjection $\alpha : P \twoheadrightarrow J$, J is generated by n elements, it follows from (5.8), that for every generic surjection $\beta : P_f \twoheadrightarrow J'_f$, J'_f is generated by n elements. Hence, by Cases 1 and 2, P_f has a unimodular element. Therefore, by (5.7), there exists a surjection $\gamma : P \twoheadrightarrow J_1$, where $J_1 \subset A$ is an ideal of height n such that $(J_1)_f = A_f$. Since $m_2, \cdots, m_r$ are the only real maximal ideals containing f, it follows that $N(P, \gamma) = r - 1$. Repeating this process we see that $t(P) \leqslant 1$. This proves the theorem. $\square$

6. The Weak Euler Class Group of a Noetherian Ring

Let A be a Noetherian ring with $\dim A = n \geqslant 2$. Let L be a projective module of rank 1. We define now the *weak Euler class group* $E_0(A, L)$ of A (with respect to L) as follows:

Let S be the set of ideals of $\mathcal{N} \subset A$ which have the property that $\mathcal{N}/\mathcal{N}^2$ is generated by n elements, where $\mathcal{N}$ is $\mathcal{M}$-primary ideal for some maximal ideal $\mathcal{M}$ of height n. Let G be the free Abelian group on the set S.

Let $J = \cap \mathcal{N}_i$ be the intersection of finitely many ideals $\mathcal{N}_i$, where $\mathcal{N}_i$ is $\mathcal{M}_i$-primary ($\mathcal{M}_i$ being distinct maximal ideals of height n). Assume that J/J^2 is generated by n elements.

We associate to J, the element $\sum \mathcal{N}_i$ of G. By abuse of notation, we denote this element by (J).

Let H be the subgroup of G generated by elements of the type (J), where $J \subset A$ is an ideal of height n such that there exists a surjection α from $L \oplus A^{n-1}$ to J.

We set $E_0(A, L) = G/H$.

Let P be a projective A-module of rank n with determinant L and $\lambda : P \twoheadrightarrow J_0$ be a surjection, where $J_0 \subset A$ is an ideal of height n. We define $e(P) = (J_0)$ in $E_0(A, L)$. We show that this assignment is well defined.

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Let $\mu : P \twoheadrightarrow J_1$ be another surjection, where J_1 is an ideal of height n. Then, by (3.0), there exists a surjection $\alpha(T) : P[T] \twoheadrightarrow I$ (where $I \subset A[T]$ is an ideal of height n) with $\alpha(0) = \lambda$ and $\alpha(1) = \mu$. Now, as before, using (3.1), we see that that $(J_0) = (J_1)$ in $E_0(A, L)$.

We note that there is a canonical surjective homomorphism from $E(A, L)$ to $E_0(A, L)$ obtained by forgetting the orientations. If $L = A$, we denote the group $E_0(A, A)$ by $E_0(A)$. Thus, there is a canonical surjection $E(A) \twoheadrightarrow E_0(A)$.

The following four propositions can be proved by using (5.1),(5.4),(5.5) of this paper and adapting the proofs of ([B-RS 2],(3.8),(3.9),(3.10),(3.11)).

PROPOSITION 6.1. *Let A be a Noetherian ring of even dimension n. Let $J_1, J_2 \subset A$ be comaximal ideals of height n and $J_3 = J_1 \cap J_2$. If any two of J_1, J_2 and J_3 are surjective images of projective A-modules of rank n, which are stably isomorphic to $L \oplus A^{n-1}$, then so is the third.*

PROPOSITION 6.2. *Let A be a Noetherian ring of even dimension n. Let $J \subset A$ be an ideal of height n. Then $(J) = 0$ in $E_0(A, L)$ if and only if J is a surjective image of a projective A-module of rank n which is stably isomorphic to $L \oplus A^{n-1}$.* $\square$

PROPOSITION 6.3. *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with determinant L. Then $e(P) = 0$ in $E_0(A, L)$ if and only if $[P] = [Q \oplus A]$ in $K_0(A)$ for some projective A-module Q of rank $n - 1$.*

PROPOSITION 6.4. *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with determinant L. Suppose that $e(P) = (J)$, in $E_0(A, L)$, where $J \subset A$ is a ideal of height n. Then, there exists a projective A-module Q of rank n, such that $[Q] = [P]$ in $K_0(A)$ and J is a surjective image of Q.*

We record, for use in the next section, the following

PROPOSITION 6.5. *Let A be a Noetherian ring of even dimension n and let $J \subset A$ be an ideal of height n such that J/J^2 is generated by n elements. Let $\widetilde{\omega}_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection. Suppose that the element $(J, \widetilde{\omega}_J)$ of $E(A)$ belongs to the kernel of the canonical homomorphism $E(A) \twoheadrightarrow E_0(A)$. Then, there exists a stably free A module P_1 of rank n and a generator χ_1 of $\wedge^n(P_1)$ such that $e(P_1, \chi_1) = (J, \widetilde{\omega}_J)$ in $E(A)$.*

Proof. Since $(J, \widetilde{\omega}_J)$ of $E(A)$ belongs to the kernel of the canonical homomorphism from $E(A)$ to $E_0(A)$, it follows that $(J) = 0$ in $E_0(A)$. Hence, by (6.2), there exists a surjection $\alpha : P \twoheadrightarrow J$ where P is a stably free A-module of rank n. Let χ be a generator of $\wedge^n(P)$. Suppose that (J, ω_J) is obtained from (α, χ). By (2.2), there exists $a \in A$ such that $\overline{a} \in A/J$ is a unit and $\widetilde{\omega}_J = \overline{a}\omega_J$. By (5.1), there exists a projective A-module P_1 of rank n with $[P_1] = [P]$ in $K_0(A)$ and a generator χ_1 of $\wedge^n P_1$, such that $e(P_1, \chi_1) = (J, \overline{a^{n-1}}\omega_J)$ in $E(A)$. Since n is even, by (5.4) we have $(J, \overline{a^{n-1}}\omega_J) = (J, \overline{a}\omega_J)$ in $E(A)$. Hence, $e(P_1, \chi_1) = (J, \widetilde{\omega}_J)$ in $E(A)$. $\square$

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Following the method of ([MK 2], Theorem 1), we prove

PROPOSITION 6.6. *Let A be a Noetherian ring of dimension 2 and $J \subset A$ an ideal of height 2. Let L be a projective module of rank 1. Let P, P_1 be two projective A-modules of rank 2 and let $\alpha : P \twoheadrightarrow J$, $\beta : P_1 \twoheadrightarrow J$ be surjections. Then we have*

(i) If P has determinant L and P_1 is free, then $[P] = [L \oplus A]$ in $K_0(A)$.
(ii) If P has trivial determinant and $P_1 \xrightarrow{\sim} L \oplus A$, then $[P] = [A^2]$ in $K_0(A)$.

