Can. J. Math., Vol. XXX, No. 6, 1978, pp. 1319–1330

POINCARÉ TRANSVERSALITY FOR DOUBLE COVERS

I. HAMBLETON AND R. J. MILGRAM

Let $\pi: X' \to X$ be a double cover of $2n$-dimensional Poincaré duality (PD) spaces. The double cover is a fibering so it is classified by a map $f: X \to RP^{l+1} (l \gg n)$. If the homotopy class of f contains a representative which is Poincaré transverse [**5**] to $RP^l \subset RP^{l+1}$, we say that π is *Poincaré splittable*. In this paper we prove that an invariant $A(X, f)$ in $Z/2$ (defined in [**4**]) is the complete obstruction to finding a Poincaré splittable double cover bordant to (X, f). In addition we give an explicit example in each dimension $2n \geqq 4$ which is not Poincaré splittable. These examples are used to recover information on secondary operations in Thom spaces of certain spherical fibrations and reprove Browder's results [**1**] on the maps in the Levitt exact sequence.

The basic information about the invariant $A(X, f)$ is given in Sections 1 and 2. It is the Arf invariant of a quadratic map

$$q: H^n(X', Z/2) \to Z/2$$

refining the non-singular bilinear form

$$l(a, b) = \langle a \cup T^*b, [X'] \rangle$$

where $a, b \in H^n(X', Z/2)$ and $T: X' \to X'$ is the free involution. This quadratic map is shown to be the same as the Browder–Livesay map (Theorem 1.4) used in [**2**] to define a desuspension obstruction for smooth free involutions on homotopy spheres.

In § 3 we describe a basic example in dimension 4 with A-invariant non-zero. This PD space is the orbit space of a free simplicial involution on a finite simplicial complex with the homotopy type of $S^2 \times S^2$. A non-splittable example in each even dimension is then given (Theorem 3.1) by forming the product with suitable smooth manifolds so that in each case the covering space has the homotopy type of a manifold. Some variations of the construction are described and in § 4, we show how to use the A-invariant to compute some secondary operations (Corollaries 4.4, 4.5).

In the final section, § 5, we state the results about Poincaré transversality needed to identify the A-invariant with the obstruction to Poincaré splittability (Theorem 5.6). These results were obtained by Levitt [**8**], Jones [**7**] and Quinn

Received August 19, 1977 and in revised form, March 30, 1978. The work of the first author was partially supported by the National Research Council of Canada, Grant A4000, and that of the second author was partially supported by NSF Grant MPS 74-07491A01.

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

[**11**] and take the form of an exact sequence measuring the obstructions to transversality for a map $S^{n+k} \to T(\eta^k)$ where $T(\eta^k)$ is the Thom space of a $(k-1)$-spherical fibration.

It is a pleasure to thank the Mathematics Institute of Aarhus University for its support during the writing of this paper.

1. A quadratic map for double covers. In this section, we recall the definition of the quadratic map q and prove that it equals the Browder–Livesay map. We use $Z/2$ coefficients throughout and $[X]$ denotes the fundamental class of a PD space X.

Let $\pi: X' \to X$ be a double cover of $2n$-dimensional PD spaces classified by $f: X \to RP^\infty$. We denote the involution on X' by T and the map covering f by $f' : X' \to S^\infty$. Form $S^\infty \times_{Z/2}(X' \times X')$ where $Z/2$ acts on $X' \times X'$ by interchanging the factors and define $F: X \to S^\infty \times_{Z/2}(X' \times X')$ to be the quotient of the equivariant map

$$F' : X' \to S^\infty \times (X' \times X')$$

given by $F'(x) = (f'(x), (x, Tx))$. Now if $a_\#$ is a cocycle on X' representing $a \in H^n(X')$, then $1 \otimes a_\# \otimes a_\#$ is an equivariant cocycle on $S^\infty \times (X' \times X')$ and so represents a class $\alpha \in H^{2n}(S^\infty \times_{Z/2}(X' \times X'))$.

Definition 1.1. $q(a) = \langle F^*(\alpha), [X] \rangle$.

Let $Y = S^\infty \times_{Z/2} X'$ and define $\lambda: Y \to S^\infty \times_{Z/2}(X' \times X')$ by $\lambda[u, x] = [u, (x, Tx)]$. If $\rho: Y \to X$ is given by $\rho[u, x] = \pi(x)$ then ρ is a homotopy equivalence and $F \circ \rho \simeq \lambda$. We now describe a chain approximation for λ. Suppose that $T: X' \to X'$ is simplicial and $T\sigma \cap \sigma = \emptyset$ for all simplices $\sigma \in X'$. Partially order the simplices so that $T(a \cup_i b) = Ta \cup_i Tb$ where $\cup_i$ denotes the Steenrod cup-sub-i-product. We give S^∞ its usual equivariant cellular decomposition with cells e_i and Te_i in each dimension. In the statement below, $\Delta_j : C_k(X') \to C_{k+j}(X' \times X')$ is the jth Steenrod map [**12**] and $\tau: C_k(X' \times X') \to C_k(X' \times X')$ is defined by $\tau(a \otimes b) = b \otimes a$. We recall the formulas:

$$\partial\Delta_j = (1 + \tau)\Delta_{j-1} + \partial_j\Delta \quad \text{and} \quad \Delta_j \circ T = (T \otimes T)\Delta_j.$$

THEOREM 1.2. *The map given by*

$$\lambda_\#(e_i \otimes c) = \sum_{0 \leqq j \leqq i} e_j \otimes (1 \otimes T)\tau^j\Delta_{i-j}(c)$$

and

$$\lambda_\#(Te_i \otimes c) = (T \otimes \tau)\lambda_\#(e_i \otimes Tc), \quad \text{for } c \in C_k(X'),$$

is a chain approximation to λ.

