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QUANTIZATION AND GROUP REPRESENTATIONS

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1. Introduction. A quantization of a fixed classical mechanical system is firstly an association between quantum mechanical observables (preferably self-adjoint operators on Hilbert space) and classical mechanical observables (i.e. real-valued functions on phase space). Secondly, a quantization should permit an interpretation of the correspondence principle that 'classical mechanics is the limit of quantum mechanics as Planck's constant approaches zero'. With these two underlying precepts, Section 2 states the four basic requirements, I to IV, of a quantization along with an additional requirement V that characterizes the subclass of special quantizations. These requirements are then illustrated by the Weyl correspondence that gives a 1-1 association between functions on phase space and operators in the case of a single particle with one degree of freedom. This example has a beautiful interpretation, outlined in Section 3, in terms of Kirillov theory [**9**; **12**; **13**] of representations of nilpotent Lie groups—specifically of the Heisenberg group.

With the Weyl correspondence as a guide, this paper develops a theory of quantization by means of representation theory of Lie groups. For the classical mechanical systems considered, phase space is an orbit in the coadjoint representation of a real Lie group $\mathscr{G}$. The quantization is not unique, but depends on the representation used.

If $\mathscr{G}$ is a connected, simply connected nilpotent Lie group, the Kirillov map between orbits and irreducible representations of $\mathscr{G}$ produces a special quantization given in Section 3. Although the association is in general not 1-1 as it is for the Heisenberg group, the family of operators needed for the correspondence principle is sufficient to determine the function on the orbit uniquely (Theorem 4.3). The method of Section 3 can be generalized to other Lie groups. Section 5 and 6 examine the quantizations obtained in this manner of the sphere and the upper-half plane respectively.

Both Kostant [**10**] and Berezin [**4**] propose a general theory of quantization of orbits. In each case a 1-1 association is first constructed and then a representation of $\mathscr{G}$ is extracted. In this sense, their viewpoint is the opposite to the one adopted here. However, the spirit of this paper is very close to [**4**] and, indeed, the two definitions of a quantization are essentially the same. In this regard, the valuable criticisms and suggestions of Dr. W. Rossmann must also be acknowledged for their contribution to the final version of this paper.

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2. Definition of quantization. A classical mechanical system is a symplectic manifold $(\mathcal{M}, \omega)$ consisting of a real differentiable manifold $\mathcal{M}$ together with a closed nondegenerate two form ω on $\mathcal{M}$. Let $C^\infty(\mathcal{M})$ denote the set of C^∞ complex-valued functions on $\mathcal{M}$. $C^\infty(\mathcal{M})$ is a commutative algebra under pointwise addition and multiplication of functions and a Lie algebra under the Poisson bracket $\{\ ,\ \}$ defined by ω[**1**].

With reference [**4**] in mind, we define a quantization of $(\mathcal{M}, \omega)$.

Definition 2.1. Let A_λ be a family of algebras with involution over the complex numbers indexed by a set E of positive real numbers that has 0 as a limit point. Let $\mathscr{A}$ be an involutive subalgebra of the direct product of the A_λ's. An element of $\mathscr{A}$ is then a map F defined on E such that $F(\lambda)$ is in A_λ. Algebra operations are defined componentwise and the same symbols are used in $\mathscr{A}$ as in A_λ. Specifically, if F and G are in $\mathscr{A}$ and α is a complex number, then $(\alpha F + G)(\lambda) \equiv \alpha F(\lambda) + G(\lambda)$, $(F * G)(\lambda) \equiv F(\lambda) * G(\lambda)$ and $F^*(\lambda) \equiv F(\lambda)^*$ are also in $\mathscr{A}$. $\mathscr{A}$ is a *quantization* of $(\mathcal{M}, \omega)$ if the following hold.