Proof. We only prove (i), the proof of (ii) being similar.

By (2.4), there exists an injective homomorphism $\Psi : P \to P_1$ such that $\beta\Psi = \alpha$. By (2.2), the map $\Psi \otimes A/J$ is an isomorphism. Therefore, it follows that $\Psi(P) + JP_1 = P_1$. Hence, using Nakayama's lemma, it follows that there exists $x \in A$ with $x = 1$ modulo J such that $xP_1 \subset \Psi(P)$. Let $K = \operatorname{coker}\Psi$. There exists an exact sequence

$$0 \to P \xrightarrow{\Psi} P_1 \twoheadrightarrow K \to 0.$$

From the above exact sequence, it follows that $\operatorname{hd}_A K = 1$. Further, since $xP_1 \subset \Psi(P)$, it follows that $xK = 0$. Let bar denote reduction modulo x. There exists an exact sequence

$$\overline{P} \xrightarrow{\overline{\Psi}} \overline{P_1} \twoheadrightarrow K \to 0.$$

Since $x = 1$ modulo J, it follows that $J/xJ = A/x$. We choose an element $\overline{p} \in \overline{P}$ such that $\overline{\alpha}(\overline{p}) = \overline{1}$. Since $\beta\Psi = \alpha$, it follows that the element $\overline{\Psi}(\overline{p}) \in \overline{P_1}$ is unimodular. Since P_1 is free of rank 2, it follows that $\overline{P_1}/(\overline{\Psi}(\overline{p}))$ is a free $\overline{A}$ module of rank 1. Thus, the $\overline{A}$ module $K = \overline{P_1}/\overline{\Psi}(\overline{P})$ is generated by a single element. Hence, there exists an exact sequence

$$0 \to Q \to A \twoheadrightarrow K \to 0.$$

Since $\operatorname{hd}_A K = 1$, it follows that Q is projective. Further, using Schanuel's lemma, it follows that $P \oplus A \xrightarrow{\sim} Q \oplus P_1$. Since P_1 is free, comparing determinants we see that $Q \xrightarrow{\sim} L$ and hence $[P] = [L \oplus A]$ in $K_0(A)$. This proves the proposition. □

Even though, as shown in (4.8), the groups $E(A, L)$ may vary with L, the following theorem shows that the groups $E_0(A, L)$ are independent of L. In order to prove this we need

PROPOSITION 6.7. *Let A be a Noetherian ring of dimension n and P, P_1 projective A-modules of rank n such that $[P] = [P_1]$ in $K_0(A)$. Then, there exists an ideal $J \subset A$ of height $\geqslant n$ such that J is a surjective image of both P and P_1.*

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Proof. Since dim $A = n$ and $[P] = [P_1]$ in $K_0(A)$, it follows that $P \oplus A \xrightarrow{\sim} P_1 \oplus A$. Therefore, there exists a short exact sequence

$$0 \to P_1 \to A \oplus P \stackrel{(b,-\alpha)}{\to} A \to 0.$$

Further, without loss of generality, we may replace α by $\alpha + b\gamma$ (since this will not change the isomorphism class of $\ker((b, -\alpha)) = P_1$). Therefore, using (2.13), we may assume that the ideal $\alpha(P) = J$ is such that height $(J) \geqslant n$. By (2.8 (i)), J is also a surjective image of P_1. □

THEOREM 6.8. *The groups $E_0(A, L)$ and $E_0(A)$ are canonically isomorphic.*

Proof. We show that we have a well defined map $\alpha : E_0(A) \to E_0(A, L)$ sending the class of (J) in $E_0(A)$ to the class of (J) in $E_0(A, L)$ and $\beta : E_0(A, L) \to E_0(A)$ which sending the class of (J) in $E_0(A, L)$ to the class of (J) in $E_0(A)$. It is then immediate that α and β are isomorphisms and are inverses of each other. We show that the map α is well defined, the proof that the map β is well defined being similar.

Let $J \subset A$ be an ideal of height n generated by n elements. We show that $(J) = 0$ in $E_0(A, L)$. Let $J = (a_1, \cdots, a_n)$. By performing elementary transformations, we may assume that the ideal $J_1 = (a_3, \cdots, a_n)$ has height $n - 2$. Let bar denote reduction modulo J_1.

The ring $\overline{A} = A/J_1$ has dimension 2. Since $\overline{J}$ is generated by two elements $\overline{a_1}, \overline{a_2}$, it follows, from (2.5), that there exists a projective $\overline{A}$-module $\widetilde{P}$ of rank 2 with determinant $\overline{L}$ and a surjection from $\widetilde{P}$ to $\overline{J}$. Since $\overline{J}$ is generated by two elements, by (6.6), $[\widetilde{P}] = [\overline{L} \oplus \overline{A}]$ in $K_0(\overline{A})$. Now by (6.7), there exists an ideal J' of A containing J_1 such that (1) $\overline{J'}$ has height $\geqslant 2$ and (2) $\overline{J'}$ is the surjective image of the projective $\overline{A}$-modules $\widetilde{P}$ and $\overline{L} \oplus \overline{A}$. If $\overline{J'} = \overline{A}$, then $\widetilde{P} \xrightarrow{\sim} \overline{L} \oplus \overline{A}$ and since $\overline{J}$ is a surjective image of $\widetilde{P}$, it follows that J is a surjective image of $L \oplus A^{n-1}$ and hence (J) is trivial in $E_0(A, L)$. We may therefore assume, that height $\overline{J'} = 2$. Since there exist surjections from $\widetilde{P}$ to $\overline{J}$ and $\overline{J'}$, it follows from (3.0), that there exists an ideal I of $A[T]$ containing $J_1 A[T]$ and a surjection from $\widetilde{P}[T]$ to $\overline{I}$, where (1) height $\overline{I} = 2$ and (2) $\overline{I(0)} = \overline{J}, \overline{I(1)} = \overline{J'}$. Now by (3.1), there exists an ideal K of A containing J_1 and a surjection from $\overline{L[T]} \oplus \overline{A}[T]$ to $\overline{I} \cap \overline{KA}[T]$ where:

(1) $\overline{K} \subset \overline{A}$ is an ideal of height $\geqslant 2$ with $\overline{K}/\overline{K^2}$ generated by 2 elements and
(2) $\overline{I} + \overline{KA}[T] = \overline{A}[T]$. Thus, $I \cap KA[T]$ is a surjective image of $L[T] \oplus A[T]^{n-1}$. Specialising at $T = 0, 1$, it follows that the ideals $J \cap K$ and $J' \cap K$ are surjective images of $L \oplus A^{n-1}$. Hence $(J) = (J')$ in $E_0(A, L)$. Since $\overline{J'}$ is a surjective image of $\overline{L} \oplus \overline{A}$, it follows that J' is a surjective image of $L \oplus A^{n-1}$. Therefore $(J) = (J') = 0$ in $E_0(A, L)$. This concludes the proof of the theorem. □

Let A be Noetherian ring of dimension n. Let P be a projective A-module of rank n with determinant L and $\lambda : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ is an ideal of height n. We define $e'(P) = (J)$ in $E_0(A)$. As we have seen above, if we define $e(P)$ to be (J) in

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$E_0(A,L)$, then $e(P)$ is well defined. Now since the isomorphism from $E_0(A,L)$ to $E_0(A)$ sends the class of J in $E_0(A,L)$ to the class of J in $E_0(A)$ it follows that $e'(P)$ is a well defined element of $E_0(A)$. We therefore can drop the superscript prime and define $e(P)=(J)$ in $E_0(A)$. It follows, that the results in Propositions ((6.2), (6.3), (6.4)) are valid if the group $E_0(A,L)$ is replaced by the group $E_0(A)$.

