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FREE CENTRE-BY-NILPOTENT-BY-ABELIAN LIE RINGS OF RANK 2

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Abstract

We study the free Lie ring of rank 2 in the variety of all centre-by-nilpotent-by-abelian Lie rings of derived length 3. This is the quotient $L/([\gamma_c(L'), L] + L''')$ with $c \geqslant 2$ where L is the free Lie ring of rank 2, $\gamma_c(L')$ is the cth term of the lower central series of the derived ideal L' of L, and L''' is the third term of the derived series of L. We show that the quotient $\gamma_c(L') + L'''/[\gamma_c(L'), L] + L'''$ is a direct sum of a free abelian group and a torsion group of exponent c. We exhibit an explicit generating set for the torsion subgroup.

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1. Introduction

In this paper we study free centre-by-nilpotent-by-abelian Lie rings of derived length 3. These are the quotients

$$G = L/([\gamma_c(L'), L] + L''') \quad (c \geqslant 2) \tag{1.1}$$

where L is an (absolutely) free Lie ring, $\gamma_c(L')$ is the cth term of the lower central series of the derived ideal L' of L, and L''' is the third term of the derived series of L. In view of the short exact sequence

$$0 \to \gamma_c(G') \to G \to G/\gamma_c(G') \to 0,$$

this is a free central extension of $G/\gamma_c(G') = L/(\gamma_c(L') + L''')$, the free nilpotent-(of class $c - 1$)-by-abelian Lie ring of derived length 3. The additive structure of the latter is well understood. Its underlying abelian group is a free $\mathbb{Z}$-module. Things are different for the central term $\gamma_c(G')$. That ideal contains elements of finite

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order. This has been known for a long time if $c = 2$ and $c = 3$. If $c = 2$, (1.1) turns into $G = L/[L'', L]$, the free centre-by-metabelian Lie ring. Its second-derived ideal $G'' = \gamma_2(G')$ contains an elementary abelian 2-subgroup. This was proved in [7] where a basis for this 2-subgroup was exhibited. However, as pointed out by Zerck [13], some of the details in [7] needed correction, and this was eventually carried out in [9] and [6]. In fact, in the case of rank 2, which turned out to be considerably easier than the case of higher ranks, a similar result on torsion in the context of the lower central quotients of the free centre-by-metabelian group was proved much earlier by Ridley [12]. If $c = 3$, (1.1) turns into $L/[\gamma_3(L'), L]$, the free centre-by-(nilpotent of class 2)-by-abelian Lie ring. Drensky [2] proved that its central ideal $\gamma_3(L')/[\gamma_3(L'), L]$ contains elements of order 3 if the rank of L is at least 7. More generally, he showed that for any prime p the free Lie ring $L/[\gamma_p(L'), L]$ contains nontrivial multilinear elements of degree $2p + 1$ which have order p. These investigations of torsion in relatively free Lie rings were motivated and inspired by parallel results on relatively free groups. In particular, in 1973 Gupta [3] discovered elements of order 2 in the free centre-by-metabelian groups $F/[F'', F]$ (here F denotes a free group of rank at least 4). This was the starting point for further work on torsion in free central extensions of the form $F/[\gamma_c(F'), F]$ (see [5] for the latest results, a brief survey and further references), and also free central extensions of the form $F/[\gamma_c(F'), F]F'''$, the exact counterpart of (1.1) in group theory (see [4, 10, 11]).