Proof. First we define an equivariant acyclic carrier $\mathscr{C}$ from $C_*(S^\infty \times X')$ to $C_*(S^\infty \times X' \times X')$ by $\mathscr{C}(e_i \otimes \sigma) = C_*(e_i \times \sigma \times T\sigma)$. Since $\lambda_\#$ is carried

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

by $\mathscr{C}$ and agrees with λ on the 0-skeleton, by the acyclic models theorem it remains to show that $\lambda_\#$ is a chain map.

First:

$$\begin{aligned}\partial\lambda_\#(e_i \otimes c) &= \sum_{0\leqq j\leqq i} (1+T)e_{j-1} \otimes (1\otimes T)\tau^j\Delta_{i-j}(c)\\ &\qquad + \sum_{0\leqq j\leqq i} e_j \otimes (1\otimes T)\tau^j[(1+\tau)\Delta_{i-j-1}(c) + \Delta_{i-j}(\partial c)]\\ &= \lambda_\#(e_i\otimes \partial c) + [1 + T\otimes (T\otimes T)\tau]\lambda_\#(e_{i-1}\otimes c).\end{aligned}$$

However,

$$\begin{aligned}\lambda_\#((1+T)e_{i-1}\otimes c) &= \lambda_\#(e_{i-1}\otimes c) + T\otimes \tau\lambda_\#(e_{i-1}\otimes Tc)\\ &= \lambda_\#(e_{i-1}\otimes c) + T\otimes\tau\left[\sum_{0\leqq j\leqq i-1} e_j\otimes (1\otimes T)\tau^j(T\otimes T)\Delta_{i-j-1}(c)\right]\\ &= [1+T\otimes (T\otimes T)\tau]\lambda_\#(e_{i-1}\otimes c).\end{aligned}$$

Therefore $\partial\lambda_\# = \lambda_\#\partial$ as required.

COROLLARY 1.3. *For $a \in H^n(X')$,*

$$q(a) = \left\langle \sum_{i=0}^{n} e^i \otimes (a_\# \cup_i Ta_\#), [Y]\right\rangle$$

where $a_\#$ is a cocycle representing a, e^i is dual to e_i and $\rho_*[Y] = [X]$.

Proof. Let c be any chain in $C_*(X')$. Then

$$\begin{aligned}\langle\lambda^\#(1\otimes a_\#\otimes a_\#), e_i\otimes c\rangle &= \langle 1\otimes a_\#\otimes a_\#, \lambda_\#(e_i\otimes c)\rangle\\ &= \left\langle 1\otimes a_\#\otimes a_\#, \sum_{0\leqq j\leqq i} e_j\otimes(1\otimes T)\tau^j\Delta_{i-j}(c)\right\rangle\\ &= \langle a_\#\cup_i Ta_\#, c\rangle.\end{aligned}$$

Now

$$\lambda^\#(1\otimes a_\#\otimes a_\#) = \sum_{i=0}^{n} e^i\otimes (a_\#\cup_i Ta_\#)$$

and the result follows from the relation $\lambda^\# = \rho^\# F^\#$.

With this explicit cochain formula, we can relate q to the Browder–Livesay map $\psi : H^n(X') \to Z/2$. First we summarize its definition [**2**].

Let x be a cocycle in $C^n(X')$ representing a cohomology class a. Cochains v^{n+j} are constructed for $0 \leqq j \leqq n$ such that

$$x \cup_{n-j} Tx + \delta v^{n+j-1} = (1+T)v^{n+j}.$$

The cochain v^{2n} turns out to be determined by x modulo $\delta C^{2n-1}(X') \oplus (1+T)C^{2n}(X')$ and the equivariant cohomology class $\{(1+T)v^{2n}\} \in$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

${H^{2n}}_{Z/2}(C_*(X')) \cong H^{2n}(X)$ depends only on a. They set

$$\psi(a) = \langle \{(1 + T)v^{2n}\}, [X] \rangle.$$

THEOREM 1.4. *For all* $a \in H^n(X')$, $\psi(a) = q(a)$.

Proof. By construction, $(1 + T)v^{2n} = x \cup Tx + \delta v^{2n-1}$ where x is a cocycle representing a. Set

$$v = \sum_{i=0}^{n-1} e^{n-i-1} \otimes v^{n+1}$$

and compute

$$\delta v = \sum_{i=0}^{n} e^i \otimes (x \cup_i Tx) + e^0 \otimes (1 + T)v^{2n}.$$

Therefore $\delta v = \lambda^{\#}(e^0 \otimes x \otimes x) + \rho^{\#}(1 + T)v^{2n}$, so

$$\langle \lambda^{\#}(e^0 \otimes x \otimes x), [Y] \rangle = \langle e^0 \otimes (1 + T)v^{2n}, [Y] \rangle$$

and the result follows.

