Canad. Math. Bull. Vol. **27** (4), 1984

SEPARATING POINTS AND COLORING PRINCIPLES

BY
W. STEPHEN WATSON

ABSTRACT. In the mid 1970's, Shelah formulated a weak version of $\diamondsuit$. This axiom Φ is a prediction principle for colorings of the binary tree of height ω_1. Shelah and Devlin showed that Φ is equivalent to $2^{\aleph_0}<2^{\aleph_1}$.

In this paper, we formulate Φ_p, a "Φ for partial colorings", show that both $\diamondsuit^*$ and Fleissner's "$\diamondsuit$ for stationary systems" imply Φ_p, that $\diamondsuit$ does not imply Φ_p and that Φ_p does not imply *CH*.

We show that Φ_p implies that, in a normal first countable space, a discrete family of points of cardinality $\aleph_1$ is separated.

In the mid 1970's, Shelah [1] formulated a weak version of $\diamondsuit$. This axiom Φ is a prediction principle for colorings of the binary tree of height ω_1. This tree may be identified with Ω, the set of functions from a countable ordinal into 2.

$$(\Phi): \forall\, F:\Omega\to 2\ \exists\, g:\omega_1\to 2: \forall\, f:\omega_1 \to 2\ \exists \text{ stationary set } S: \forall\, \alpha\in S\ \ F(f\restriction\alpha)=g(\alpha)$$

The axiom states that however we color the nodes of the binary tree of height ω_1 with two colors, there is a coloring of ω_1 with two colors which coincides with the coloring of each branch in the tree on a stationary set. Shelah and Devlin [1] showed that Φ is equivalent to $2^{\aleph_0}<2^{\aleph_1}$.

In this paper, we formulate Φ_p, a "Φ for partial colorings", show that both $\diamondsuit^*$ and Fleissner's "$\diamondsuit$ for stationary systems" imply Φ_p and that $\diamondsuit$ does not imply Φ_p. Fleissner [2] formulated "$\diamondsuit$ for stationary systems" in 1972 in order to show that, in a normal first countable space, a discrete family of points of cardinality $\aleph_1$ is separated and asked whether this axiom was implied by $\diamondsuit^+$. Shelah [4] showed in 1976 that $\diamondsuit^+$ does not imply Fleissner's axiom but that $\diamondsuit^+$, nevertheless, implies that, in a normal first countable space, a discrete family of points of cardinality $\aleph_1$ is separated. We show that Φ_p implies that, in

Received by the editors January 12, 1983 and, in final revised form, March 14, 1984.

1980 AMS Subject Classification. Primary: 03E05, 54D15; Secondary: 54A35, 03E35, 03E45

This work has been supported by the Natural Sciences and Engineering Research Council of Canada.

a normal first countable space, a discrete family of points of cardinality $\aleph_1$ is separated.

To formulate Φ_p, we observe that a good partial coloring F (a coloring to which Φ_p can apply) must be such that, whenever $f:\omega_1 \to 2$, $\{\alpha : F(f \restriction \alpha)$ is not defined$\}$ does not contain a closed unbounded set. We require a little more: a *good* partial coloring F is a partial coloring which is such that, for any $\{f_n : \omega_1 \to 2 \ (n \in \omega)\}$, $\bigcup_{n\in\omega} \{(\alpha : F(f_n \restriction \alpha)$ is not defined)$\}$ does not contain a closed unbounded set.

$$(\Phi_p): \forall F : \Omega \xrightarrow{\text{good}} 2 \ \exists g : \omega_1 \to 2 : \forall f : \omega_1$$
$$\to 2 \ \exists \text{ stationary set } S : \forall \alpha \in S \ F(f \restriction \alpha) = g(\alpha).$$

Φ_P is a strengthening of Φ and therefore implies $2^{\aleph_0} < 2^{\aleph_1}$.

We need an equivalent formulation of $\Diamond^*$.