I. Whenever F and G are in $\mathscr{A}$ so is the map $\lambda \rightarrow (1/\lambda)(F(\lambda) * G(\lambda) - G(\lambda) * F(\lambda))$. This element is denoted $(1/\lambda)(F * G - G * F)$.

II. There is an algebra homomorphism ϕ from $\mathscr{A}$ into $C^\infty(\mathcal{M})$ that takes multiplication in $\mathscr{A}$ into pointwise multiplication of functions in $C^\infty(\mathcal{M})$ and involution into complex conjugation.

III. The homomorphism satisfies $\phi((1/\lambda)(F * G - G * F)) = i\{\phi(F), \phi(G)\}$ where $i = \sqrt{-1}$.

IV. Given two distinct points x_1, x_2 in $\mathcal{M}$, there is an $F \in \mathscr{A}$ such that $\phi(F)(x_1) \neq \phi(F)(x_2)$.

Definition 2.2. Suppose every A_λ is a subalgebra of bounded operators on a Hilbert space. As ω is nondegenerate there is a natural volume element dm on $\mathcal{M}$ formed from ω [**1**]. $\mathscr{A}$ is called a *special quantization* if there is a family of positive constants c_λ such that, for all $F \in \mathscr{A}$, $F(\lambda)$ is a trace class operator in A_λ and

$$\text{V. } \lim_{\lambda \rightarrow 0} c_\lambda \operatorname{Tr}(F(\lambda)) = \int_{\mathcal{M}} \phi(F) dm.$$

The motivation for the above conditions is best explained by the fundamental example of a quantization. For a single particle moving on a line, phase space is the plane R^2 with position and momentum coordinates (q, p). The Poisson bracket is $\{f, g\} = (\partial f/\partial q)(\partial g/\partial p) - (\partial f/\partial p)(\partial g/\partial q)$. If h is a fixed real-valued Hamiltonian function and f a function of the observables q and p, we have the equation of motion [**1**]

$$(2.1) \quad \frac{df}{dt} = \{f, h\}.$$

The quantization is the one originally suggested by Weyl [**15**]. As an element $\lambda \in E$ is interpreted as Planck's constant divided by 2π, E comprises all positive real numbers. As in [**2**; **3**], let A_λ be all operators on $L^2(R)$ of the form $T(Q, \lambda P)f$ where f is in the Schwartz space $\mathcal{S}(R^2)$ of rapidly decreasing test functions on the plane and

$$(2.2) \qquad T(Q, \lambda P)f = \frac{1}{2\pi}\int_{R^2} \mathcal{F}f(q, p) \exp\,(iqQ + ip(\lambda P))dqdp.$$

In this formula, $\exp\,(iqQ + ip(\lambda P))$ is the unitary operator given by Stone's theorem from the essentially self-adjoint operators Q and λP on $L^2(R)$ that denote the multiplier x and the differential operator $-i\lambda\, d/dx$ respectively. $\mathcal{F}$ is the usual Fourier transform. Let $\mathcal{A}$ be the algebra generated (generate always means in the sense of condition I) by the maps $F_f(\lambda) = T(Q, \lambda P)f$ for some $f \in \mathcal{S}(R^2)$ and let ϕ be the extension of $\phi(F_f) = f$. That ϕ is well-defined and produces a special quantization can be readily deduced from [**3**]. The trace is given by

$$(2.3) \quad \mathrm{Tr}\,(F_f(\lambda)) = \frac{1}{2\pi\lambda}\int_{R^2} f(q, p)dqdp.$$