For example we have

COROLLARY 6.9. *Let A be a Noetherian ring of even dimension n. Let P be a projective A-module of rank n with determinant L. Let $\alpha : P \twoheadrightarrow J$ be a surjection where $J \subset A$ is an ideal of height n. Then J is a surjective image of a stably free A-module of rank n if and only if $[P]=[Q\oplus A]$ in $K_0(A)$ for some projective A-module Q of rank $n-1$.*

7. Relations Between $E(A)$ and $Um_{d+1}(A)/SL_{d+1}(A)$

Let A be a Noetherian ring of dimension 2. Let $\widetilde{K_0}Sp(A)$ be the set of isometry classes of (P,s), where P is a projective A module of rank 2 and $s : P\times P\to A$ a non-degenerate skew-symmetric bilinear form. We note that there is (upto isometry) a unique non-degenerate alternating form on A^2, which we denote by h.

We define a binary operation $*$ on $\widetilde{K_0}Sp(A)$ as follows. Let (P_1,s_1) and (P_2,s_2) be two elements of $\widetilde{K_0}Sp(A)$ (where P_1,P_2 have rank 2 and trivial determinant). Since $\dim A=2$ and $P_1\oplus P_2$ has rank 4, $(P_1,s_1)\perp(P_2,s_2)$ is isometric to $(P_3,s_3)\perp(A^2,h)$, where P_3 is a projective A-module of rank 2 and, by a theorem of Bass ([Ba 2],4.16)), (P_3,s_3) is determined uniquely upto isometry. We define $(P_1,s_1)*(P_2,s_2)=(P_3,s_3)$. Then $\widetilde{K_0}Sp(A)$ is a group under $*$ with the isometry class of (A^2,h) as the identity element. Since $\dim A=2$, this coincides with the usual notion of $\widetilde{K_0}Sp(A)$.

Remark 7.1. Let P be a projective A module of rank 2. Then, having a non-degenerate alternating form on P is equivalent to giving an isomorphism $\lambda : \wedge^2(P)\xrightarrow{\sim} A$. Thus, we can identify the pair (P,s) with (P,χ), where χ is the generator of $\wedge^2(P)$ given by $\lambda^{-1}(1)$. It is easy to see that the isometry classes of (P,s) coincide with the isomorphism classes of (P,χ).

The proof of the following theorem is motivated by ([Bg-O],Theorem (6.2)).

THEOREM 7.2. *Let A be a Noetherian ring of dimension 2. The map from $\widetilde{K_0}Sp(A)$ to $E(A)$ sending (P,χ) to $e(P,\chi)$ is an isomorphism.*

Proof. We first show that the map is a homomorphism.

Step 1. Let P_1,P_2 be projective A-modules of rank 2 with trivial determinant and χ_1,χ_2 be generators of $\wedge^2(P_1)$ and $\wedge^2(P_2)$ respectively. Let $\alpha_1 : P_1\twoheadrightarrow J_1$ and $\alpha_2 : P_2\twoheadrightarrow J_2$ be surjections, where J_1 and J_2 are ideals of height 2 which are comaximal. Let (J_i,ω_{J_i}), $i=1,2$ be obtained from the pair (α_i,χ_i) respectively. Let

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$\gamma_1 : \wedge^2(P_1) \to P_1$ be defined as follows: $\gamma_1(p \wedge q) = \alpha_1(q)p - \alpha_1(p)q$. Then $\mathrm{im}(\gamma_1) \subset \ker(\alpha_1)$. One can similarly define $\gamma_2 : \wedge^2(P_2) \to P_2$ with $\mathrm{im}(\gamma_2) \subset \ker(\alpha_2)$. Let $\gamma_i(\chi_i) = p_i, i = 1, 2$. An easy local checking, shows that $O(p_i) = J_i$ (where $O(p) = \{f(p) | f \in P^*\}$).

Step 2. It follows from an easy local computation, that if $q \in P_1$, then

$$p_1 \wedge q = \alpha_1(q)\chi_1. \tag{$*$}$$

A similar equation holds for P_2.

The element χ_i induces a non-degenerate alternating form $s_i : P_i \times P_i \to A$. Using $(*)$ we have $s_1(p_1, q) = \alpha_1(q)$. We choose $a_1 \in J_1$ and $a_2 \in J_2$ such that $a_1 + a_2 = 1$. Let $q_1 \in P_1, q_2 \in P_2$ be such that $\alpha_1(q_1) = a_1, \alpha_2(q_2) = a_2$. Let $P_4 = P_1 \oplus P_2$. Then $s = s_1 \perp s_2$ defines a non-degenerate alternating form on P_4. Let $e_1 = (p_1, p_2)$ and $e_2 = (q_1, q_2)$. We have $s(e_1, e_2) = s_1(p_1, q_1) + s_2(p_2, q_2) = \alpha_1(q_1) + \alpha_2(q_2) = 1$. It follows, that the restriction of $s = s_1 \perp s_2$ to the submodule $Ae_1 \oplus Ae_2$ is hyperbolic.

Step 3. Let $\alpha_1 + \alpha_2 : P_1 \oplus P_2 \twoheadrightarrow J_1 + J_2 = A$ be the map defined by $(\alpha_1 + \alpha_2)(q, q') = \alpha_1(q) + \alpha_2(q')$. Let $Q = \ker(\alpha_1 + \alpha_2)$ and $e = (q, q')$. Since $s(e_1, e) = \alpha_1(q) + \alpha_2(q')$, it follows that Q is the submodule of P_4, consisting of those elements which are perpendicular under s to e_1. Let P_3 be the submodule of Q, consisting of those elements which are perpendicular under s to e_2. Then $P_4 = P_3 \oplus (Ae_1 \oplus Ae_2)$ and $(P_1, s_1) \perp (P_2, s_2)$ is isometric to $(P_3, s_3) \perp (A^2, h)$ (where $A^2 = Ae_1 \oplus Ae_2$ and s_3 is the restriction of the form $s_1 \perp s_2$ to P_3).