To motivate this formulation, we can say that $\Diamond$-principles predict subsets A of ω_1 and that a $\Diamond$-sequence consists of guessing $A \cap \alpha$ for each $\alpha \in \omega_1$. $\Diamond$ states that there is a $\Diamond$-sequence such that, for each $A \subset \omega_1$, there is a stationary set S such that, for each $\alpha \in S$, the $\Diamond$-sequence guesses $A \cap \alpha$ correctly. ZFC implies that there is no $\Diamond$-sequence such that, for each $A \subset \omega_1$, there is a closed unbounded set C such that, for each $\alpha \in C$, the $\Diamond$-sequence guesses $A \cap \alpha$ correctly but $\Diamond^*$ states that there *are* countably many $\Diamond$-sequences such that, for each $A \subset \omega_1$, there is a closed unbounded set C such that, for each $\alpha \in C$, at least *one* of the $\Diamond$-sequences guesses $A \cap \alpha$ correctly.

We need a principle which predicts countable sequences $(A_n : n \in \omega)$ of subsets of ω_1. the equivalent formulation of $\Diamond^*$ states that there is a $\Diamond$-sequence for each $n \in \omega$ (the nth $\Diamond$-sequence consists of guessing $A_n \cap \alpha$ for each $\alpha \in \omega_1$) such that, for each sequence $\{A_n : n \in \omega\}$ of subsets of ω_1, there is a closed unbounded set C such that, for each $\alpha \in C$, some $A_n \cap \alpha$ is guessed correctly by *its* $\Diamond$-sequence. The difference between $\Diamond^*$ and its equivalent formulation is that, in the former we know which set we're guessing correctly but we don't know which $\Diamond$-sequence is guessing it, whereas in the latter we don't know which set we're guessing correctly but we do know which $\Diamond$-sequence is guessing it.

LEMMA 1. $\Diamond^*$ *if and only if* $\exists \{S^n_\alpha : n \in \omega, \alpha \in \omega_1\} : \forall \{S^n : n \in \omega, S^n \subset \omega_1\} \ \exists$ *closed unbounded set* $C : \forall \alpha \in C \exists \ n \in \omega : S^n \cap \alpha = S^n_\alpha$.

Proof of Lemma 1. $\Diamond^*$ states that $\exists \{S^n_\alpha : n \in \omega, \ \alpha \in \omega_1\} : \forall S \subset \omega_1 \ \exists$ closed unbounded set $C : \forall \alpha \in C \ \exists \ n \in \omega : S \cap \alpha = S^n$. By a standard coding argument (ω_1^2 may be coded by ω_1), $\Diamond^*$ implies that $\exists \{S^{n,m}_\alpha : m, n \in \omega, \alpha \in \omega_1\} : \forall \{S^n : n \in \omega\} \ \exists$ closed unbounded set $C : \forall \alpha \in C \ \exists \ m \in \omega : \forall \ n \in \omega \ S^{n,m}_\alpha = S^n \cap \alpha$. Letting $S^m_\alpha = S^{m,m}_\alpha$ for each $m \in \omega$, we get that $\Diamond^*$ implies that $\exists \{S^n_\alpha : n \in \omega, \alpha \in$

https://doi.org/10.4153/CMB-1984-061-3 Published online by Cambridge University Press

$\omega_1\}:\forall\{S^n:n\in\omega\}\exists$ closed unbounded set $C:\forall\alpha\in C\,\exists\, m\in\omega:S^m_\alpha=S^m\cap\alpha$ as required. The other direction is obtained by letting $S^n=S$ for each $n\in\omega$.

THEOREM 1. *$\diamondsuit^*$ implies Φ_p.*

We need a preliminary lemma.