The final result of this section is a formula for evaluating $q(b)$ when $b \in \operatorname{im}\pi^*$. In the statement $\mathrm{Sq}_i : H^n(X) \to H^{2n-i}(X)$ is Steenrod's operation [**12**].

PROPOSITION 1.5. *Let* $\pi: X' \to X$ *be a double cover of* $2n$*-dimensional* PD *spaces and* $\bar{b} \in H^n(X)$. *Then*

$$q(\pi^*\bar{b}) = \left\langle \sum_{i=0}^{n} f^*(u^i) \cup \mathrm{Sq}_i(\bar{b}), [X] \right\rangle$$

where u *generates* $H^1(RP^\infty)$.

Proof. The following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "F"] \arrow[d, "f \times 1"'] & S^\infty \times_{Z/2} (X' \times X') \arrow[d, "1 \times \pi \times \pi"] \\
RP^\infty \times X \arrow[r, "1 \times_{Z/2} \Delta"] & S^\infty \times_{Z/2} (X \times X)
\end{tikzcd}$$

Since $(1 \times \pi \times \pi)^*(1 \otimes \bar{b} \otimes \bar{b}) = 1 \otimes \pi^*\bar{b} \otimes \pi^*\bar{b}$ and

$$(1 \times {}_{Z/2}\Delta)^*(1 \otimes \bar{b} \otimes \bar{b}) = \sum_{i=0}^{n} e^i \otimes \mathrm{Sq}_i(\bar{b}),$$

we obtain,

$$q(\pi^*\bar{b}) = \langle F^*(1 \times \pi \times \pi)^*(1 \otimes \bar{b} \otimes \bar{b}), [X] \rangle$$
$$= \left\langle \sum_{i=0}^{n} f^*(u^i) \cup \mathrm{Sq}_i(\bar{b}), [X] \right\rangle.$$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

2. A product formula. We now recall the definition of $A(X, f)$ where $f : X \to RP^\infty$ classifies the double cover $\pi : X' \to X$ of $2n$-dimensional PD spaces. According to [**2**] or [**4**].

$$q(a + b) - q(a) - q(b) = \langle a \cup T^*b, [X'] \rangle$$

for all $a, b \in H^n(X')$. The bilinear form defined by the formula on the right-hand side is non-singular and hence there exists a symplectic basis for $H^n(X')$ with respect to this form. $A(X, f)$ is the Arf invariant [**2**] associated to any such base. The first observation is:

PROPOSITION 2.1. *$A(X, f)$ defines a homomorphism $A : \mathcal{N}_{2n}{}^{\mathrm{PD}}(RP^\infty) \to Z/2$.*

Proof. Since A is additive on disjoint unions, it is enough to check that if $(X, f) = \partial(W, h)$ then $A(X, f) = 0$. But this is obvious from the definition of q by the usual argument, namely that q vanishes on im $(i^* : H^n(W') \to H^n(X'))$, an isotropic subspace of half the rank of $H^n(X')$.

The second observation, a result of [**4**], is:

PROPOSITION 2.2. *If $\pi : X' \to X$ is a Poincaré splittable double cover of $2n$-dimensional* PD *spaces, then $A(X, f) = 0$ where $f : X \to RP^\infty$ classifies π.*

For the later calculations, we need to compute the A-invariant of product covers

$$X' \times N \xrightarrow{\pi \times 1} X \times N$$

where N is a PD space of dimension $2m$. Our main applications are the cases $N = CP^2$ and $N = RP^2$.

THEOREM 2.3. *Let $\pi \times 1 : X' \times N \to X \times N$ be a product covering where N is a $2m$-dimensional* PD *space. Let $a \in H^p(X')$ and $b \in H^r(N)$ with $p + r = n + m$, then*

$$q(a \otimes b) = \left\langle \sum_{0 \leqq j \leqq r} F^*(1 \otimes a \otimes a) \cup f^*(u^j) \otimes \mathrm{Sq}_j(b), [X] \otimes [Y] \right\rangle$$

where u generates $H^1(RP^\infty)$.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
X' \times N \arrow[r, "F'"] \arrow[d, "\Delta \times 1"'] & S^\infty \times (X' \times N) \times (X' \times N) \arrow[d, "\Delta \times 1 \times 1"'] \\
(X' \times X') \times N \arrow[d, "1 \times f' \times 1"'] & S^\infty \times S^\infty \times (X' \times N) \times (X' \times N) \arrow[d, "\text{shuffle}"'] \\
(X' \times S^\infty \times N) \arrow[r, "F' \times 1 \times \Delta"] & S^\infty \times (X' \times X') \times S^\infty \times (N \times N)
\end{tikzcd}$$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

If we give $(X' \times X' \times N)$ the involution $T \times T \times 1$, then all the maps are equivariant so we get a diagram on the quotient spaces. Let G be the composite of the vertical right-hand maps, then

$$F^*G^*(1 \otimes a \otimes a \otimes 1 \otimes b \otimes b) = F^*(1 \otimes a \otimes b \otimes a \otimes b),$$

and the proof is completed by setting this equal to the other composite.

Two easy applications are:

COROLLARY 2.4. *If* $b \in H^m(N)$, *then* $q(a \otimes b) = q(a) \langle b^2, [N] \rangle$.