In the present paper we focus on the relatively free Lie ring (1.1), but we restrict ourselves to the case of rank 2. In this case we extend the results on (1.1) for $c = 2$ to arbitrary values of c. We show that, for arbitrary $c \geqslant 2$, the central ideal $\gamma_c(G')$ is a direct sum of a free abelian group and a torsion group of exponent c. Our main result (Theorem 6.1) refers to the torsion subgroup. We exhibit an explicit independent generating set, that is, a generating set such that the torsion subgroup is the direct sum of the cyclic subgroups generated by the members of this set. If c is a prime, this generating set takes a particularly simple form (see Corollary 6.2), and for $c = 2$ we recover the known results about torsion elements in free centre-by-metabelian Lie rings (Corollary 6.3). Our approach to the problem is based on an isomorphism identifying $\gamma_c(G')$ with the tensor product $M^c(L'/L'') \otimes_U \mathbb{Z}$, where $M^c(L'/L'')$ is the cth homogeneous component of the free metabelian Lie ring on L'/L'', regarded as a module for the universal envelope U of L/L', and $\mathbb{Z}$ is a trivial U-module. We use homological techniques to get the torsion subgroup of this tensor product. A crucial advantage in restricting ourselves to rank 2 is that the quotient L'/L'' is a free cyclic U-module in this case. We hope, however, that our results will be a first step towards understanding the additive structure of (1.1) for arbitrary ranks.

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2. Reduction to metabelian Lie powers

In this section L denotes a free Lie ring of finite rank. By the Shirshov–Witt theorem, the quotient L'/L''' is the free metabelian Lie ring on L'/L'': $L'/L''' =$

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$M(L'/L'')$. This is a graded Lie ring and its degree-c homogeneous component $M^c(L'/L'')$ is isomorphic to the lower central quotient

$$\gamma_c(L'/L''')/\gamma_{c+1}(L'/L''') = (\gamma_c(L') + L''')/(\gamma_{c+1}(L') + L''').$$

The adjoint representation induces on these lower central quotients the structure of an L/L'-module, and hence a module for its universal envelope $U = U(L/L')$. The latter is a polynomial ring. In fact, if X is a free generating set for L, and hence a $\mathbb{Z}$-basis for L/L', then U may be identified with the polynomial ring on X: $U = \mathbb{Z}[X]$. Thus we have a U-module isomorphism

$$(\gamma_c(L') + L''')/(\gamma_{c+1}(L') + L''') \cong M^c(L'/L''). \tag{2.1}$$

In view of the canonical isomorphism

$$(\gamma_c(L') + L''')/(\gamma_{c+1}(L') + L''') \otimes_U \mathbb{Z} \cong (\gamma_c(L') + L''')/([\gamma_c(L'), L] + L'''),$$

trivializing the U-action on both sides of (2.1) gives an isomorphism

$$(\gamma_c(L') + L''')/([\gamma_c(L'), L] + L''') \cong M^c(L'/L'') \otimes_U \mathbb{Z}. \tag{2.2}$$

We will exploit this isomorphism to investigate the additive structure of the quotient on the left-hand side by examining the tensor product on the right-hand side.

3. Metabelian Lie powers

We use the left normed convention for Lie products, that is, $[a_1, a_2, \ldots, a_c]$ stands for $[[a_1, a_2, \ldots, a_{c-1}], a_c]$, and $U = \mathbb{Z}[X]$ denotes the polynomial ring on a finite set X. Let A be an arbitrary U-module whose underlying abelian group is a free $\mathbb{Z}$-module. We denote by A^k the kth symmetric power of A. Symmetric powers and tensor powers of U-modules will be regarded as U-modules under the derivation action. For example, for $x \in X$, $a_i \in A$,

$$(a_1 \circ a_2 \circ \cdots \circ a_c)x = \sum_{i=1}^{c} a_1 \circ \cdots \circ a_i x \circ \cdots \circ a_c, \quad (a_1 \otimes a_2)x = a_1 x \otimes a_2 + a_1 \otimes a_2 x.$$

The metabelian Lie power $M^c(A)$, that is, the degree-c homogeneous component of the free metabelian Lie algebra $M(A)$ on A, is generated by the simple Lie products $[a_1, a_2, \ldots, a_c]$ with $a_i \in A$. The U-action on A extends to the whole of $M(A)$ via

$$[a_1, a_2, \ldots, a_c]x = \sum_{i=1}^{c} [a_1, \ldots, a_i x, \ldots, a_c].$$

For $c \geqslant 2$ the metabelian Lie power $M^c(A)$ fits into a short exact sequence

$$0 \to M^c(A) \xrightarrow{\nu_A} A \otimes A^{c-1} \xrightarrow{\pi_A} A^c \to 0 \tag{3.1}$$

where the maps ν_A and π_A are given by

$$[a_1, a_2, \ldots, a_c] \mapsto a_1 \otimes (a_2 \circ \cdots \circ a_c) - a_2 \otimes (a_1 \circ \cdots \circ a_c)$$

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and $a_1 \otimes (a_2 \circ \cdots \circ a_c) \mapsto a_1 \circ a_2 \circ \cdots \circ a_c$, respectively (see [4, Corollary 3.2]). The maps are actually compatible with the U-action on the three terms in (3.1), so this is a short exact sequence of U-modules.