Condition II is a mathematical formulation of the statements 'Planck's constant measures the extent to which the operators fail to commute' and 'real-valued functions correspond to self-adjoint operators (observables)'. To interpret III, let h be a fixed real-valued Hamiltonian function with $H = T(Q, \lambda P)h$. The dynamical equation in the Heisenberg picture [**11**] of fixed states and varying observables is

$$\frac{dB}{dt} = \frac{i}{\lambda}(HB - BH).$$

Formally, applying ϕ and condition III, we have essentially (2.1); namely,

$$\frac{d\phi(B)}{dt} = ii\{\phi(H), \phi(B)\} = \{\phi(B), h\}.$$

Together, conditions II and III give an appropriate translation of the correspondence principle. Condition IV insures that phase space does not "collapse" in the quantization. This is of little significance in the quantization of orbits through representations but would play a role if covering spaces of orbits were to be quantized (see section 5 of [**10**]). The physical impact of V lies in the correspondence of classical and quantum statistical mechanics [**7**; **11**]. It is mathematically relevant to the problem of obtaining functions from operators. That is, if $F \in \mathcal{A}$, let $f_{(F,\lambda)}$ be the unique function satisfying

$$\frac{1}{2\pi\lambda}\int_{R^2} f_{(F,\lambda)}(q, p)f(q, p)dqdp = \mathrm{Tr}\,(F(\lambda)F_f(\lambda))$$

for all $f \in \mathcal{S}(R^2)$. Then $f_{(F,\lambda)}$ is in $\mathcal{S}(R^2)$ and, in fact, $T(Q, \lambda P)f_{(F,\lambda)} = F(\lambda)$.

3. Quantization of orbits—nilpotent Lie groups. Let $\mathscr{G}$ be any real Lie group with Lie algebra $\mathfrak{G}$ of dimension n. Let $\mathscr{O}$ be any orbit in the coadjoint representation that represents $\mathscr{G}$ as linear transformations acting on the vector space $\mathfrak{G}'$ of all real-valued linear functionals of $\mathfrak{G}$. This action produces a natural non-degenerate 2-form ω on $\mathscr{O}$ [**5**; **10**; **12**; **13**]. The classical mechanical system $(\mathscr{O}, \omega)$ include such examples as the plane in Section 2 and the sphere and Lobachevskii plane in [**4**].

In order to generalize (2.2), some preliminaries are required. Let $[x, y]$ denote the Lie bracket of two elements of $\mathfrak{G}$. Let (x, l) be the canonical bilinear form $\mathfrak{G} \times \mathfrak{G}' \rightarrow R$ denoting the linear functional l applied to x. If dx is a fixed but arbitrary translation invariant measure on $\mathfrak{G}$, there is a unique measure dl on $\mathfrak{G}'$ such that the Fourier inversion formula is valid. Formally, for functions on $\mathfrak{G}'$, the Fourier transform is

$$\mathscr{F}f(x) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}'} f(l) e^{-i(x,l)} dl$$

and the inversion formula is

$$f(l) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F}f(x) e^{i(x,l)} dx.$$

The measures can be realized by picking a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{G}$, identifying $\mathfrak{G}$ with R^n, setting dx equal to Lebesgue measure and dl Lebesgue measure on $\mathfrak{G}'$ with respect to the dual basis $\{e_1', \ldots, e_n'\}$. The Fourier transform establishes a homeomorphism between $\mathscr{S}(\mathfrak{G}')$ and $\mathscr{S}(\mathfrak{G})$—the spaces of rapidly decreasing test functions on the vector spaces $\mathfrak{G}'$ and $\mathfrak{G}$ respectively.

Let us restrict our attention to the case of a connected, simply connected nilpotent Lie group $\mathscr{G}$. The exponential map, $\exp: \mathfrak{G} \rightarrow \mathscr{G}$, is a diffeomorphism such that dx induces left (and right) invariant Haar measure on $\mathscr{G}$. To every orbit, there is a unique irreducible unitary representation of $\mathscr{G}$ [**9**; **12**; **13**]. To quantize $\mathscr{O}$, notice that $(1/\lambda)\mathscr{O} = \{1/\lambda)l : l \in \mathscr{O}\}$ is also an orbit if λ is in the set E of all positive real numbers. Consider the irreducible representations U_λ given by the family of orbits $(1/\lambda)\mathscr{O}$. Let A_λ be all operators of the form