COROLLARY 2.5. *If* $b \in H^r(N)$, *where* $r < m$, *then* $q(a \otimes b) = 0$.

The product formula we need is:

COROLLARY 2.6. *If* $\pi \times 1 : X' \times N \to X \times N$ *is the product covering with* $N = CP^2$ *or* RP^2, *then* $A(X \times N, f.p_1) = A(X, f)$ *where* $p_1 : X \times CP^2 \to X$ *is the projection.*

Proof. We give the proof only for $N = CP^2$ as the other case is similar. Consider the equivariant decomposition

$$H^{n+2}(X' \times CP^2) \cong H^n(X') \otimes H^2(CP^2) \oplus H^{n+2}(X') \otimes H^0(CP^2) \oplus H^{n-2}(X') \otimes H^4(CP^2).$$

With respect to the bilinear form above, the first summand is self-dual and orthogonal to the other two. The second and third summands are dually paired by Poincaré duality. Since, by Corollary 2.5, q vanishes on the second summand, the result is established by Corollary 2.4.

3. Examples of non-splittable double covers. In this section we will describe a basic example in dimension 4 with $\pi_1(X^4) = Z/2$ and $A(X^4, f) = 1$ where $f : X^4 \to RP^\infty$ classifies the universal cover. The product formula is then applied to give examples in each even dimension of non-splittable double covers. Some variations of the construction are also sketched, and an orientable example X^6 given in dimension 6. The result that X^4 and X^6 have non-zero A-invariant is used in the next section to give information on secondary operations in certain Thom complexes associated to them.

The example X^4 in dimension 4 is among those constructed in [**13**, p. 240]. Let K be the 3-skeleton of $RP^2 \times S^2$ in a normal cell decomposition, fixed for the remainder of the discussion. Note that the universal cover $K' \simeq S_1^2 \vee S_2^2 \vee S^3$. We obtain X^4 by attaching the 4-cell σ^4 by a different map than that used to get $RP^2 \times S^2$. To describe the map we denote generators of $\pi_2 S_i^2$, $\pi_3 S^3$ and $\pi_3 S_i^2$ by I_i, J and η_i respectively for $i = 1, 2$. Note that by construction the 3-skeleton K of X^4 will be the same as that of $RP^2 \times S^2$. Then, according to the Hilton–Milnor theorem, $\pi_3 K$ is generated by J, η_1, η_2 and $[I_1, I_2]$. The $Z/2$ action on these is given by

$$TJ = J - [I_1, I_2], \quad T\eta_i = \eta_i, \quad T[I_1, I_2] = -[I_1, I_2]$$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

and the attaching map used to obtain $RP^2 \times S^2$ has class J. To construct X^4 we use a map in the class $J + \eta_1$, where the notation is chosen so that S_1^2 is the sphere covering RP^2 in the universal cover of $(RP^2 \times S^2)_{(3)}$. Since $(1 - T)\sigma^4$ is then attached with class $[I_1, I_2]$, $X' \simeq S^2 \times S^2$. Observe that X^4 is non-orientable. This PD space has $A(X^4, f) = 1$ where the map $f \colon X \to RP^\infty$ induces the universal covering $\pi \colon X' \to X$. To see this we need a description for the generators of $H^2(X')$. By construction, $X^4 \simeq (RP^2 \vee S^2) \cup \sigma^3 \cup \sigma^4$. A basis for $H^2(X') \cong \mathrm{Hom}(H_2(X'), Z/2)$ is given by duality from the basis of H_2X' represented by covers of $RP^2 \subset RP^2 \vee S^2 \subset X^4$ and $S^2 \subset RP^2 \vee S^2 \subset X^4$. Denote these classes by a and b respectively. Then $b = \pi^*\bar{b}$ for some $\bar{b} \in H^2(X^4)$ and from Proposition 1.5.

$$q(b) = \left\langle \sum_{i=0}^{2} f^*(u^i) \cup \mathrm{Sq}_i(\bar{b}), [X] \right\rangle = \langle \bar{a} \cup \bar{b}, [X] \rangle = 1$$

where $\bar{a}$ is dual to the class represented by $RP^2 \subset RP^2 \vee S^2 \subset X^4$. Since $\{a, b\}$ form a symplectic basis of $H^2(X')$, to prove that $A(X^4, f) = 1$ it is enough to check $q(a) = 1$.

For this recall that by Corollary 1.3 we must compute $a_\# \cup_i Ta_\#$ where $a_\#$ is the obvious cochain representing a. Clearly $a_\# \cup_2 Ta_\# = 0$. but for the others it is convenient to construct a complex L by collapsing $* \vee S^2$ and then the resulting S^3 in X^4 to a point (the notation again refers to the normal cell decomposition as above). Let $p \colon X \to L$ be the quotient map and note that $L \simeq RP^2 \cup \sigma^4$ so $H^2(L') = Z/2$ generated by a class c such that $p^*c = a$. Also $p^* \colon H^4(L) \to H^4(X)$ is an isomorphism. It is important to observe that the attaching map of the 4-cell in L represents $\eta_1 \in \pi_3(RP^2) \cong \pi_3(S_1^2)$. Since η_1 mod 2 is detected by Sq^2, we have $c_\# \cup_0 Tc_\# = (1 + T)\sigma^4$ where σ^4, $T\sigma^4$ are the generators of $C^4(L')$. Now $c_\# \cup_2 Tc_\# = 0$ and $c_\# \cup_1 Tc_\# = 0$ hence $q(c) = 1$. By naturality, $q(a) = q(p^*c) = 1$.