Let $\beta : P_4 (= P_1 \oplus P_2) \to A$ be the map given by $\beta(q, q') = \alpha_1(q)$. Then, it is easy to see that the restriction of β to P_3 gives a surjection $\alpha_3 : P_3 \twoheadrightarrow J_3 = J_1 \cap J_2$

The form s_3 corresponds to a generator χ_3 of $\wedge^2(P_3)$. Let (J_3, ω_{J_3}) be obtained from (α_3, χ_3). We show that $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2}) = (J_3, \omega_{J_3})$ in $E(A)$. This will prove that the map $\widetilde{K_0}Sp(A)$ to $E(A)$ is a homomorphism.

Step 4. Let $\lambda_1 : P_3 \to P_1$ be the first projection. Consider the following commutative diagram:

$$\begin{array}{ccccc} P_3 & \xrightarrow{\alpha_3} & J_3 & \to & 0 \\ \lambda_1 \downarrow & & \downarrow & & \\ P_1 & \xrightarrow{\alpha_1} & J_1 & \to & 0 \end{array}$$

CLAIM. $\wedge^2 \lambda_1(\chi_3) = u\chi_1$, where $u - 1 \in J_1$.

Let $(\sum_i (p_{1i}, p_{2i}) \wedge (q_{1i}, q_{2i})) = \chi_3$ (where (p_{1i}, p_{2i}) and $(q_{1i}, q_{2i}) \in P_3 \subset P_1 \oplus P_2$). Then, since χ_3 corresponds to the alternating form s_3 (which is the restriction of the form $s_1 \perp s_2$ to P_3), it follows that $\sum_i s_1(p_{1i}, q_{1i}) + s_2(p_{2i}, q_{2i}) = 1$. Let $s_1(p_{1i}, q_{1i}) = \delta_i$ and $s_2(p_{2i}, q_{2i}) = \mu_i$. Then $p_{1i} \wedge q_{1i} = \delta_i \chi_1$, $p_{2i} \wedge q_{2i} = \mu_i \chi_2$ and $\sum_i (\delta_i + \mu_i) = 1$. Therefore, since $\lambda_1(p_{1i}, p_{2i}) = p_{1i}$ and $\lambda_1(q_{1i}, q_{2i}) = q_{1i}$, to prove the claim, it is enough to show that $\mu_i \in J_1$.

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We have

$$\gamma_2(p_{2i} \wedge q_{2i}) = \alpha_2(q_{2i})p_{2i} - \alpha_2(p_{2i})q_{2i} = \mu_i\gamma_2(\chi_2) = \mu_i p_2. \tag{**}$$

Since (p_{1i}, p_{2i}) and $(q_{1i}, q_{2i}) \in P_3$, it follows that $\alpha_2(q_{2i}), \alpha_2(p_{2i}) \in J_3 = J_1 \cap J_2$ and hence $\mu_i p_2 \in J_3P_2$. Therefore $O(\mu_i p_2) \subset J_3$. But $O(p_2) = J_2$ (Step 1) and $J_1 + J_2 = A$. Hence $\mu_i \in J_1$ and the claim is proved.

By a similar argument, we can prove that $\wedge^2\lambda_2(\chi_3) = v\chi_2$ where $v - 1 \in J_2$. Therefore, $(J_1, \omega_{J_1}) + (J_2, \omega_{J_2}) = (J_3, \omega_{J_3})$ in $E(A)$. Thus, the map from $\widetilde{K_0}Sp(A)$ to $E(A)$ is a homomorphism.

Step 5. Now we show that the map is surjective. Let $J \subset A$ be an ideal such that J/J^2 is generated by two elements and ω_J a surjection $(A/J)^2 \twoheadrightarrow J/J^2$. By (2.5), there exists a projective module P of rank 2 with trivial determinant mapping onto J. Let χ be a generator of $\wedge^2(P)$. Then $e(P, \chi) = (J, \widetilde{\omega}_J)$. Since $(J, \omega_J) = (J, \overline{a}\widetilde{\omega}_J)$, where $\overline{a}$ is a unit in A/J, by (5.1), there exists a projective A-module P_1 and a generator χ_1 of $\wedge^2(P_1)$ such that $e(P_1, \chi_1) = (J, \omega_J)$. Thus, the map is surjective.

If $e(P, \chi) = 0$, then by (4.4), P is free. It follows, that the map from $\widetilde{K_0}Sp(A)$ to $E(A)$ is injective and hence an isomorphism. This proves the theorem. $\square$

Let G be the set of isometry classes of non-degenerate alternating forms on A^4. Let $H(A^2) = (A^2, h)\perp(A^2, h)$. As before, we can define a group structure on G as follows: We set $(A^4, s_1) * (A^4, s_2) = (A^4, s_3)$, where s_3 is the unique (upto isometry) alternating form on A^4 satisfying the property that $(A^4, s_1)\perp(A^4, s_2)$ is isometric to $(A^4, s_3)\perp H(A^2)$. Then G is a group with $H(A^2)$ as the identity element. Let s be a non-degenerate alternating form on A^4. Then, since dim $A = 2, (A^4, s) \xrightarrow{\sim} (P, s')\perp(A^2, h)$. The assignment sending (A^4, s) to (P, s') gives rise to an injective homomorphism from G to $\widetilde{K_0}Sp(A)$. In view of the above theorem, we have the following

THEOREM 7.3. *Let A be a Noetherian ring of dimension* 2. *Then, we have the following exact sequence*

$$0 \to G \to \widetilde{K_0}Sp(A)(\xrightarrow{\sim} E(A)) \to E_0(A) \to 0.$$

Proof. Let s be a non-degenerate alternating form on A^4 and let $(A^4, s) \xrightarrow{\sim} (P, s')\perp(A^2, h)$. Then P is a stably free A-module of rank 2 and hence, by (6.7), there exists a surjection $\psi : P \twoheadrightarrow J$, where $J \subset A$ is an ideal of height 2 generated by 2 elements. Therefore, the image of $(P, s') = (J) = 0$ in $E_0(A)$.