$$(3.1) \quad T_\lambda f = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} M_\lambda f(x) U_\lambda(\exp x) dx$$

where $f \in \mathscr{S}(\mathfrak{G}')$ and M_λ is the dilation $M_\lambda f(l) = f(\lambda l)$. (The dilation is present to move the function up to the orbit $(1/\lambda)\mathscr{O}$.) Let $\mathscr{A}$ be the algebra generated (subject to requirement I) by the maps

$$F_f(\lambda) \equiv T_\lambda f.$$

Let ϕ be defined on the generators by $\phi(F_f) = f_{|\mathscr{O}}$, the restriction of f to $\mathscr{O}$. As is seen in Section 4, ϕ is well-defined and can be extended to the homomorphism of a special quantization of $\mathscr{O}$.

Remark 1. A subtle notational change from Section 2 has occurred in that f is no longer a function on the manifold. To produce an operator from a function on $\mathcal{O}$, it is first necessary to extend it to all of $\mathfrak{G}'$ in a suitable manner. In general, the operator will depend on the extension chosen.

Remark 2. A few comments are in order to interpret the quantization in Section 2 in terms of Kirillov theory. $\mathcal{G}$ is the 3 dimensional Heisenberg group with Lie algebra spanned by $\{e_0, e_1, e_2\}$ and brackets $[e_2, e_1] = e_0$, $[e_1, e_0] = [e_2, e_0] = 0$. The orbits in $\mathfrak{G}'$ are either

i) planes indexed by λ whose first coordinate with respect to the dual basis is the non-zero constant $1/\lambda$, or

ii) single points in the plane with first coordinate zero.

The quantization of Section 2 is the same as the quantization of the orbit of index 1 given in this section. Indeed, for $f \in \mathcal{S}(\mathfrak{G}')$, the operator $T_\lambda f$ of (3.1) is actually $T(Q, \lambda P)\tilde{f}$ where $\tilde{f}(q, p) = f(1, q, p)$. The important feature of the Heisenberg group that is missing in general is that the method of extending a function from the orbit to all of $\mathfrak{G}'$ is irrelevant since the same operator is always produced.

Remark 3. For the special quantizations of this section, requirement V can be exploited to realize the elements of A_λ as tempered distributions. If $F \in \mathcal{A}$ and $\lambda \in E$ are fixed, consider the map $\mathcal{S}(\mathfrak{G}') \rightarrow C$ given by $f \rightarrow c_\lambda \operatorname{Tr}(F(\lambda)T_\lambda f)$. This defines a continuous linear functional on $\mathcal{S}(\mathfrak{G}')$, so there is a unique tempered distribution $h_{(F,\lambda)} \in \mathcal{S}'(\mathfrak{G}')$ such that $c_\lambda \operatorname{Tr}(F(\lambda)T_\lambda f) = h_{(F,\lambda)}(f)$ (the right side is the action of the distribution on the test function). Varying F over $\mathcal{A}$, we obtain for each λ an algebra of tempered distributions. These algebras play the role of the algebras of differentiable functions on the orbit used in the term "special quantization" present in [**4**]. In fact,

$$\lim_{\lambda\to 0} h_{(F,\lambda)}(f) = \lim_{\lambda\to 0} c_\lambda \operatorname{Tr}(F(\lambda)T_\lambda f) = \int_{\mathcal{O}} \phi(F) f dm.$$

Therefore, given $F \in \mathcal{A}$, $\lim_{\lambda\to 0} h_{(F,\lambda)}$ exists in the distributional sense and is the measure $\phi(F)dm$ that is supported on $\mathcal{O}$.

4. Proofs.

The purpose of this section is to demonstrate the claims made implicitly in Section 3. The terminology is that of Section 3.