Our main result on the existence of non-splittable covers is:

THEOREM 3.1. *In each dimension $2n \geqq 4$ there exists a PD space X^{2n} and a map $f \colon X^{2n} \to RP^\infty$ such that $A(X^{2n}, f) = 1$ and X' has the homotopy type of a smooth manifold.*

Proof. The method is clear. We simply form the product of $X' \to X^4$ with suitably many copies of CP^2 and RP^2. The result follows from Corollary 2.6.

Another example X^6 can be constructed in a similar way to X^4. One takes the 4-skeleton of $RP^3 \times S^3$ and reattaches the 5-cell using the generator of π_4S^3 corresponding to the first summand of

$$\pi_4((RP^3 \times S^3)_{(4)}) \cong \pi_4(RP^3) \oplus \pi_4(S^3) \oplus \pi_4(S^4)$$

together with a generator of $\pi_4(S^4)$. This PD space was described in [**6**] and motivated the construction of X^4. One sees that X^6 is orientable, $\pi_1(X^6) = Z/2$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

and $A(X^6, f) = 1$, however X' is homotopy equivalent to the non-trivial S^3 fibration over S^3 which is not the homotopy type of a manifold.

The construction given here can also be attempted in higher dimensions. The resulting examples will not be used in the rest of the paper so some details are omitted. Let $K_0 = (RP^n \times S^n)_{(n+1)}$ be the $(n + 1)$-skeleton in a normal cell decomposition. Since

$$\pi_{n+1}(K_0) \cong \pi_{n+1}(RP^n) \oplus \pi_{n+1}(S^n) \oplus \pi_{n+1}(S^{n+1})$$

we can construct a complex K by attaching an $(n + 2)$-cell to K_0 using a map representing $\eta + \alpha$ where $\eta \in \pi_{n+1}(RP^n)$ is the generator and $\alpha \in \pi_{n+1}(K_0)$ is the class used to get the normal $(n + 2)$-skeleton of $RP^n \times S^n$. From [**10**], η is a projective element if and only if $n \equiv 2 \pmod 4$. If we now use a projective η then we will be able to choose a map ϕ: $S^{n+1} \to K'$ whose class generates a summand of $\pi_{n+2}(K)$ such that $T\phi \simeq \pm\phi$. This is the main difficulty in the proof of the following result.

PROPOSITION 3.2. *If* $n \equiv 2 \pmod 4$, *there exists a* PD *space* X^{2n} *with* $\pi_1(X) = Z/2, X' \simeq S^n \times S^n, X_{(n+2)} \simeq K$ *in a normal cell decomposition and* $A(X, f) = 1$.

4. The Spivak normal bundles to X^4 and X^6.

The purpose of this section is to clarify the structure of these non-splittable covers by finding the stable homotopy types of certain Thom complexes.

Define an injection $\rho : \mathcal{N}_*^{\mathrm{PD}}(pt) \to \mathcal{N}_*^{\mathrm{PD}}(RP^\infty)$ by $\rho[X^n] = [X^n, w_1]$ where $w_1 : X^n \to RP^\infty$ classifies the first Stiefel–Whitney class of X.

The following result contains the result of Browder [**1**] on the maps in the Levitt exact sequence and will be proved in the next section. Another proof can be found in [**5**, Theorem 10.6].

THEOREM 4.1. *The Pontrjagin–Thom map*

$$p: \mathcal{N}_m^{\mathrm{PD}}(RP^\infty) \to \pi_m^s(RP^\infty \wedge MG)$$

is an injection for $m \geqq 4$, *so characteristic numbers detect each bordism class.*

Consider the class $[X^4]$ in $\mathcal{N}_4^{\mathrm{PD}}(pt)$. We calculate that the Stiefel–Whitney class of X^4 is $1 + e^1$ and hence $[X^4, f] = \rho[X^4]$. Therefore $[X^4] \neq 0$ and we have

COROLLARY 4.2. *The characteristic number* $k_3.e^1 \neq 0$ *on* X^4 *and* $[X^4]$ *generates the cokernel of* im $(\mathcal{N}_4^{\mathrm{Diff}}(pt))$ *in* $\mathcal{N}_4^{\mathrm{PD}}(pt)$.

Here k_3 is the first exotic characteristic class and the result follows from checking the classes in dimension $\leqq 4$. (see [**9**]).

Let

$$\kappa: X^4 \xrightarrow{\tau} S^3 \cup_2 \sigma^4 \xrightarrow{\lambda} BG$$

be the composition where τ is the pinching map and λ satisfies $\lambda^*(k_3) \neq 0$, $\lambda^*(\omega_4) = 0$. Let (κ) be the induced fibre space (of some large fibre dimension).

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

COROLLARY 4.3. *The Spivak normal fibre space of* X^4 *is* $\eta_1 \oplus (\kappa)$ *where* η_1 *is the non-trivial line bundle.*

Proof. One checks easily that the set of homotopy classes of maps $[X^4, BG]$ are distinguished by their induced cohomology maps so the assertion follows from Corollary 4.2.