Now let P_1 be a projective A-module of rank 2 and let s_1 be a non-degenerate alternating form on P_1. Let χ_1 be a generator of $\wedge^2(P_1)$ corresponding to s_1. Let $\psi : P_1 \twoheadrightarrow J_1$ be a surjection, where $J_1 \subset A$ is an ideal of height 2. Then $e(P_1, \chi_1) = (J_1, \omega_{J_1})$, where (J_1, ω_{J_1}) is obtained from (ψ, χ_1). Suppose that the image of $(P_1, s_1) = 0$ in $E_0(A)$. Then (J_1, ω_{J_1}) is an element of the kernel of the canonical map from $E(A)$ to $E_0(A)$. By (6.5), $(J_1, \omega_{J_1}) = e(P_2, \chi_2)$, where P_2 is a stably free

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projective A-module of rank 2. Let s_2 be the non-degenerate form on P_2 corresponding to χ_2. Since $e(P_1, \chi_1) = e(P_2, \chi_2)$, the images of (P_1, s_1) and (P_2, s_2) in $E(A)$ are the same. Hence, by (7.2), (P_2, s_2) is isometric to (P_1, s_1). Since P_2 is stably free, it follows that $(P_2, s_2) \perp (A_2, h) = (A^4, s)$, for some non-degenerate alternating form s on A^4. This proves the theorem. $\square$

Let A be a Noetherian ring of dimension $n \geqslant 2$ and let $[a_0, a_1, \cdots, a_n] \in Um_{n+1}(A)$. Let $\theta : A^{n+1} \twoheadrightarrow A$ be the surjection given by $\theta(e_i) = a_i$, where $(e_0, e_1, \cdots, e_n)$ is the standard basis of A^{n+1} and let $P = \ker \theta$. Let $b_0, b_1, \cdots, b_n \in A$ be such that $\sum_{i=0}^n a_i b_i = 1$. Let $p_i = a_i f - e_i$, where $f = \sum_{i=0}^n b_i e_i$. Then P is generated by p_i and we have $\sum_{i=0}^n b_i p_i = 0$

Let $\omega_i \in \wedge^n(P)$ be defined by

$$\omega_i = p_0 \wedge p_1 \wedge \cdots \wedge p_{i-1} \wedge p_{i+1} \wedge \cdots \wedge p_n, 0 \leqslant i \leqslant n.$$

Then, since $\sum_{i=0}^n b_i p_i = 0$, it is easy to see that $b_i \omega_0 = (-1)^i b_0 \omega_i, 0 \leqslant i \leqslant n$. Therefore, if $\chi = \sum_{i=0}^n (-1)^i a_i \omega_i$, then χ is a generator of $\wedge^n(P)$ and $b_0 \chi = \omega_0$. Moreover, it can be seen easily that χ is independent of the choice of b_i.

To $[a_0, a_1, \cdots, a_n]$, we associate the element $e(P, \chi)$ of the Euler class group $E(A)$, where χ is as above.

Note that this association induces a set theoretic map $\Psi : Um_{n+1}(A)/SL_{n+1}(A) \to E(A)$. Hence, it also induces a set theoretic map from $Um_{n+1}(A)/E_{n+1}(A)$ to $E(A)$ which we continue to denote by Ψ.

We give an explicit description of Ψ.

Let $[a_0, a_1, \cdots, a_n] \in Um_{n+1}(A)/SL_{n+1}(A)$ or $Um_{n+1}(A)/E_{n+1}(A)$. Assume, by performing elementary transformations, that that height $(a_1, \cdots, a_n) = n$. Now consider the element (P, χ) associated to $[a_0, a_1, \cdots, a_n]$ as above.

Let $\beta : P \twoheadrightarrow (a_1, \cdots, a_n)$ be the surjection defined as follows:

(1) $\beta(p_0) = b_0 a_0 - 1$,
(2) $\beta(p_i) = b_0 a_i$ if $i > 0$.

Let $J = (a_1, \cdots, a_n)$ and $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be defined by sending the ith coordinate function to $\overline{a_i}$. Then, if we compute $e(P, \chi)$ using β, we see that we see that $e(P, \chi) = (J, \overline{b_0}^{n-1} \omega_J)$. Therefore, if n is even, by (5.4), $e(P, \chi) = (J, \overline{b_0} \omega_J) = (J, \overline{a_0^2 b_0} \omega_J) = (J, \overline{a_0} \omega_J)$ in $E(A)$.

Thus $\Psi([a_0, a_1, \cdots, a_n]) = (J, \overline{a_0} \omega_J)$. This explicit description of Ψ will be used in what follows.

We make the following remark before stating the next result.

Remark 7.4. Let A be a Noetherian ring with $\dim A = n \geqslant 2$ and $(a_3, \cdots, a_n)$ an ideal such that height $(a_3, \cdots, a_n) = n - 2$ and $\dim A/(a_3, \cdots, a_n) = 2$. Let bar denote reduction modulo $(a_3, \cdots, a_n)$. Let $\overline{J} \subset \overline{A}$ be an ideal of of height 2 such that $\overline{J}/\overline{J}^2$ is generated by two elements, $\overline{f}$ and $\overline{g}$. Then $J \subset A$ is an ideal of height n and

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J/J^2 is generated by the n elements $\overline{f}, \overline{g}, \overline{a_3}, \cdots, \overline{a_n}$. In view of this, there exists a canonical homomorphism $E(\overline{A}) \to E(A)$.

PROPOSITION 7.5. *Let A be a Noetherian ring of dimension $n \geqslant 2$. The map $\Psi : Um_{n+1}(A)/SL_{n+1}(A) \to E(A)$ is a group homomorphism.*

Proof. First assume that n is odd. Let $[a_0, a_1, \cdots, a_n] \in Um_{n+1}(A)$ and P be the projective A-module associated with $[a_0, a_1, \cdots, a_n]$ as above. Then P has a unimodular element and hence, by (4.4), $e(P, \chi) = 0$ in $E(A)$. Therefore Ψ is the zero group homomorphism.

Now let n be even.

Case (1) $n = 2$.

In this case, first note that, by a theorem of Vaserstein ([Su-V], Corollary 7.4), there exists a bijection from $Um_3(A)/SL_3(A)$ with the group G (defined above) and in fact the group structure on $Um_3(A)/SL_3(A)$ is the one induced by this bijection. By (7.3), there is a (injective) group homomorphism from G to $E(A) = \widetilde{K_0}Sp(A)$. It is easy to check that Ψ is just the composite of these two maps. Therefore Ψ is a homomorphism. It also follows, that the map $\Psi : Um_3(A)/E_3(A) \to E(A)$ is also a group homomorphism.

Case (2) $n > 2$. We first show that the set theoretic map $\Psi : Um_{n+1}(A)/E_{n+1}(A) \to E(A)$ is a homomorphism. Let $[v_1] = [a_0, a_1, \cdots, a_n]$ and $[v_2] = [d_0, d_1, \cdots, d_n]$ be two elements of $Um_{n+1}(A)/E_{n+1}(A)$. Let $[v_3] = [v_1] * [v_2]$ where $*$ is the group operation on $Um_{n+1}(A)/E_{n+1}(A)$ defined in ([VK 1]). By performing elementary transformations, we may assume by ([VK 1],3.4), that $a_i = d_i, i \geqslant 3$ and $a_3, \cdots, a_d$ are in general position i.e. a_i is not contained in any prime ideal that is minimal over $a_{i+1}, \cdots, a_n$ for $i \geqslant 3$. Suppose that $\Psi([v_1]) = e(P_1, \chi_1)$, $\Psi([v_2]) = e(P_2, \chi_2)$, and $\Psi([v_3]]) = e(P_3, \chi_3)$. In order to prove the proposition, it is sufficient to prove that $e(P_1, \chi_1) + e(P_2, \chi_2) = e(P_3, \chi_3)$ in $E(A)$. Since $a_3, \cdots, a_d$ are in general position, it follows that dim $A/(a_3, \cdots, a_d) \leqslant 2$. If dim $A/(a_3, \cdots, a_d) \leqslant 1$, then $[v_1]$ and $[v_2]$ are completable to elementary matrices and hence, so is $[v_3]$. Hence P_1, P_2, P_3 are all free and there is nothing to prove.