LEMMA 2. *$\diamondsuit^*$ implies $\forall F:\Omega\to 2\,\exists\{g_n:n\in\omega\}\subset{}^{\omega_1}2:\forall\{f_n:n\in\omega\}\subset{}^{\omega_1}2\,\exists$ closed unbounded set $C:\forall\alpha\in C\,\exists\, n\in\omega:g_n(\alpha)=F(f_n\upharpoonright\alpha)$.*

Proof of Lemma 2. We can state the equivalent formulation of $\diamondsuit^*$ in terms of functions in ${}^{\omega_1}2$ by identifying subsets of ω_1 with their characteristic functions. $\diamondsuit^*$ implies $\exists\{f^n_\alpha:n\in\omega,\alpha\in\omega_1\}:\forall\{f_n:n\in\omega\}\subset{}^{\omega_1}2\,\exists$ closed unbounded set $C:\forall\alpha\in C\,\exists\, n\in\omega:f^n\upharpoonright\alpha=f^n_\alpha$. Letting $g_n\in{}^{\omega_1}2$ be defined, for each $n\in\omega$, by $g_n(\alpha)=F(f^n_\alpha)$ and applying F to the equation $f_n\upharpoonright\alpha=f^n_\alpha$, we get that $\forall\{f^n:n\in\omega\}\subset{}^{\omega_1}2\,\exists$ closed unbounded set $C:\forall\alpha\in C\,\exists\, n\in\omega:F(f^n\upharpoonright\alpha)=g_n(\alpha)$ as required.

Proof of Theorem 1. Let $F:\Omega\xrightarrow{\text{good}}2$. Extend f to $F':\Omega\to 2$ arbitrarily. By Lemma 2, $\diamondsuit^*$ implies that $\exists\{g_n:n\in\omega\}\subset{}^{\omega_1}2:\forall\{f_n:n\in\omega\}\subset{}^{\omega_1}2\,\exists$ closed unbounded set $C:\forall\alpha\in C\,\exists\, n\in\omega:g_n(\alpha)=F'(f_n\upharpoonright\alpha)$. If Φ_p fails, then, for each $n\in\omega$, there is $f_n\in{}^{\omega_1}2$ and a closed unbounded set C_n such that $\forall\alpha\in C_n$ $g_n(\alpha)\neq F(f_n\upharpoonright\alpha)$ or $F(f_n\upharpoonright\alpha)$ is not defined. F is good implies that $D=\bigcup\{\{\alpha:F(f_n\upharpoonright\alpha)\text{ is not defined}\}:n\in\omega\}$ does not contain a closed unbounded set and so $E=(\omega_1-D)\cap C\cap\bigcap\{C_n:n\in\omega\}$ is nonempty. Let $\alpha\in E$. $\alpha\in C$ implies that, for some $n\in\omega$, $g_n(\alpha)=F'(f_n\upharpoonright\alpha)$, $\alpha\in\omega_1-D$ implies that $F(f_n\upharpoonright\alpha)$ is defined and $\alpha\in C_n$ implies $g_n(\alpha)\neq F(f_n\upharpoonright\alpha)=F'(f_n\upharpoonright\alpha)$.

THEOREM 2. *$\diamondsuit$ for stationary systems implies Φ_P.*

Proof. $\diamondsuit$ for stationary systems states that, whenever $\{S_f:f\in{}^{\omega_1}2\}$ is a stationary system (that is, whenever $\{S_f:f\in{}^{\omega_1}2\}$ is such that each S_f is a stationary set and such that, whenever $\alpha\in\omega_1$, $f\upharpoonright\alpha=g\upharpoonright\alpha$ implies $S_f\cap(\alpha+1)=S_g\cap(\alpha+1)$), $\exists\{f_\alpha:\alpha\in\omega_1\}:\forall f:\omega_1\to 2\,\exists$ stationary set $S\subset S_f:\forall\alpha\in S\, f\upharpoonright\alpha=f_\alpha$. Let $F:\Omega\to 2$. For each $f\in{}^{\omega_1}2$, let $S_f=\{\alpha:F(f\upharpoonright\alpha)\text{ is defined}\}$. F is good implies that $\{S_f:f\in{}^{\omega_1}2\}$ is a stationary system. $\diamondsuit$ for stationary systems implies that $\exists\{f_\alpha:\alpha\in\omega_1\}:\forall f:\omega_1\to 2\,\exists$ stationary set $S:\forall\alpha\in S$ $f\upharpoonright\alpha=f_\alpha$ and $F(f\upharpoonright\alpha)$ is defined. Define $g:\omega_1\to 2$ so that $g(\alpha)=F(f_\alpha)$ whenever $F(f_\alpha)$ is defined. Whenever $f:\omega_1\to 2$, there is a stationary set S such that, for each $\alpha\in S$, $g(\alpha)=F(f\upharpoonright\alpha)$.