There is also a U-module homomorphism $\rho_A : A \otimes A^{c-1} \to M^c(A)$ given by

$$a_1 \otimes (a_2 \circ \cdots \circ a_c) \mapsto [a_1, a_2, a_3, \ldots, a_c] + [a_1, a_3, a_2, \ldots, a_c] \\ + \cdots + [a_1, a_c, a_2, \ldots, a_{c-1}].$$

Indeed, this is just the composite

$$A \otimes A^{c-1} \to A \otimes A \otimes A^{c-2} \to M^c(A)$$

where the two maps are given by

$$a_1 \otimes (a_2 \circ \cdots \circ a_c) \mapsto a_1 \otimes a_2 \otimes (a_3 \circ \cdots \circ a_c) + \cdots + a_1 \otimes a_c \otimes (a_2 \circ \cdots \circ a_{c-1})$$

and

$$a_1 \otimes a_2 \otimes (a_3 \circ \cdots \circ a_c) \mapsto [a_1, a_2, a_3, \ldots, a_c],$$

respectively. The second map is correctly defined because a simple Lie product $[a_1, a_2, a_3, \ldots, a_c]$ in a metabelian Lie algebra is symmetric with respect to the entries $a_3, \ldots, a_c$. Using this, antisymmetry in the first two entries and the Jacobi identity, one easily calculates that the composite

$$M^c(A) \xrightarrow{\nu_A} A \otimes A^{c-1} \xrightarrow{\rho_A} M^c(A) \tag{3.2}$$

amounts to multiplication by the integer c on $M^c(A)$:

$$\nu_A \rho_A = c|_{M^c(A)}. \tag{3.3}$$

By applying the homology functor to (3.2) we find, in view of (3.3), that the kernel of the induced homomorphism $H_k(M^c(A)) \to H_k(A \otimes A^{c-1})$ is annihilated by c for all $k \geqslant 0$. Indeed, if $w \in \ker H_k(\nu_A)$, then

$$0 = wH_k(\nu_A)H_k(\rho_A) = wH_k(\nu_A\rho_A) = cw.$$

So we have the following result.

Lemma 3.1. *Let A be a $\mathbb{Z}$-free U-module, $c \geqslant 2$. Then, for all $k \geqslant 0$, the kernel of the homomorphism $H_k(M^c(A)) \to H_k(A \otimes A^{c-1})$ is an abelian group of exponent dividing c.* □

Now suppose that A is *free* as a U-module. Then the tensor product $A \otimes A^{c-1}$ is also a free U-module under derivation action (see, for example, [9, Lemma 5.2]). Hence $H_0(A \otimes A^{c-1}) = (A \otimes A^{c-1}) \otimes_U \mathbb{Z}$ is a free abelian group, and $H_k(A \otimes A^{c-1}) = 0$ for $k \geqslant 1$. The initial part of the long exact homology sequence stemming from the short exact sequence (3.1) is given by

$$0 \to H_1(A^c) \to M^c(A) \otimes_U \mathbb{Z} \to (A \otimes A^{c-1}) \otimes_U \mathbb{Z} \to A^c \otimes_U \mathbb{Z} \to 0. \tag{3.4}$$

From this we can read off the following result.