Let $\phi_{2;2,1}$ be the secondary operation based on the relation $\mathrm{Sq}^2(\mathrm{Sq}^2\,\mathrm{Sq}^1) = 0$. It is defined on the kernel of $\mathrm{Sq}^2\,\mathrm{Sq}^1$ with values in $H^{*+4}(-)/\mathrm{Sq}^2(H^{*+2}(-))$.

COROLLARY 4.4. *Let* U *be the Thom class of* $T(\eta_1)$ *over* X^4. *Then* $\phi_{2;2,1}(U) = e^4 \cup U$ *is non-zero with zero indeterminacy.*

Proof. Let V be the Thom complex of η_1 over the 3-skeleton of X^4. Then V is

$$(S^0 \cup_2 e^1) \vee (S^2 \cup_2 e^3)$$

and $\pi_3(V) = Z/2 \oplus Z/2 \oplus Z/2$ with generators $\nu(I_0)$, $\eta\langle\eta, 2, I_0\rangle$, ηI_2.

Now Steenrod squares detect the first and third generators, hence the attaching map of the top cell of $T(\eta_1)$ is $\epsilon\eta\langle\eta, 2, I_0\rangle$ where $\epsilon = 0$ or 1. If $\epsilon = 0$, then $T(\eta_1)$ is reducible contradicting 4.3. Therefore $\epsilon = 1$ and $\phi_{2;2,1}$ is defined on $T(\eta_1)$ and detects $\eta\langle\eta, 2, I_0\rangle$ so the result follows.

We can also apply the same ideas to X^6. The class $[X^6, e^1]$ in $\mathcal{N}_6^{\mathrm{PD}}(RP^\infty)$ is non-zero since $A(X^6, e^1) = 1$ and hence a characteristic number is non-zero. But the Stiefel–Whitney class of X^6 is 1 and the only indecomposable exotic classes in dimensions $\leqq 6$ are k_3, $\mathrm{Sq}^1 k_3$, $\mathrm{Sq}^2 k_3$, $\mathrm{Sq}^2\mathrm{Sq}^1 k_3$. Thus $(e^1)^3 \cup k_3$ is the only possible non-zero characteristic number.

COROLLARY 4.5. *The secondary operation* $\phi_{2,2}$ *based on the relation* $\mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^3\mathrm{Sq}^1 = 0$ *is defined on all of* $H^3(X^6)$ *with zero indeterminacy and is non-zero.*

Proof. It follows from the definition of k_3 (see [**9**]) that if $\phi_{2,2}$ is defined on all of $H^3(X^6)$ with zero indeterminacy then

$$\langle k_3 \cup a, [X^6]\rangle = \langle \phi_{2,2}(a), [X^6]\rangle.$$

5. The transversality obstruction. Let η^k be a $(k-1)$-spherical fibre space over a space B and consider triples (Y, g, b) such that Y is an m-dimensional PD space, $g: Y \to B$ a map and $b: \nu^k \to \eta^k$ covers g where ν^k is the normal fibration of a Poincaré embedding of Y in S^{m+k}. Define bordism of triples in the obvious way and call the set of bordism classes $T(S^{m+k}, T(\eta^k))$.

The general theory of [**7**], [**8**] or [**11**] gives an exact sequence (for $k \geqq 3$ and $m \geqq 4$)

$$\ldots \to L_m(\pi_1(B), w) \to T(S^{m+k}, T(\eta^k)) \xrightarrow{p} \pi_{m+k}(T(\eta^k)) \xrightarrow{\theta} L_{m-1}(\pi_1(B), w) \to \ldots$$

where $w = \omega_1(\eta^k)$, $L_m(\pi_1(B), w)$ are the surgery obstruction groups ($L = L^h$) of C.T.C. Wall [**14**] and θ is a "transversality obstruction". The sequence is natural under maps $h: B \to B'$ covered by $\hat{h}: \eta \to \eta'$.

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

Let l, k be large compared to n (so that the groups below are stable) and $\eta_1 \to RP^l$ the canonical line bundle. Then if $\omega_k \to BG(k)$ is the universal $(k-1)$-spherical fibration $T(\omega_1 \times \eta_1) = MG(k) \wedge RP^{l+1}$ and we can combine two special cases of the Levitt sequence into one diagram,

(5.1)
$$
\begin{tikzcd}
 & 0 \arrow[d] & \\
 & T(S^{2n+k}, T(\omega_k \times \eta_1)) \arrow[d, "p_0"] & \\
\ldots \arrow[r] & \mathcal{N}_{2n}^{PD}(RP^{l+1}) \arrow[r, "p"] \arrow[d, dotted, "A"] & \pi_{2n+k}(MG(k) \wedge RP^{l+1}) \arrow[r] \arrow[d, "\theta"] & 0 \\
 & Z/2 \arrow[r, "\cong"] & L_{2n-2}(Z/2 \times Z/2^-) &
\end{tikzcd}
$$

though at this point we cannot yet assert that the square above commutes. (The full diagram is defined for $n \geqq 3$; for $n = 2$ we will use the Pontrjagin–Thom maps). Since $A : \mathcal{N}_{2n}^{\mathrm{PD}}(RP^{l+1}) \to Z/2$ vanishes on im $(\mathcal{N}_{2n}^{\mathrm{PD}}(pt) \to \mathcal{N}_{2n}^{\mathrm{PD}}(RP^{l+1}))$ (Proposition 2.2) it can be regarded as a homomorphism $\tilde{\mathcal{N}}_{2n}^{\mathrm{PD}}(RP^{l+1}) \to Z/2$. We will now prove Theorem 4.1 and use this to prove that $A = \theta p$, i.e. that the square in 5.1 commutes.