THEOREM 3. *Φ_p implies that, in a normal first countable space, discrete families of points of cardinality $\aleph_1$ are separated.*

To prove Theorem 3, we need an equivalent formulation of Φ_p.

https://doi.org/10.4153/CMB-1984-061-3 Published online by Cambridge University Press

LEMMA 3. Φ_p *if and only if* $\forall F: {}^{\omega_1}c \to 2\ \exists g: \omega_1 \to 2: \forall f: \omega_1 \to c\ \exists$ *stationary set* $S: \forall \alpha \in S\ F(f \restriction \alpha) = g(\alpha)$.

Proof. A standard device is to code each function $f: \omega_1 \to 2^\omega$ by a function $f^*: \omega_1 \to 2$ by letting $f(\alpha)(n) = f^*(\alpha \cdot \omega + n)$ for each $\alpha \in \omega_1$. Let $F: {}^{\omega_1}c \to 2$. Let $F^*: {}^{\omega_1}2 \to 2$ be defined so that $F^*(f^* \restriction \alpha) = F(f \restriction \alpha)$ when α is a limit ordinal (F^* is well-defined since, if α is a limit ordinal $f^* \restriction \alpha = g^* \restriction \alpha$ implies $f \restriction \alpha = g \restriction \alpha$ and F^* is good since $\{\alpha : F^*(f^* \restriction \alpha)$ is defined$\} = \{\alpha : F(f \restriction \alpha)$ is defined and α is a limit ordinal$\}$). By Φ_p, there is a $g: \omega_1 \to 2$ such that, for each $f^*: \omega_1 \to 2$, $\exists$ stationary set $S: \forall \alpha \in S\ F^*(f^* \restriction \alpha) = g(\alpha)$. For each $f: \omega_1 \to c$, there is a stationary set S of limit ordinals $: \forall \alpha \in S\ F(f \restriction \alpha) = F^*(f^* \restriction \alpha) = g(\alpha)$.

Proof of Theorem 3. Let X be a first countable normal space (X need only have character C) with a discrete unseparated family D of points of cardinality $\aleph_1$. We use a result of Taylor [6]:

Claim: There is a discrete family of points $\{X_\alpha : \alpha \in \omega_1\}$ such that there does not exist a closed unbounded set C such that $\{X_\alpha : \alpha \in C\}$ is separated.

Proof of Claim. Suppose otherwise. Whenever $\varphi: D \to \omega_1$ is such that each $\varphi^{-1}(\alpha)$ is countable, there is a closed unbounded set C such that $\varphi^{-1}(C)$ is separated (otherwise, enumerating each $\varphi^{-1}(\alpha)$ by $\{d_n^\alpha : n \in \omega\}$, there are closed unbounded sets $\{C_n : n \in \omega\}$ such that each $\{d_n^\alpha : \varphi(\alpha) \in C_n\}$ is separated and a closed unbounded set $C = \cap \{C_n : n \in \omega\}$ such that, applying $\aleph_0$-collectionwise normality, $\varphi^{-1}(C)$ is separated). Define $\varphi_n: D \to \omega_1$ such that each $\varphi_n^{-1}(\alpha)$ is countable and closed unbounded sets C_n by induction on $n \in \omega$: define $\varphi_0: D \to \omega_1$ to be a bijection and when $\varphi_n: D \to \omega_1$ is defined, define C_n to be a closed unbounded set such that $\varphi_n^{-1}(C_n)$ is separated and $\varphi_{n+1}: D \to \omega_1$ by $\varphi_{n+1}(p) = \max\{\alpha \in C_n : \alpha \le \varphi_n(p)\}$. Each $\varphi_n^{-1}(C_n)$ is separated; by $\aleph_0$-collectionwise normality, $\bigcup \{\varphi_n^{-1}(C_n) : n \in \omega\}$ is separated and thus there is a $p \in D$ such that, for each $n \in \omega$, $\varphi_n(p) \notin C_n$. $\{\varphi_n(p) : n \in \omega\}$ is an infinite descending sequence of ordinals and the claim is established.