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LEMMA 3.2. *Let A be a free U-module, $c \geqslant 2$. Then the tensor product $M^c(A) \otimes_U \mathbb{Z}$ is a direct sum of a free abelian group and a torsion group of exponent dividing c. The latter is isomorphic to the homology group $H_1(A^c)$.*

PROOF. The exactness of (3.4) gives that the image of $M^c(A) \otimes_U \mathbb{Z}$ in $(A \otimes A^{c-1}) \otimes_U \mathbb{Z}$ is free abelian, and the kernel is isomorphic to $H_1(A^c)$. By Lemma 3.1, this kernel is of exponent dividing c. □

4. Homology of symmetric powers

In this section we calculate the homology group $H_1(A^c)$ in the case where U is the polynomial ring in two indeterminates x and y, $U = \mathbb{Z}[x, y]$, and A is a free cyclic U-module of rank 1, so $A = U$. In this case the Koszul complex, a free resolution of the trivial U-module $\mathbb{Z}$ of length 2, is

$$\mathcal{P}:\ 0 \to U(e_1 \wedge e_2) \xrightarrow{\partial_2} Ue_1 \oplus Ue_2 \xrightarrow{\partial_1} U \to \mathbb{Z} \to 0,$$

where the differentials are given by $e_1 \wedge e_2 \mapsto ye_1 - xe_2$ and $e_1 \mapsto x$, $e_2 \mapsto y$, respectively (see, for example, [8, Ch. VII]). For an arbitrary U-module B, the homology groups $H_k(B)$ are the homology groups of the complex $B \otimes_U \mathcal{P}$. We first focus on $k = 2$. The terms of our complex in degrees 2 and 1 can be identified with B and $B \oplus B$, respectively, and then the differential in degree 2 is the map

$$\delta : B \to B \oplus B, \quad a \mapsto (ay, -ax) \quad (a \in B). \tag{4.1}$$

The homology group $H_2(B)$ can be identified with the kernel of the homomorphism (4.1). We consider the case where $B = U^c$, the cth symmetric power of U. The polynomial ring $U = \mathbb{Z}[x, y]$ has a $\mathbb{Z}$-basis consisting of all monomials $x^p y^q$ where $p, q \geqslant 0$. We denote this basis by $\mathcal{U}$, and we order this set by setting $x^{p_1} y^{q_1} > x^{p_2} y^{q_2}$ if and only if $p_1 > p_2$ or $p_1 = p_2$ and $q_1 > q_2$. For $u \in U$ and a positive integer k it will be convenient to set $u^{\circ k} = \underbrace{u \circ u \circ \cdots \circ u}_{k}$. The symmetric power U^c has a $\mathbb{Z}$-basis $\mathcal{B}$ consisting of all symmetric tensors

$$b = u_1^{\circ r_1} \circ u_2^{\circ r_2} \circ \cdots \circ u_s^{\circ r_s} \tag{4.2}$$

where $u_1 > \cdots > u_s$ and $r_1, \ldots, r_s$ are positive integers such that $r_1 + \cdots + r_s = c$. We shall use the lexicographic order on $\mathcal{B}$ induced by the ordering on $\mathcal{U}$, that is, for $u_1 \geqslant \cdots \geqslant u_c$ and $u'_1 \geqslant \cdots \geqslant u'_c$, $u_1 \circ u_2 \circ \cdots \circ u_c > u'_1 \circ \cdots \circ u'_c$ if and only if there is a k such that $u_k > u'_k$ and $u_i = u'_i$ for $i < k$. Any element in $w \in U^c$ can be written as a unique linear combination $w = \sum_{b \in \mathcal{B}} \alpha_b b$ where $\alpha_b \in \mathbb{Z}$ and only finitely many of the α_b are not zero. The leading term of w is the $\alpha_c c$ where $c \in \mathcal{B}$ is the largest basis element occurring with nonzero coefficient in the decomposition of w. Now the action of $x \in U$ on a basis element (4.2) is given by

$$(u_1^{\circ r_1} \circ u_2^{\circ r_2} \circ \cdots \circ u_s^{\circ r_s})x = \sum_{i=1}^{s} r_i (u_1^{\circ r_1} \circ \cdots \circ u_i x \circ u_i^{\circ (r_i - 1)} \circ \cdots \circ u_s^{\circ r_s}). \tag{4.3}$$

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It is easily seen from (4.3) that if $w \in U^c$ with leading term $\alpha_b b$ where $b \in \mathcal{B}$ is as in (4.2), then the leading term of wx is $\alpha_b r_1(u_1 x \circ u_1^{\circ(r_1-1)} \circ u_2^{\circ r_2} \circ \cdots \circ u_s^{\circ r_s})$. This implies that the map $w \mapsto wx$ with $w \in U^c$ is injective on U^c. Hence the kernel of the map (4.1) is zero. But this kernel is isomorphic to $H_2(U^c)$. So we have the following result.