PROPOSITION 5.2. *For the examples* (X^{2n}, f) *of Theorem* 3.1., $(n \geqq 2)$
$A(X^{2n}, f) = \theta p(X^{2n}, f) = 1.$

Proof. There is a cohomological formula for $\theta p(X^{2n}), f$ from Theorem F [**5**] involving a class $\tilde{\kappa} = (k_*) \in H^{4*-1}(BSG)$. (The component in dimension 3 is just k_3, cf. § 4). For $n = 2$, we interpret the right-hand side by this formula.

$$
\begin{aligned}
\theta p(X^{2n}, f) &= \langle p(f)^* \Phi \tilde{\kappa}(\omega_k \times \eta_1), [S^{2n+k}] \rangle \quad (\Phi = \text{Thom isomorphism}) \\
&= \langle p(f)^* \Phi(\tilde{\kappa}(\omega_k) \otimes V^2(\eta_1)), [S^{2n+k}] \rangle \\
&= \left\langle (\nu_X \times f)^* (\tilde{\kappa}(\omega_k) \otimes \sum_{i=1}^{\infty} (e^1)^{2^i - 1}), [X^{2n}] \right\rangle \\
&= \left\langle \sum_{i=1}^{\infty} k_{2n-2^i+1} \cdot (e^1)^{2^i - 1}, [X^{2n}] \right\rangle
\end{aligned}
$$

For the examples of Theorem 3.1, this formula reduces to $\theta p(X^{2n}, f) = \langle k_3 . e^1, [X^4] \rangle$. In order to give an alternate proof that this is non-zero (without using Theorem 4.1), consider the commutative diagram:

$$
\begin{tikzcd}
L_4(Z/2^-) \arrow[r] \arrow[d, "i_*"] & \mathcal{N}_4^{PD}(pt) \arrow[r] \arrow[d, "i_*"] & \pi_{4+k}(MG(k)) \arrow[d] \\
L_4(Z/2 \times Z/2^-) \arrow[r] & \mathcal{N}_4^{PD}(RP^{l+1}) \arrow[r, "p"] & \pi_{4+k}(MG(k) \wedge RP_+^{l+1})
\end{tikzcd}
$$

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

Since i_* is an isomorphism on the Wall groups [**14**], if $\langle k_3.e^1, [X^4] \rangle = 0$ then $[X^4, e^1] = i_*[Y]$. But this contradicts $A[X^4, e^1] = 1$.

Proof of Theorem 4.1. The classifying map of $\omega_k \times \eta_1$ induces a map of Levitt sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(S^{2n+k}, T(\omega_k \times \eta_1)) & \xrightarrow{p_0} & \pi_{2n+k}(MG(k) \wedge RP^{l+1}) & \longrightarrow & L_{2n-2}(Z/2 \times Z/2^-) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T(S^{2n+k}, T(\omega_{k+1})) & \longrightarrow & \pi_{2n+k}(MG(k+1)) & \xrightarrow{\theta} & L_{2n-2}(Z/2^-) & & \end{array}$$

From Proposition 5.2, θ in the upper sequence is onto ($n \geqq 3$) and the map on Wall groups is an isomorphism. Hence the map θ in the lower sequence is onto. (This is the result of Browder [**1**]). Finally, the naturality of the Levitt sequence for the inclusion $pt \subset RP^{l+1}$ (as in the proof of Proposition 5.2) yields Theorem 4.1.

COROLLARY 5.3. *The Pontrjagin–Thom map*

$$p: \tilde{\mathcal{N}}_{2n}{}^{\mathrm{PD}}(RP^{l+1}) \rightarrow \pi_{2n+k}(MG(k) \wedge RP^{l+1})$$

is an isomorphism for $n \geqq 2$ (*and* $l, k \gg n$).

These results will now be applied to show that $A = \theta p$.

Definition 5.4. Let $\mathcal{G}_{2n}$ be the subgroup of $\tilde{\mathcal{N}}_{2n}{}^{\mathrm{PD}}(RP^{l+1})$ containing those bordism classes with a Poincaré splittable representative.

PROPOSITION 5.5. *There is a monomorphism* s: $T(S^{2n+k}, T(\omega_k \times \eta_1)) \rightarrow \mathcal{G}_{2n}$ (*for* $n \geqq 3$ *and* $k \gg n$).