Assume therefore that dim $A/(a_3, \cdots, a_d) = 2$. Let bar denote reduction modulo $(a_3, \cdots, a_d)$. Using the explicit description of the map $\Psi : Um_{n+1}(A)/E_{n+1}(A) \to E(A)$ for n even, one verifies that the following diagram is commutative.

$$\begin{array}{ccc} Um_3(\overline{A})/E_3(\overline{A}) & \xrightarrow{\overline{\Psi}} & E(\overline{A}) \\ \downarrow & & \downarrow \\ Um_{n+1}(A)/E_{n+1}(A) & \xrightarrow{\Psi} & E(A) \end{array}$$

The vertical maps are canonical. By ([VK 1], (3.6)), the canonical map $Um_3(\overline{A})/E_3(\overline{A}) \to Um_{n+1}(A)/E_{n+1}(A)$ is a group homomorphism. Since the map

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$Um_3(\overline{A})/E_3(\overline{A}) \to E(\overline{A})$ is a homomorphism and the canonical map $E(\overline{A}) \to E(A)$ is a homomorphism (see (7.4)), it follows by a diagram chase, that the map $Um_{n+1}(A)/E_{n+1}(A) \to E(A)$ is a homomorphism when restricted to the image of $Um_3(\overline{A})/E_3(\overline{A}) \to Um_{n+1}(A)/E_{n+1}(A)$. Since $[v_1]$ and $[v_2]$ belong to this image, it follows that $e(P_1, \chi_1) + e(P_2, \chi_2) = e(P_3, \chi_3)$ in $E(A)$. Therefore, the map $\Psi : Um_{n+1}(A)/E_{n+1}(A) \to E(A)$ (and hence the map $\Psi : Um_{n+1}(A)/SL_{n+1}(A) \to E(A)$) is a homomorphism. This proves the proposition. □

THEOREM 7.6. *Let A be a Noetherian ring of even dimension $n \geqslant 2$. Then, there exists an exact sequence of groups*

$$Um_{n+1}(A)/SL_{n+1}(A) \xrightarrow{\Psi} E(A) \to E_0(A) \to 0.$$

Proof. If $n = 2$, this is proved in (7.3). As in the proof of (7.3), it follows, by (6.7), that the sequence is a complex.

Let $(J, \omega_J) \in E(A)$ belong to the kernel of the canonical surjection $E(A) \to E_0(A)$. By (6.5), there exists a stable free module P of rank n and a generator χ of $\wedge^n(P)$ such that $e(P, \chi) = (J, \omega_J)$. Since P is stably free, by (6.7), there exists a surjection from P to J_1 where $J_1 \subset A$ is an ideal of height n such that J_1 is generated by n elements $a_1, \cdots, a_n$. Let $e(P, \chi) = (J_1, \omega_{J_1})$. Let $\widetilde{\omega}_{J_1} : (A/J_1)^n \twoheadrightarrow J_1/J_1{}^2$ be the surjection which sends the ith coordinate function to $\overline{a_i}$. Then, by (5.0), $(J_1, \omega_{J_1}) = (J_1, \overline{a_0}\widetilde{\omega}_{J_1})$ where $\overline{a_0} \in A/J$ is a unit. But $\Psi([a_0, a_1, \cdots, a_n]) = (J_1, \overline{a_0}\widetilde{\omega}_{J_1})$ and $(J_1, \omega_{J_1}) = e(P, \chi) = (J, \omega_J)$. Therefore, the sequence is exact. □

COROLLARY 7.7. *Let A be a Noetherian ring of even dimension $n \geqslant 2$. Let H be a subset of $Um_{n+1}(A)/SL_{n+1}(A)$ consisting of elements $[v] = [a_0, a_1, \cdots, a_n]$ such that the projective A-module corresponding to $[v]$ has a unimodular element. Then H is a subgroup of $Um_{n+1}(A)/SL_{n+1}(A)$.*

Proof. This follows from the fact that $\Psi : Um_{n+1}(A)/SL_{n+1}(A) \to E(A)$ is a group homomorphism and ker $\Psi = H$. □

COROLLARY 7.8. *Let $X = \operatorname{Spec} A$ be a smooth affine variety of even dimension n over the field $\mathbf{R}$ of real numbers such that the canonical module $K_A = \wedge^n \Omega_{A/\mathbf{R}}$ is trivial. Let $X(\mathbf{R})$ denote the topological space consisting of the set of real points of X and t denote the number of compact connected components of $X(\mathbf{R})$. Then $Um_{n+1}(A)/SL_{n+1}(A) = H \oplus F$ where F is a free Abelian group of rank t and H is as in (7.7).*

Proof. In view of (7.6) and (7.7), it is enough to show that the kernel of the canonical map $E(A) \to E_0(A)$ is a free Abelian group of rank t.

Let S denote the multiplicatively closed subset of A, consisting of all elements which do not have any real zeroes and let $\mathbf{R}(X) = A_S$. Then we have canonical surjective homomorphisms Γ from $E(A)$ to $E(\mathbf{R}(X))$ and β from $E_0(A)$ to $E_0(\mathbf{R}(X))$ respectively, such that the following diagram commutes (see ([B-RS 2],

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$$\begin{array}{ccc} E(A) & \overset{\Gamma}{\twoheadrightarrow} & E(\mathbf{R}(X)) \\ \downarrow & & \downarrow \\ E_0(A) & \overset{\beta}{\twoheadrightarrow} & E_0(\mathbf{R}(X)) \end{array}$$

By ([B-RS 2], (5.4)), the induced map from kernel(Γ) to kernel(β) is an isomorphism. Moreover, by ([B-RS 2], (4.10) and (4.12)), $E_0(\mathbf{R}(X)) = (\mathbf{Z}/\mathbf{2})^t$ and $E(\mathbf{R}(X))$ is a free Abelian group of rank t. Hence, the kernel of the canonical map $E(A) \to E_0(A)$ is a free Abelian group of rank t. □

We now state some further consequences of (7.5).