Lemma 4.1. *If $U = \mathbb{Z}[x, y]$ is the integer polynomial ring in two indeterminates, then, for any $c \geqslant 1$, $H_2(U^c) = 0$.* □

Things are different if we reduce modulo c. For a basis element $b \in \mathcal{B}$ as in (4.2) we let $\omega(b) = (r_1, \ldots, r_s)$ denote the greatest common divisor of the exponents $r_1, \ldots, r_s$.

Lemma 4.2. *If $U = \mathbb{Z}[x, y]$ and $c \geqslant 2$, then $H_2(U^c \otimes \mathbb{Z}_c)$ is a direct sum of the cyclic subgroups generated by the cycles*

$$\frac{c}{\omega(b)}\, b \otimes (e_1 \wedge e_2) \tag{4.4}$$

where $b \in \mathcal{B}$ with $\omega(b) > 1$. The order of a cycle (4.4) is $\omega(b)$.

Proof. We need to determine the kernel of the map (4.1) with $B = U^c$. Evidently, this is equal to the kernel of the map $a \mapsto ax$ where $a \in U^c$. We use the basis $\mathcal{B}$, but now we are working modulo c. We claim that for $a = \sum_{b \in \mathcal{B}} \alpha_b b$ (with only finitely many α_b nonzero modulo c), $ax = 0$ if and only if $c | \alpha_b \omega(b)$ for all $b \in \mathcal{B}$. This implies that the kernel is generated by the elements (4.4) as required. In order to verify the claim we introduce some new orderings on the set $\mathcal{B}$ as follows. Let d be a positive divisor of c. For $b \in \mathcal{B}$ as in (4.2), we define $\kappa_d(b)$ as the smallest subscript i such that d does not divide r_i, and we set $\kappa_d(b) = \infty$ if $d | r_i$ for all $i = 1, 2, \ldots, s$. We set $b_1 >_d b_2$ if and only if $\kappa_d(b_1) < \kappa_d(b_2)$ or $\kappa_d(b_1) = \kappa_d(b_2)$ and $b_1 > b_2$. Now assume that our claim is not true. In other words, $ax = 0$ for some $a = \sum_{b \in \mathcal{B}} \alpha_b b$ such that $c \nmid \alpha_b \omega(b)$ for some $b \in \mathcal{B}$. Suppose that b is as in (4.2). Then there exists a subscript k such that $c \nmid \alpha_b r_k$, but $c | \alpha_b r_i$ for $i < k$. Observe that $k > 1$, since the arguments used in the proof of Lemma 4.1 give an immediate contradiction if $k = 1$. Our assumption implies that there exists a positive divisor d of c such that $d \nmid r_k$, but $d | r_i$ for $i < k$. Take the largest such b (with respect to the ordering $>_d$) that occurs with nonzero coefficient in the decomposition of a, say

$$b = u_1^{\circ r_1} \circ u_2^{\circ r_2} \circ \cdots \circ u_k^{\circ r_k} \circ \cdots \circ u_s^{\circ r_s} \tag{4.5}$$

with $\kappa_d(b) = k$. Then

$$\alpha_b bx = \alpha_b r_k (u_1^{\circ r_1} \circ u_2^{\circ r_2} \circ \cdots \circ u_k x \circ u_k^{\circ r_k - 1} \circ \cdots \circ u_s^{\circ r_s}) + \cdots . \tag{4.6}$$