PROPOSITION 4.1. *ϕ is well-defined on the generators of $\mathcal{A}$.*

Proof. Suppose F_f is the zero element of $\mathcal{A}$. Obviously, it is enough to show that $f(l_0) = 0$ for a fixed but arbitrary $l_0 \in \mathcal{O}$. The form of the representations U_λ is required. Only an outline is provided here until the kernel function is presented in (4.1)—for the complete details see [**13**]. Let $\mathfrak{h} \subset \mathfrak{G}$ be a maximal subordinate subalgebra of l_0. That is, $([x_1, x_2], l_0)$ is identically zero for x_1, x_2 in $\mathfrak{h}$ and $\mathfrak{h}$ is maximal with this property. Then U_λ is the representation of $\mathcal{G}$ induced by the character $\chi_\lambda(\exp x) = e^{i(x, l_0/\lambda)}$ on $\exp \mathfrak{h}$. Choose a basis

$\{e_1, \ldots, e_n\}$ of $\mathfrak{G}$ so that $\{e_1, \ldots, e_m\}$ spans $\mathfrak{h}$ and $\{e_1, \ldots, e_j\}$ spans a subalgebra for $m \leqq j \leqq n$. Let

$$\Gamma = \{\gamma = \exp^{-1} (\exp (t_{m+1} e_{m+1}) \ldots \exp (t_n e_n)) : t_j \in R, m < j \leqq n\}.$$

Γ may be identified with the set of representatives of the right cosets of $\mathscr{G}$ with respect to $\exp \mathfrak{h}$. The operator $T_\lambda f$ is a Hilbert-Schmidt operator on $L^2(\Gamma)$—the space of functions on Γ that are square integrable with respect to the measure on Γ coming from $dt_{m+1} \ldots dt_n$. The kernel of $T_\lambda f$, a function on $\Gamma \times \Gamma$, is

$$(4.1) \quad \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{h}} \mathscr{F} M_\lambda f(\exp^{-1} ((\exp \gamma)^{-1} \exp x \exp \gamma')) \chi_\lambda (\exp x) dx.$$

In particular, if F_f is zero, then for all $\gamma \in \Gamma$ and $\lambda \in E$,

$$\int_{\mathfrak{h}} \int_{\mathfrak{G}'} f(\lambda l) e^{-i(\exp^{-1}(\exp x \exp \gamma), l)} e^{i(x, l_0/\lambda)} dl dx = 0.$$

Fix $y = t_{m+1} e_{m+1} + \ldots + t_n e_n$ and vary γ with λ to obtain

$$\int_{\mathfrak{h}} \int_{\mathfrak{G}'} f(l) e^{-i(\exp^{-1}(\exp(\lambda x) \exp(\lambda t_{m+1} e_{m+1}) \ldots \exp(\lambda t_n e_n))/\lambda, l)} e^{i(x, l_0)} dl dx = 0.$$

Since $\mathscr{G}$ is nilpotent, $\exp^{-1} (\exp z_1 \exp z_2)$ is a polynomial in z_1, z_2 whose first terms by the Campbell-Baker-Hausdorff formula [**8**] are

$$(4.2) \quad \exp^{-1} (\exp z_1 \exp z_2) = z_1 + z_2 + \tfrac{1}{2}[z_1, z_2] + \ldots$$

Thus, as $\lambda \to 0$,

$$\int_{\mathfrak{h}} \int_{\mathfrak{G}'} f(l) e^{-i(x+y, l)} e^{i(x, l_0)} dl dx = 0.$$

Decompose $\mathfrak{G}'$ into a direct sum $\mathfrak{G}_1' \oplus \mathfrak{G}_2'$ as $l = l_1 + l_2$ where $(y, l_1) = 0$ for all y in the span of $\{e_{m+1}, \ldots, e_n\}$ and $(x, l_2) = 0$ for all x in $\mathfrak{h}$. Then

$$0 = \int_{\mathfrak{G}_2'} e^{-i(y, l_2)} \int_{\mathfrak{h}} \int_{\mathfrak{G}_1'} f(l_1 + l_2) e^{-i(x, l_1)} e^{i(x, l_0)} dl_1 dx dl_2.$$