COROLLARY 7.9. *Let A be a Noetherian ring of dimension $n \geqslant 2$ and (J, ω_J) an element of $E(A)$ such that its image (which is independent of ω_J) in $E_0(A)$ is zero. Then, the element $(J, \omega_J) + (J, -\omega_J) = 0$ in $E(A)$.*

Proof. We first settle the case where J is generated by n elements $(a_1, \cdots, a_n)$. We may assume by performing elementary transformations, that $a_3, \cdots, a_n$ are such that $\dim A/(a_3, \cdots, a_n) = 2$. Let bar denote reduction modulo $(a_3, \cdots, a_n)$. In view of the canonical homomorphism of $E(\overline{A})$ to $E(A)$, it follows that, in this case, it suffices to prove the proposition for $\overline{A}$. Thus, one may assume that $\dim A = 2$ and that J is generated by 2 elements a_1, a_2. Let $\widetilde{\omega}_J : (A/J)^2 \twoheadrightarrow J/J^2$ be the surjection which sends the coordinate functions to $\overline{a_1}$ and $\overline{a_2}$. By (5.0), $(J, \omega_J) = (J, \overline{a_0}\widetilde{\omega}_J)$ where $\overline{a_0} \in A/J$ is a unit. Let b_0 be chosen such that $a_0 b_0 = 1$ modulo J. Using ([VK 1],(3.6)), it follows that $[a_0, a_1, a_2] * [-b_0, a_1, a_2] = 0$ in $Um_3(A)/SL_3(A)$. Therefore, using ([VK 1], (3.16, (iii))), it follows that $[a_0, a_1, a_2] * [-a_0, a_1, a_2] = 0$ in $Um_3(A)/SL_3(A)$. Taking images in $E(A)$ under Ψ (see (7.5)), it follows that $(J, \overline{a_0}\widetilde{\omega}_J) + (J, -\overline{a_0}\widetilde{\omega}_J) = 0$ in $E(A)$. Thus $(J, \omega_J) + (J, -\omega_J) = 0$ in $E(A)$.

We prove the general case as follows: To avoid repetition, we shall assume that any ideal K considered in the proof has height n and satisfies the property that K/K^2 is generated by n elements.

We consider the set of all ideals I of A of which satisfy the following property (p): For any choice of ω_I,

$$(I, \omega_I) + (I, -\omega_I) = 0.$$

First note that addition and subtraction principles hold for property (p), i.e. given any two comaximal ideals $J_1, J_2 \subset A$ of height n, if any two of the ideals J_1, J_2 and $J_1 \cap J_2$ satisfy property (p), then so does the third.

Note that the kernel of the homomorphism $E(A)$ to $E_0(A)$ is generated by elements of the type (J_1, ω_{J_1}), where $J_1 = (a_1, \cdots, a_n)$ (see ([B-RS 2], (3.3)) for details). Thus, if

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(J, ω_J) belongs to the kernel of $E(A)$ to $E_0(A)$, then

$$(J, \omega_J) + \sum_{l=r+1}^{r+s} (J_l, \omega_l) = \sum_{t=1}^{r} (J_t, \omega_t),$$

where J_l, J_t are generated by n elements. Now, adapting the proof of (4.2), using the addition and subtraction principles for property (p) and the fact that property (p) holds for ideals generated by n elements, it follows that J satisfies property (p) i.e. $(J, \omega_J) + (J, -\omega_J) = 0$ in $E(A)$. This proves the corollary. □

COROLLARY 7.10. *Let A be a Noetherian ring of odd dimension n. Let P be a projective A-module of rank n with trivial determinant. Assume that the kernel of the canonical surjection $E(A) \twoheadrightarrow E_0(A)$ has no non trivial 2-torsion. Suppose that $e(P) = 0$ in $E_0(A)$. Then P has a unimodular element.*

Proof. Let $\alpha : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ is an ideal of height n. Let χ be a generator of $\wedge^n(P)$. Suppose that $e(P, \chi) = (J, \omega_J)$ in $E(A)$ (where (J, ω_J) is obtained from the pair (α, χ)). Since n is odd, scalar multiplication by -1 is an automorphism θ of P of determinant -1. Now, computing $e(P, \chi)$ using the surjection $\alpha\theta$, we see that $e(P, \chi) = (J, -\omega_J)$. Hence $2(e(P, \chi)) = (J, \omega_J) + (J, -\omega_J)$ in $E(A)$. Since $e(P) = 0$ in $E_0(A)$, it follows from (7.9), that $2(e(P, \chi)) = 0$ in $E(A)$. Since the kernel of the surjection $E(A) \twoheadrightarrow E_0(A)$ has no non trivial 2-torsion, $e(P, \chi) = 0$ in $E(A)$. Hence, by (4.4), P has a unimodular element. □

In ([B-RS 2], (6.3)), an example is given to show that $E(A)$ can have nontrivial 2-torsion. However, in this example, the kernel of the canonical homomorphism $E(A) \twoheadrightarrow E_0(A)$ is zero. Therefore, in view of (7.10), one can ask:

QUESTION 7.11. *Let A be a Noetherian ring of dimension $n \geqslant 2$. Can the kernel of the canonical homomorphism $E(A) \twoheadrightarrow E_0(A)$ have non trivial 2-torsion ?*

QUESTION 7.12. *Let A be a Noetherian ring of odd dimension n and P a projective module of rank n having determinant L. Suppose that $e(P) = 0$ in $E_0(A, L)(= E_0(A))$. Does P have a unimodular element?*

Let A be a Noetherian ring of dimension n and P a projective A-module of rank n with determinant L. Suppose that there exists a projective A module Q of rank $n - 1$ such that $[P] = [Q \oplus A]$ in $K_0(A)$. Then, it is easy to show using (6.7) and (4.4), that $e(P) = 0$ in $E_0(A, L)$. In this case we can answer (7.12). In fact, we prove the following theorem (see also ([RS 2], Theorem 4.2)).

THEOREM 7.13. *Let A be a Noetherian ring of odd dimension n. Let P_1 be a projective A-module of rank n with determinant L. Suppose that there exists a projective*

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A module Q of rank $n-1$ such that $[P_1] = [Q \oplus A]$ in $K_0(A)$. Then P_1 has a unimodular element.

Proof. Let $P = Q \oplus A$. Since $[P_1] = [P]$ in $K_0(A)$, we have $P_1 \oplus A \xrightarrow{\sim} P \oplus A$. Hence, there exists an exact sequence

$$0 \to P_1 \to A \oplus P \overset{(b,-\alpha)}{\to} A \to 0.$$

By (2.13), we may assume that height $(J = \alpha(P)) \geqslant n$. By (2.8), the map $\beta : P_1 \to A$ given by $\beta(q) = c$ where $q = (c, p)$, is such that $\beta(P_1) = \alpha(P) = J$. Hence, if $J = A$, P_1 has a unimodular element and there is nothing to prove. Therefore, we may assume that height $(J) = n$.