Proof. Let (Z^{2n-1}, g) represent a class in $T(S^{2n+k}, T(\omega_k \times \eta_1))$. (Since k is large, b is unique so can be ignored). By low-dimensional surgery on g, we may assume that $g_\#: \pi_1(Z) \rightarrow \pi_1(BG(k) \times RP^l)$ is an isomorphism. Let $E(\eta) \rightarrow Z$ be the pull-back of the disk bundle of η_1, $S(\eta)$ be the pull-back of η_1, then $(E(\eta), S(\eta))$ is a PD pair of dimension $2n$ with $S(\eta) \simeq Z'$ ($Z' \rightarrow Z$ is the 2-fold cover induced by $Z \xrightarrow{g} BG(k) \times RP^l \xrightarrow{p_2} RP^l$). If h: $E(\eta) \rightarrow E(\eta_1)$ is a bundle map covering $p_2.g$ and π: $E(\eta_1) \rightarrow RP^l$ the projection, we can assume $\pi.h|S(\eta) = *$ (a base point). Define $s(Z, g) = (E(\eta) \cup_{S(\eta)} Y, \pi h \cup *) = (X, f)$ where (Y, Z') is a PD pair bordant (rel. Z') to $(E(\eta), Z')$. It is easy to see that s is a well-defined homomorphism into $\tilde{\mathcal{N}}_{2n}{}^{\mathrm{PD}}(RP^{l+1})$ whose image, by construction, lies in $\mathcal{G}_{2n}$. If we surger $(E(\eta), Z')$ (rel. Z') to (Y, Z') with

$$\pi_1(Z') \xrightarrow{\text{incl.}} \pi_1(Y)$$

an isomorphism, and use this pair in the construction of $s(Z, g)$, it is evident that $(\nu_X) \times f_\#: \pi_1(X) \rightarrow \pi_1(BG(k) \times RP^{l+1})$ is an isomorphism (ν_X: $X \rightarrow BG(k)$ classifies the Spivak normal fibre space of X).

Suppose that $s(Z, g) \sim 0$, or equivalently that $(X, f) = \partial(W, h)$. Again we may assume that the inclusion map $X \rightarrow W$ induces an isomorphism on π_1.

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press

The map $h: W \to RP^{l+1}$ is now transverse to RP^l when restricted to X. The obstruction to making it transverse on all of W lies in LN_{2n-1} $(Z/2 \to Z/2 \times Z/2^-)$ according to § 5 [**7**]. Since this group is zero (by the exact sequence 12.9.2 [**15**]) $(Z, g) = \partial(V, \bar{h})$ where $\bar{h} \simeq h$ and $V = \bar{h}^{-1}(RP^l) \subset W$.

THEOREM 5.6. *The homomorphism* $A : \tilde{\mathcal{N}}_{2n}{}^{\mathrm{PD}}(RP^{l+1}) \to Z/2$ *can be identified with* θp *for* $n \geqq 2$.

Proof. For $n = 2$, we interpret $\theta.p$ by the cohomological formula used in Proposition 5.2. Since our example (X^4, f) generates the cokernel of im $(\mathcal{N}_4{}^{\mathrm{Diff}}(RP^{l+1}) \to \mathcal{N}_4{}^{\mathrm{PD}}(RP^{l+1}))$ (cf. 4.2) the result follows from 5.2.

For $n \geqq 3$, we note that $\mathcal{G}_{2n} \subseteq \ker A$ (Proposition 2.2) and that $\mathcal{G}_{2n} \subseteq \ker \theta.p$ since the Pontrjagin–Thom construction on a Poincaré splittable cover gives a Poincaré transverse map in $\pi_{2n+k}(MG(k) \wedge RP^{l+1})$. However, by Propositions 5.2, 5.3, 5.5, rank $\mathcal{G}_{2n}$ = rank ker A (as $Z/2$-vector spaces) so $\mathcal{G}_{2n} = \ker A = \ker \theta.p$.

REFERENCES

1. W. Browder, *The Kervaire invariant, products and Poincaré transversality*, Topology *12* (1973), 145–158.

2. W. Browder and R. Livesay, *Free involutions on homotopy spheres*, Tohoku Math. J. *25* (1972), 69–88.

3. G. Brumfiel, I. Madsen, and R. J. Milgram, *PL characteristic classes and cobordism*, Ann. of Math. *97* (1973), 83–159.

4. G. Brumfiel and R. J. Milgram, *Normal maps, covering spaces and quadratic functions*, Duke Math. J. *44* (1977), 663–694.

5. G. Brumfiel and J. Morgan, *Homotopy theoretic consequences of N. Levitt's obstruction theory to transversality for spherical fibrations*, Pac. J. Math. *67* (1976), 1–100.

6. I. Hambleton, *Free involutions on 6-manifolds*, Michigan Math. J. *22* (1975), 141–149.

7. L. Jones, *Patch spaces*, Ann. of Math. *97* (1973), 306–343.

8. N. Levitt, *Poincaré duality cobordism*, Ann. of Math. *96* (1972), 211–244.

9. R. J. Milgram, *The* mod(2) *spherical characteristic classes*, Ann. of Math. *92* (1970), 238–261.

10. R. J. Milgram, J. Strutt, and P. Zvengrowski, *Projective stable stems of spheres*, to appear.

11. F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. *78* (1972), 262–267.

12. N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Annals of Mathematics Studies *50* (Princeton, 1964).

13. C. T. C. Wall, *Poincaré complexes I*, Ann. of Math. *86* (1967), 213–245.

14. ——— *Classification of hermitian forms VI: Group rings*. Ann. of Math. *103* (1976), 1–80.

15. ——— *Surgery on compact manifolds* (Academic Press, 1970).

McMaster University,
Hamilton, Ontario;
Stanford University,
Stanford, California

https://doi.org/10.4153/CJM-1978-109-3 Published online by Cambridge University Press