We use Fleissner's proof in [2]: We shall define a partial function F mapping functions from a countable ordinal into $\omega \times 2$ to 2. For each $\alpha \in \omega_1$, let $\{U_n(\alpha) : n \in \omega\}$ be a neighborhood base for x_α. Whenever $f: \alpha \to (\omega \times 2)$, f assigns a color and a neighborhood to each x_β (If $f(\beta) = \langle n, 1\rangle$, x_β is assigned the ith color and the nth neighborhood) and we can define $V_i(f)$ to be $\bigcup \{U_n(\beta) : f(\beta) = \langle n, i\rangle\}$. For each $f: \alpha \to (\omega \times 2)$, let

$$F(f) = \begin{cases} \text{undefined} & \text{if } x_\alpha \notin \overline{V_0(f) \cup V_1(f)} \\ 1 & \text{if } x_\alpha \in \overline{V_0(f)} \\ 0 & . \ \text{otherwise} \end{cases}$$

We shall obtain a contradiction.

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Suppose F is good. An application of Lemma 3 yields $g:\omega_1\to 2$ such that, for each $f:\omega_1\to(\omega\times 2)$, there is an α (we do not need a stationary set of α) such that $F(f\upharpoonright\alpha)=g(\alpha)$. The normality of X implies that there is $n:\omega_1\to\omega$ such that, whenever $\alpha,\alpha'\in\omega_1$, $g(\alpha)\neq g(\alpha')$ implies $U_{n(\alpha)}(\alpha)\cap U_{n(\alpha')}(\alpha')=\emptyset$. Let $f:\omega_1\to(\omega\times 2)$ be defined by $f(\alpha)=\langle n(\alpha), g(\alpha)\rangle$. Let $\alpha\in\omega_1$ be such that $F(f\upharpoonright\alpha)=g(\alpha)$. $g(\alpha)=1$ implies that $x_\alpha\in\overline{V_0(f)}$, that $U_{n(\alpha)}(\alpha)\cap V_0(f)\neq\emptyset$ and so that, for some $\alpha'\in\omega_1$, $U_{n(\alpha)}(\alpha)\cap U_{n(\alpha')}(\alpha')\neq\emptyset$ while $g(\alpha)=1$ and $g(\alpha')=0$. $g(\alpha)=0$ implies that $x_\alpha\in\overline{V_1(f)}$ and a similar contradiction.

Suppose f is not good. There are functions $f_n:\omega_1\to(\omega\times 2)(n\in\omega)$ and a closed unbounded set C such that $\forall\,\alpha\in C\ \exists\, n\in\omega: F(f_n\upharpoonright\alpha)$ is undefined. $\{x_\alpha:\alpha\in C\}$ is not separated by hypothesis. In a normal space, countable discrete families of closed sets are separated. This implies that there is $A\subset\omega_1$ and $n\in\omega$ such that $\{x_\alpha:\alpha\in A\}$ is not separated and such that, for each $\alpha\in A$, $F(f_n\upharpoonright\alpha)$ is undefined. For each $\alpha\in A$, let $m\in\omega$ be such that $U_m(\alpha)\cap U_n(\beta)=\emptyset$ whenever $\beta\in A$ and $\beta<\alpha$. $\{U_m(\alpha)\cap U_n(\alpha):\alpha\in A\}$ is a separation of $\{x_\alpha:\alpha\in A\}$.

COROLLARY. *$\diamondsuit$ does not imply Φ_p.*

Proof. Shelah [5] has shown that the existence of a normal first countable space with a discrete unseparated family of points of cardinality $\aleph_1$ is consistent with $\diamondsuit$.