Since $ax = 0$, the first summand on the right-hand side of (4.6) must cancel with terms in ax stemming from other $\alpha_b b$ in the decomposition of a. The basis elements b such that bx involves the first basis element on the right-hand side of (4.6) are of the form

$$u_1^{\circ r_1} \circ \cdots \circ u_i^{\circ r_i - 1} \circ \frac{u_i}{x} \circ \cdots \circ u_k x \circ u_k^{\circ r_k - 1} \circ \cdots \circ u_s^{\circ r_s} \tag{4.7}$$

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with $2 \leqslant k \leqslant s$, or

$$u_1^{\circ r_1} \circ \cdots \circ u_k x \circ u_k^{\circ r_k - 1} \circ \cdots \circ u_i^{\circ r_i - 1} \circ \frac{u_i}{x} \circ \cdots u_s^{\circ r_s} \tag{4.8}$$

with $1 \leqslant k \leqslant s-1$. However, the basis elements (4.8) cannot occur in the basis decomposition of a as they are larger (with respect to $>_d$) than the element (4.5). Now consider the basis elements (4.7). If $u_k x = u_i$, then (4.7) is equal to (4.5), and hence we may disregard this case. Finally, if $u_k x \neq u_i$, then (4.7) is larger than (4.5) (with respect to $>_d$) since $d \nmid r_i - 1$. Hence these basis elements cannot occur in the decomposition of a. This completes the proof of the lemma. □

Now consider the short exact sequence

$$0 \to U^c \xrightarrow{c} U^c \to (U^c \otimes \mathbb{Z}_c) \to 0.$$

Part of the associated long exact homology sequence is

$$\cdots \to H_2(U^c) \to H_2(U^c \otimes \mathbb{Z}_c) \to H_1(U^c) \xrightarrow{c} H_1(U^c) \to \cdots.$$

Since $H_2(U^c) = 0$ by Lemma 4.1, and multiplication by c annihilates $H_1(U^c)$ by Lemma 3.2, the exactness of this sequence implies that the connecting homomorphism $H_2(U^c \otimes \mathbb{Z}_c) \to H_1(U^c)$ is an isomorphism. The image of a generator (4.4) under this connecting homomorphism is easily calculated. We record the result as follows.

Lemma 4.3. *For $U = \mathbb{Z}[x, y]$ and $c \geqslant 2$, there is an isomorphism*

$$H_2(U^c \otimes \mathbb{Z}_c) \cong H_1(U^c).$$

The image in $H_1(U^c)$ of a generator (4.4) with b as in (4.2) and $r_i = \omega(b) r_i'$ $(1 \leqslant i \leqslant s)$ is the homology class of the cycle

$$\sum_{i=1}^{s} r_i' (u_1^{r_1} \circ \cdots \circ u_i y \circ u_i^{r_i - 1} \circ \cdots \circ u_s^{r_s} \otimes e_1$$
$$- u_1^{r_1} \circ \cdots \circ u_i x \circ u_i^{r_i - 1} \circ \cdots \circ u_s^{r_s} \otimes e_2) \tag{4.9}$$

in $U^c \otimes_U \mathcal{P}$. □

Of course, by Lemma 3.2, the torsion subgroup of $M^c(U) \otimes \mathbb{Z}$ is isomorphic to $H_1(U^c)$. In the next section we calculate the images of the cycles in Lemma 4.3 under the connecting homomorphism in (3.4).

5. The connecting homomorphism $H_1(U^c) \to M^c(U) \otimes_U \mathbb{Z}$

In order to carry out the computation of this connecting homomorphism, we tensor the short exact sequence (3.1) with the free resolution $\mathcal{P}$. Here is the relevant part of

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the resulting commutative diagram.

$$\begin{tikzcd}
U^c \otimes_U (Ue_1 \oplus Ue_2) & & \\
(U \otimes U^{c-1}) \otimes_U (Ue_1 \oplus Ue_2) \arrow[u] \arrow[r] & (U \otimes U^{c-1}) \otimes_U U & \\
& M^c(U) \otimes_U U \arrow[u] \arrow[r] & M^c(U) \otimes_U \mathbb{Z}
\end{tikzcd}$$