Let $\chi : \wedge^n(L \oplus A^{n-1}) \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Let $e(P, \chi) = (J, \omega_J)$ in $E(A, L)$ (where (J, ω_J) is obtained from (α, χ)). Since $P = Q \oplus A$, by (4.4), $(J, \omega_J) = 0$ in $E(A, L)$. By (5.1), there exists an isomorphism $\chi_1 : \wedge^n(L \oplus A^{n-1}) \xrightarrow{\sim} \wedge^n(P_1)$ such that $e(P_1, \chi_1) = (J, \overline{a^{n-1}}\omega_J)$ in $E(A, L)$ (where $(J, \overline{a^{n-1}}\omega_J)$ is obtained from (β, χ_1) and a satisfies the property that $ab = 1$ modulo J). Since n is odd, by (5.4), $(J, \overline{a^{n-1}}\omega_J) = (J, \omega_J)$ in $E(A, L)$. Therefore $e(P_1, \chi_1) = 0$ in $E(A, L)$. Hence, by (4.4), P_1 has a unimodular element. □

We conclude this section by exhibiting some relations between the Grothendieck–Witt group of quadratic forms of the residue fields of the maximal ideals of A and $E(A)$.

PROPOSITION 7.14. *Let A be a smooth affine domain of dimension $n \geqslant 2$ over a field of charactereristic zero. Let m be a maximal ideal of A and ω_m a generator of $\wedge^n(m/m^2)$. Then, the following relations hold*

(1) *If $\alpha \in (A/m)^*$ then $(m, \omega_m) = (m, \alpha^2\omega_m)$ in $E(A)$.*
(2) *If $\alpha, \beta \in (A/m)^*$ such that $\alpha + \beta$ is not zero, then $(m, \alpha\omega_m) + (m, \beta\omega_m) = (m, (\alpha + \beta)\omega_m) + (m, \alpha\beta(\alpha + \beta)\omega_m)$.*

Proof. By (5.4), (1) holds. We prove (2). Let ω_m correspond to the generator $\overline{a_1} \wedge \cdots \wedge \overline{a_n}$ of $\wedge^n(m/m^2)$. Using ([Sw], (1.3) and (1.4)), we may assume that $(a_1, \cdots, a_n) = m \cap J'$, where J' is a reduced ideal of height n. By performing elementary transformations on $(a_1, \cdots, a_n)$, we may assume that $(a_3, \cdots, a_n)$ generates a prime ideal such that $A/(a_3, \cdots, a_n)$ is a smooth affine domain of dimension 2. Let bar denote reduction modulo $(a_3, \cdots, a_n)$. Let $\overline{\omega}_{\overline{m}}$ be the generator of $\wedge^2(\overline{m}/\overline{m}^2)$ obtained from $\overline{a_1}, \overline{a_2}$. Since, under the canonical homomorphism $E(\overline{A}) \to E(A)$, $(\overline{m}, \overline{\omega}_{\overline{m}})$ is mapped to (m, ω_m), it follows that in order to prove (2), we may replace A by $\overline{A}$ and ω_m by $\overline{\omega}_{\overline{m}}$. Thus, we may assume that $\dim A = 2$.

Calling $\alpha\omega_m$ as ω_m and using (1), we see that it is enough to show that $(m, \omega_m) + (m, \lambda\omega_m) = (m, (1 + \lambda)\omega_m) + (m, \lambda(1 + \lambda)\omega_m)$ in $E(A)$ (where $\lambda \in (A/m)^*$ is such that $(1 + \lambda) \neq 0$). As before, using ([Sw], (1.3.) and (1.4)), we see that

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$(b_1, b_2) = m \cap J$ where $J \subset A$ is a reduced ideal of height 2 and ω_m is obtained from b_1, b_2. Let $m \cap J = J_1$. Let ω_J be the generator of $\wedge^2(J/J^2)$ obtained from b_1, b_2 and ω_{J_1} be the generator $\overline{b_1} \wedge \overline{b_2}$ of $\wedge^2(J_1/J_1{}^2)$. Let $\mu \in A/J$ be chosen with the property that μ, $1+\mu \in (A/J)^*$ and both are squares. Let $\delta \in (A/J_1)^*$ be chosen such that $\delta = \lambda$ modulo m and $\delta = \mu$ modulo J.

Suppose that we show that

$$(J_1, \omega_{J_1}) + (J_1, \delta\omega_{J_1}) = (J_1, (1+\delta)\omega_{J_1}) + (J_1, \delta(1+\delta)\omega_{J_1})$$

in $E(A)$, then it would follow that $(m, \omega_m) + (m, \lambda\omega_m) + (J, \omega_J) + (J, \mu\omega_J) = (m, (1+\lambda)\omega_m) + (m, \lambda(1+\lambda)\omega_m) + (J, (1+\mu)\omega_J) + (J, \mu(1+\mu)\omega_J)$ in $E(A)$. Since both μ and $1+\mu$ are squares in $(A/J)^*$, it would follow that $(m, \omega_m) + (m, \lambda\omega_m) = (m, (1+\lambda)\omega_m) + (m, \lambda(1+\lambda)\omega_m)$ in $E(A)$. Therefore, it suffices to show that $(J_1, \omega_{J_1}) + (J_1, \delta\omega_{J_1}) = (J_1, (1+\delta)\omega_{J_1}) + (J_1, \delta(1+\delta)\omega_{J_1})$.

From the definition of ω_{J_1}, it follows that $(J_1, \omega_{J_1}) = 0$ in $E(A)$. Thus, it suffices to prove that the following relation holds in $E(A)$:

$$(J_1, \delta\omega_{J_1}) = (J_1, (1+\delta)\omega_{J_1}) + (J_1, \delta(1+\delta)\omega_{J_1}). \tag{$**$}$$

Now from ([VK 2], relation 2 on p. 293) and ([VK 1], (3.16,(iii))), it follows that the following equation holds in the group $Um_3(A)/SL_3(A)$ (where b_0 is a preimage of δ in A) :

$$[b_0, b_1, b_2] = [1+b_0, b_1, b_2] * [b_0(1+b_0), b_1, b_2].$$

Taking images under $\Psi : Um_3(A)/SL_3(A) \to E(A)$, we see that $(**)$ holds and the proposition is proved. $\square$

The following theorem answers a question of Nori and follows from the previous proposition.

THEOREM 7.15. *Let A be a smooth affine domain of dimension n over a field of characteristic* 0. *Let m be a maximal ideal of A and $GW(k(m))$ denote the Grothendieck–Witt group of quadratic forms of the field $k(m) = A/m$. For each maximal ideal m of A, fix a generator ω_m of $\wedge^n(m/m^2)$. Then, there exists a surjective homomorphism*

$$\bigoplus_m GW(k(m)) \twoheadrightarrow E(A),$$

which sends the class of the one dimensional form $<\alpha> \in GW(k(m))$ to the element $(m, \alpha\omega_m)$ of $E(A)$.

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to us his work with Morel (preprint), where they give another definition of the Euler class group in the two-dimensional case which is functorial and prove among other things, results similar to (4.4) and (7.15).

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