The referee has stated the surprising result

THEOREM 4. *If CH holds and $\kappa<\lambda$ are regular uncountable cardinals and κ Cohen reals and λ Cohen subsets of ω_1 are added by product forcing to the universe, then Φ_P holds in the extension.*

COROLLARY. *It is consistent with $\neg$CH that normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

Proof (due in part to Juris Steprāns). Let V be a model of CH. Let $\kappa<\lambda$ be regular uncountable cardinals. Let $P=Fn(\kappa,2,\omega)$ be the partial order which adds κ Cohen reals. Let $Q=Fn(\lambda\times\omega_1,2,\omega_1)$ be the partial order which adds λ Cohen subsets of ω_1. Let $V^{P\times Q}\models$"$F:{}^{\omega_1}2\to 2$ is good". Assume, without loss of generality, that $1\Vdash$"$F:{}^{\omega_1}2\to 2$ is good". By the $\aleph_2$-chain condition, $\aleph_1<\lambda$ and $\kappa<\lambda$ there is $\gamma\in\lambda$ such that $F\in V^{P\times Q\upharpoonright\gamma\times\omega_1}$. Let G be the generic function from $\lambda\times\omega_1$ into 2. Let $g:\omega_1\to 2$ be defined in $V^{P\times Q}$ by $g(\alpha)=G(\gamma,\alpha)$. We must show (1) $V^{P\times Q}\Vdash$"$\forall f:\omega_1\to 2\ \exists$ stationary $S:\forall\,\alpha\in S$ $F(f\upharpoonright\alpha)=g(\alpha)$". We work in $M=V^{Q\upharpoonright(\lambda-\{\gamma\})\times\omega_1}$.

Let $R=Q\upharpoonright\{\gamma\}\times\omega_1$. Note that $F\in M^P$ and $V^{P\times Q}=M^{P\times R}$. If (1) is not true, then there is $p\in P$ and $q\in R$ such that (2) $(p,q)\Vdash$"$f:\omega_1\to 2$ and C is a closed

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unbounded set of ω_1 and $(\forall\, \alpha \in C)\ F(f \restriction \alpha) \neq g(\alpha)$". Without loss of generality, since P has the countable chain condition, $C \in M^R$. Construct a descending continuous sequence $\{q_\alpha : \alpha \in \omega_1\} \subset R$ such that $q_0 = q$;

q_α decides whether $\alpha \in C$; there is $\beta \geq \alpha$ such that $(\phi, q_\alpha) \Vdash \beta \in C$ and $(\forall\, \alpha \in \omega_1)\, \exists$ antichain $A_\alpha \subset P : \forall\, p \in A_\alpha (p, q_\alpha)$ decides $f(\alpha)$. Let $D = \{\alpha \in \omega_1 : (\varnothing, q_\alpha) \Vdash$ "$\alpha \in C$"$\}$. D is a closed unbounded set and $D \in M$. Let $E \subset D$ be a closed unbounded set of limit ordinals such that (3) $\alpha \in E$ and $\beta < \alpha$ implies dom $q_\beta \subset \{\gamma\} \times \alpha$. Let h be a P-name such that

$$1 \Vdash h : \omega_1 \to 2 \quad \text{and} \quad (4)\, (\varnothing, q_\alpha) \Vdash h \restriction \alpha = f \restriction \alpha.$$

$1 \Vdash$ "$\check{E}$ is a closed unbounded set and $F : \Omega \to 2$ is good" and $h : \omega_1 \to 2$ implies that $1 \Vdash$ "$(\exists \alpha \in E)\ F(h \restriction \alpha)$ is defined". Choose $\bar{p} \leq p$ and $\alpha \in E$ and $i \in 2$ such that $(\bar{p}, \varnothing) \Vdash$ "$F(h \restriction \alpha) = i$". This is possible since $h \restriction \alpha \in M^P$, $E \in M$ and $F \in M^P$. By (4), $(\bar{p}, q_\alpha) \Vdash$ "$F(f \restriction \alpha) = i$". Let $\bar{q} = q_\alpha \cup \{\langle \alpha, i \rangle\}$. $\bar{q}$ is defined since α is a limit ordinal, $\{q_\alpha : \alpha \in \omega_1\}$ is continuous and (3) $(\bar{p}, \bar{q}) \Vdash F(f \restriction \alpha) = g(\alpha)$ by the definition of g and since $\bar{q} \Vdash$ "$\bar{q} \subset G \restriction \{\gamma\} \times \omega_1$". $(\bar{p}, \bar{q}) \Vdash$ "$\alpha \in C$" since $\alpha \in D$ and $(\varnothing, q_\alpha) \geq (\bar{p}, \bar{q})$ by the definition of D. This contradicts (2).