We start with a cycle (4.9) and follow the standard procedure for computing the connecting homomorphism (see, for example, [8, Ch. II.4]). For a cycle (4.9), an inverse image in $(U \otimes U^{c-1}) \otimes_U (Ue_1 \oplus Ue_2)$ is

$$\begin{aligned} &+ \sum_{i=1}^{s} r_i' u_i y \otimes u_1^{r_1} \circ u_2^{r_2} \circ \cdots \circ u_i^{r_i-1} \circ \cdots \circ u_s^{r_s} \otimes e_1 \\ &- \sum_{i=1}^{s} r_i' u_i x \otimes u_1^{r_1} \circ u_2^{r_2} \circ \cdots \circ u_i^{r_i-1} \circ \cdots \circ u_s^{r_s} \otimes e_2. \end{aligned} \tag{5.1}$$

The image of (5.1) in $U \otimes U^{c-1} \otimes_U U$ is

$$\begin{aligned} &\sum_{i=1}^{s} r_i'(r_i - 1) u_i y \otimes u_i x \circ u_1^{r_1} \circ u_2^{r_2} \circ \cdots \circ u_i^{r_i-2} \circ \cdots \circ u_s^{r_s} \otimes 1 \\ &\quad + \sum_{1 \leqslant i, j \leqslant s, i \neq j} r_i' r_j u_i y \otimes u_j x \circ u_1^{r_1} \circ \cdots \circ u_i^{r_i-1} \circ \cdots \circ u_j^{r_j-1} \circ \cdots \circ u_s^{r_s} \otimes 1 \\ &\quad - \sum_{i=1}^{s} r_i'(r_i - 1) u_i x \otimes u_i y \circ u_1^{r_1} \circ u_2^{r_2} \circ \cdots \circ u_i^{r_i-2} \circ \cdots \circ u_s^{r_s} \otimes 1 \\ &\quad - \sum_{1 \leqslant i, j \leqslant s, i \neq j} r_i' r_j u_i x \otimes u_j y \circ u_1^{r_1} \circ \cdots \circ u_i^{r_i-1} \circ \cdots \circ u_j^{r_j-1} \circ \cdots \circ u_s^{r_s} \otimes 1. \end{aligned}$$

An inverse image of this element in $M^c(U) \otimes_U U$ is

$$\begin{aligned} &\sum_{i=1}^{s} r_i'(r_i - 1)[u_i y, u_i x, u_1^{r_1}, u_2^{r_2}, \ldots, u_i^{r_i-2}, \ldots, u_s^{r_s}] \otimes 1 \\ &\quad + \sum_{1 \leqslant i, j \leqslant s, i \neq j} r_i' r_j' \omega(b)[u_i y, u_j x, u_1^{r_1}, \ldots, u_i^{r_i-1}, \ldots, u_j^{r_j-1}, \ldots, u_s^{r_s}] \otimes 1 \end{aligned}$$

and the canonical image of this in $M^c(U) \otimes_U \mathbb{Z}$ is the image of (4.9) under the connecting homomorphism.

6. The main result

It remains to use the isomorphism (2.2) to translate the result from the previous section into the setting of free Lie rings. For brevity, we introduce the following

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notation. For the basis monomials $u_i = x^{p_i}y^{q_i} \in \mathcal{U}$, $i = 1, \ldots, s$, that occur in a given $b \in \mathcal{B}$ as in (4.2), we set

$$U_i = [y, x, \underbrace{x, \ldots, x}_{p_i}, \underbrace{y, \ldots, y}_{q_i}],$$

and for $w, v \in G'$ and a nonnegative integer k we write

$$[w, v^k] = [w, \underbrace{v, \ldots, v}_{k}].$$

Moreover, to each basis element $b \in \mathcal{B}$ as in (4.2) we assign the element

$$\begin{aligned} t_b = &\sum_{i=1}^{s} r_i'(r_i - 1)[[U_i, y], [U_i, x], U_1^{r_1}, U_2^{r_2}, \ldots, U_i^{r_i-2}, \ldots, U_s^{r_s}] \\ &+ \sum_{1\leqslant i,j\leqslant s, i\neq j} r_i' r_j' \omega(b)[[U_i, y], [U_j, x], U_1^{r_1}, \ldots, U_i^{r_i-1}, \ldots, U_j^{r_j-1}, \ldots, U_s^{r_s}] \end{aligned} \tag{6.1}$$

in $\gamma_c(G')$. Our main result is as follows.