Fourier inversion on $\mathfrak{h}$ yields, with $l_0 = l_{01} + l_{02}$ as the decomposition of l_0,

$$0 = \int_{\mathfrak{G}_2'} e^{-i(y, l_2)} f(l_{01} + l_2) dl_2.$$

As y is arbitrary in the dual of $\mathfrak{G}_2'$, $f(l_{01} + l_2) = 0$ for all $l_2 \in \mathfrak{G}_2'$. In particular, $f(l_{01} + l_{02}) = f(l_0) = 0$.

It is clear from the Heisenberg group that, in general, the function can only be determined on $\mathscr{O}$ by the family F_f. In this sense, Proposition 4.1 is the best possible result.

LEMMA 4.2. *Given $F \in \mathscr{A}$, there is a family of functions f_λ contained in $\mathscr{S}(\mathfrak{G}')$ such that*

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i) $F(\lambda) = T_\lambda f_\lambda$ *and*
ii) $\lim_{\lambda\to 0} f_\lambda$ *converges in the topology of* $\mathscr{S}(\mathfrak{G}')$ *to a function* f_0.

Proof. It suffices to show that, first, the result is true for F_f and, second, if F and G satisfy the properties so do $F * G$ and $(1/\lambda)(F * G - G * F)$. The result is trivial for F_f with $f_\lambda = f$.

a) Suppose $F(\lambda) = T_\lambda f_\lambda$ and $G(\lambda) = T_\lambda g_\lambda$. Then

$$F * G(\lambda) = T_\lambda f_\lambda T_\lambda g_\lambda$$
$$= \left(\frac{1}{2\pi}\right)^n \int_{\mathfrak{G}} \int_{\mathfrak{G}} \mathscr{F} M_\lambda f_\lambda(x) \mathscr{F} M_\lambda g_\lambda(y)\, U_\lambda(\exp x \exp y) dy dx.$$

Thus $F * G(\lambda) = T_\lambda h_\lambda$ where $\mathscr{F} h_\lambda(y)$ is

$$(4.3)\quad \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_\lambda(x) \mathscr{F} g_\lambda\left(\frac{\exp^{-1}(\exp(-\lambda x)\exp(\lambda y))}{\lambda}\right) dx.$$

By (4.2),

$$(4.4)\quad \lim_{\lambda\to 0} \mathscr{F} h_\lambda(y) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_0(x) \mathscr{F} g_0(y - x) dx.$$

The convergence is clearly in $\mathscr{S}(\mathfrak{G})$, so the Fourier transform establishes the result.

b) Similarly $(1/\lambda)(F * G - G * F) = T_\lambda k_\lambda$ where $\mathscr{F} k_\lambda(y)$ is

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_\lambda(x) \frac{1}{\lambda} \left\{ \mathscr{F} g_\lambda\left(\frac{\exp^{-1}(\exp(-\lambda x)\exp \lambda y)}{\lambda}\right) - \mathscr{F} g_\lambda\left(\frac{\exp^{-1}(\exp \lambda y \exp(-\lambda x))}{\lambda}\right)\right\} dx.$$

If dx is Lebesgue measure with respect to $\{e_1, \ldots, e_n\}$, $x = \sum_{i=1}^n x_i e_i$ is in $\mathfrak{G}$, and u is a function on $\mathfrak{G}$, let $D_x u$ denote the derivative $\sum_i x_i\, \partial u/\partial x_i$. With this notation, by (4.2),

$$(4.5)\quad \lim_{\lambda\to 0} \mathscr{F} k_\lambda(y) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_0(x)\,(D_{[-x,y]} \mathscr{F} g_0)(y - x) dx.$$

Again, the convergence is in $\mathscr{S}(\mathfrak{G})$.