Note: In this model, $\kappa > \aleph_1$ implies Ostaszewski's axiom ♧ is false. Moreover, whenever $\aleph_2$-many Cohen reals are added to a model M ($V^{P \times Q}$ may be obtained by adding $\aleph_2$-many Cohen reals to $V^{P \restriction (\kappa - \omega_2) \times Q}$), the principle $\exists \{S_\alpha : \alpha \in \omega_1\} \subset \mathscr{P}(\omega_1) : \forall\, S \subset \omega_1\, \exists \alpha \in \omega_1 : S \supset S_\alpha$ is false. Otherwise, by the countable chain condition, we may assume, without loss of generality, $\{S_\alpha : \alpha \in \omega_1\} \in M$. Letting S be coded by the first $\aleph_1$-many Cohen reals provides a contradiction.

A discussion is facilitated by some definitions.

A *weak* $\diamondsuit$*-sequence* is a sequence $\{S_\alpha : \alpha \in \omega_1\}$ such that each $S_\alpha \subset \mathscr{P}(\alpha)$ and such that, whenever $A \subset \omega_1$, $\{\alpha : A \cap \alpha \in S_\alpha\}$ is stationary.

A sequence $\{A_\alpha : \alpha \in \omega_1\}$ *refines* a sequence $\{B_\alpha : \alpha \in \omega_1\}$ iff $A_\alpha \subset B_\alpha$ $(\alpha \in \omega_1)$.

A weak $\diamondsuit$-sequence $\{S_\alpha : \alpha \in \omega_1\}$ is *wide* iff whenever $\{A_n : n \in \omega\}$ are subsets of ω_1, $\{\alpha \in \omega_1 : n \in \omega$ implies $A_n \cap \alpha \in S_\alpha\}$ is stationary.

Mathias [3] has formulated $\diamondsuit$ for stationary systems as: each weak $\diamondsuit$-sequence can be refined by a $\diamondsuit$-sequence. Mathias showed that, under $\diamondsuit^*$, each weak $\diamondsuit$-sequence can be refined by a weak $\diamondsuit$-sequence of countable sets.

Shelah [4] has shown that it is consistent with $\diamondsuit^+$ that there is a weak $\diamondsuit$-sequence of sets of size 2 which cannot be refined by a $\diamondsuit$-sequence. We have shown that, under $\diamondsuit^*$, each wide weak $\diamondsuit$-sequence can be refined by a wide weak $\diamondsuit$-sequence of countable sets. The difference is that, under ZFC, any wide weak $\diamondsuit$-sequence of countable sets can be refined by a $\diamondsuit$-sequence. Let Φ'_P be formulated by applying Φ_P to partial colorings f which are not necessarily good but are such that, whenever $f : \omega_1 \to 2$, $\{\alpha : F(f \restriction \alpha)$ is not defined$\}$ does not contain a closed unbounded set. Φ'_P is not used in this paper, despite its comparative simplicity, because it is not implied by $\diamondsuit^*$ (or even $\diamondsuit^+$). This is true because any weak $\diamondsuit$-sequence $\{\{S_\alpha^0, S_\alpha^1\}: \alpha \in \omega_1\}$ codes a partial coloring F defined, whenever A is a subset of ω_1 (letting χ_A be the characteristic function of A), by $F(\chi_A \upharpoonright \alpha) = i$ iff $A \cap \alpha = S_\alpha^i$ $(i \in \alpha)$ and because $g: \omega_1 \to 2$ as in Φ_P' provides the $\diamondsuit$-sequence refinement $\{S_\alpha^{g(\alpha)}: \alpha \in \omega_1\}$.

We are grateful to the referee for the preceding discussion.

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York University
Downsview, Ontario.
M3J1P3 Canada