Theorem 6.1. *Let G be the rank-2 free centre-by-nilpotent-by-abelian Lie ring (1.1) of derived length 3 with $c \geqslant 2$ on free generators x and y. Then the central ideal $\gamma_c(G')$ is a direct sum of a free abelian group and an abelian group of exponent c. The latter is the direct sum of the cyclic subgroups generated by the elements (6.1) for all $b \in \mathcal{B}$ with $\omega(b) = (r_1, \ldots, r_s) > 1$ and $r_i = \omega(b)r_i'$. The order of (6.1) is $\omega(b)$.*

Proof. By (2.2), the central ideal $\gamma_c(G')$ is isomorphic to $M^c(L'/L'') \otimes_U \mathbb{Z}$. Moreover, if L has rank 2 with free generators x and y, then $U = \mathbb{Z}[x, y]$ and L'/L'' is a free U-module of rank 1 with free generator $[y, x]$. This is because the elements $[y, x]x^p y^q = [y, x, x, \ldots, x, y, \ldots, y]$ with $p, q \geqslant 0$ form a basis of L'/L'' as a free $\mathbb{Z}$-module (see [1, Ch. 4.7]). Then Lemma 3.2 gives that $\gamma_c(G')$ is a direct sum of a free abelian group and a torsion subgroup of exponent dividing c that is isomorphic to $H_1((L'/L'')^c)$. This homology group has been determined in Lemma 4.3, and the images of its generators in $M^c(L'/L'') \otimes \mathbb{Z}$ have been calculated in Section 5. Their images in $\gamma_c(G')$ are the elements (6.1). Among them are always elements of order precisely c, namely, the ones with $s = 1$ and $r_1 = c$, and hence the exponent of the torsion subgroup is c. □

In the case where c is a prime, the generating set for the torsion subgroup takes a particularly simple form.

Corollary 6.2. *If c is a prime, then the torsion subgroup of $\gamma_c(G')$ is an elementary abelian c-group, and the elements*

$$[[u, y], [u, x], \underbrace{u, \ldots, u}_{c-2}]$$

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where $u = [y, x, \underbrace{x, \ldots, x}_{p}, \underbrace{y, \ldots, y}_{q}]$ *with* $p, q \geqslant 0$ *form a basis of the torsion subgroup as a* $\mathbb{Z}_c$*-module.* □

Proof. If c is a prime, the only basis elements (4.2) with $\omega(b) > 1$ are of the form $u^{\circ c}$ with $u \in \mathcal{U}$. Therefore, in (6.1) we only get terms with $s = 1$ and $r_1 = c$. □

For $c = 2$ we recover the description of the torsion subgroup of the free centre-by-metabelian Lie ring of rank 2.

Corollary 6.3 [6, 7]. *Let G be the free centre-by-metabelian Lie ring of rank* 2 *with free generators x and y. Then the torsion subgroup of G'' is an elementary abelian* 2*-group, and the elements*

$$[[y, x], [y, x, x, \ldots, x, y, y, \ldots, y]]$$

such that the partial degrees of both x and y are odd and each of them is at least 3*, form a basis of it as a* $\mathbb{Z}_2$*-module.*

Proof. By using the identity $[[v, z], w] = -[v, [w, z]]$ which holds for all $v, w \in G'$ and $z \in G$, the generators for the torsion subgroup in Corollary 6.2 with $c = 2$ can be rewritten in the required form. □

Remark. If $c = 3$, then $L''' \subset [\gamma_3(L'), L]$, and hence the free centre-by-(nilpotent of class 2)-by-abelian Lie ring has derived length 3. Thus $G = L/[\gamma_3(L'), L]$ in this case. As mentioned in the Introduction, it was proved by Drensky [2] that this relatively free Lie ring contains elements of order 3. His elements are multilinear and have degree 7, so they occur if the rank of G is at least 7. Our result shows that 3-torsion actually occurs already in rank 2.

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