THEOREM 4.3. *For F as in Lemma 4.2, define $\phi(F) = f_{0|\mathscr{O}}$. ϕ is well-defined.*

Proof. Suppose $F = T_\lambda f_\lambda = T_\lambda g_\lambda$ for two families such that $f_\lambda \to f_0$ and $g_\lambda \to g_0$ in $\mathscr{S}(\mathfrak{G}')$. Then $T_\lambda(f_\lambda - g_\lambda)$ is the zero operator for each λ. The argument in the proof of Proposition 4.1 remains valid with $f_\lambda - g_\lambda$ in place of f. Thus $(f_0 - g_0)(l_0) = 0$ for any $l_0 \in \mathscr{O}$. That is, ϕ is well-defined.

THEOREM 4.4. *ϕ satisfies the requirements of a special quantization.*

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Proof. a) Since the Fourier transform takes convolution into multiplication, equation (4.4) insures that $h_0(l) = f_0(l)g_0(l)$. In other words, that $\phi(F * G) = \phi(F)\phi(G)$.

b) For requirement III, the concrete form of the Poisson bracket is required. Choose a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{G}$. Define the structure constants as $[e_i, e_j] = \sum_{k=1}^{n} C_{ij}{}^k e_k$. For f, g in $\mathscr{S}(\mathfrak{G}')$, their restrictions to $\mathscr{O}$ have bracket

$$\{f_{|\mathscr{O}}, g_{|\mathscr{O}}\} = \sum_{i,j,k} C_{ij}{}^k l_k \frac{\partial f}{\partial l_j} \frac{\partial g}{\partial l_i}$$

where l is decomposed relative to the dual basis. The interested reader can verify this from the explicit form of ω in [**5**] and the general method of constructing Poisson brackets from two forms [**1**].

The right hand side of (4.5), with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, becomes

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_0(x) \sum_{i,j,k} C_{ij}{}^k y_i x_j \frac{\partial \mathscr{F} g_0}{\partial x_k} (y - x) dx.$$

Since $C_{ij}{}^k = -C_{ji}{}^k$ the product $y_i x_j$ can be replaced by $(y_i - x_i)x_j$. Taking derivatives and polynomials past the Fourier transform, we obtain

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \sum C_{ij}{}^k \mathscr{F}\left(-i \frac{\partial f_0}{\partial l_j}\right)(x) \mathscr{F}\left(-i \frac{\partial}{\partial l_i} (-i l_k g_0)\right)(y - x) dx.$$

Therefore $\lim_{\lambda \to 0} k_\lambda(l)$, for $l \in \mathscr{O}$, is

$$\text{(4.6)} \quad i \sum C_{ij}{}^k \frac{\partial f_0}{\partial l_j}(l) \frac{\partial l_k g_0}{\partial l_i}(l) = i\{f_{0|\mathscr{O}}, g_{0|\mathscr{O}}\} + i \sum_{j,k} C_{kj}{}^k \frac{\partial f_0}{\partial l_j} g_0.$$

But $\sum_k C_{kj}{}^k = \sum_k ([e_k, e_j], e_k') = -\operatorname{tr}(\operatorname{ad} e_j)$ where the trace is taken of the linear transformation $\operatorname{ad} x(y) \equiv [x, y]$. For nilpotent groups, $\operatorname{tr}(\operatorname{ad} x) = 0$ for all x. Therefore $\phi((1/\lambda)(F * G - G * F)) = i\{\phi(F), \phi(G)\}$.

c) By [**9**], $T_\lambda f$ is a trace class operator and there is a constant c_λ such that

$$c_\lambda \operatorname{Tr}(T_\lambda f) = \int_{\mathscr{O}} f_{|\mathscr{O}}\, dm.$$

It is also noted in [**9**] that dm is a tempered distribution. Thus, if $F(\lambda) = T_\lambda f_\lambda$ as in Lemma 4.2,

$$\lim_{\lambda \to 0} c_\lambda \operatorname{Tr}(F(\lambda)) = \lim_{\lambda \to 0} \int_{\mathscr{O}} f_{\lambda|\mathscr{O}}\, dm = \int_{\mathscr{O}} \phi(F) dm.$$

Remark. A crude attempt to carry out the program of Section 3 for orbits of general Lie groups is to take U_λ as the left regular representation U of $\mathscr{G}$ on $L^2(\mathscr{G}, d\mu)$ for each $\lambda \in E$. Here $d\mu$ is the left invariant Haar measure whose pullback under the exponential map is $\rho(x)dx$, ρ being an analytic function near the origin 0 of $\mathfrak{G}$ with $\rho(0) = 1$. As the exponential map is only a local diffeomorphism, operators are formed from functions f such that $\mathscr{F} f \in C_0^\infty(\mathfrak{G})$

(i.e. $\mathscr{F}f$ is a C^∞ function with compact support). As in (3.1) set

$$(4.7)\quad T_\lambda f = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} M_\lambda f(x)\, U(\exp x) K(x) dx.$$

Requirement V must be abandoned because $T_\lambda f$ is a convolution operator on the group [3]. With only the asymptotic requirements I-IV to satisfy, we may introduce the factor $K(x)$ that is a real-valued continuous function analytic near the origin and $K(0) = 1$. Let A_λ be the algebra of bounded operators generated by $T_\lambda f$ and $\mathscr{A}$ be generated by the maps

$$F_\lambda(\lambda) = T_\lambda f.$$

By means of group convolution, the analogue of Lemma 4.2 becomes the statement that for every $F \in \mathscr{A}$ there is a $\lambda_0 > 0$ such that there is a family of functions f_λ for $0 < \lambda < \lambda_0$ satisfying

i) $F(\lambda) = T_\lambda f_\lambda$ where $\mathscr{F} f_\lambda \in C_0^\infty(\mathfrak{G})$,

ii) $\mathscr{F} f_\lambda \to \mathscr{F} f_0$ as $\lambda \to 0$ in the topology of $C_0^\infty(\mathfrak{G})$, and

iii) given two families satisfying i) and ii), say f_λ and g_λ, then $f_\lambda = g_\lambda$ for λ sufficiently small.

ϕ is then defined as $\phi(F) = f_{0|\mathcal{O}}$. It is clearly well-defined by property iii) above.

An indication that ϕ is a quantization follows. Notice that the support of $\mathscr{F} M_\lambda f$ is $\{\lambda x$: x is in the support of $\mathscr{F} f\}$. If λ is so small that exponential coordinates may be used and that K, ρ are positive analytic, the expression for $\mathscr{F} h_\lambda(y)$ in (4.3) changes to

$$\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_\lambda(x) \mathscr{F} g_\lambda\left(\frac{\exp^{-1}(\exp(-\lambda x)\exp \lambda y)}{\lambda}\right)$$
$$\times \frac{K(\exp^{-1}(\exp(-\lambda x)\exp\lambda y))}{K(\lambda y)} \frac{\rho(\lambda y)}{\rho(\exp^{-1}(\exp(-\lambda x)\exp\lambda y))} K(\lambda x) dx.$$

On taking the limit as $\lambda \to 0$, this becomes exactly (4.4) since $K(\exp^{-1}(\exp(-\lambda x)\exp\lambda y))/K(\lambda y)$ and the similar expression involving ρ are analytic functions of y for small λ whose derivatives with respect to y all go to zero uniformly in compact sets as $\lambda \to 0$ while the functions themselves approach 1 uniformly. Likewise the right-hand side of (4.5) is altered, after allowing for the modular function to eliminate right multiplication in the group, to

$$(4.8)\quad \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathfrak{G}} \mathscr{F} f_0(x)\{D_{[-x,y]} \mathscr{F} g_0(y - x) - \operatorname{tr}(\operatorname{ad} x) \mathscr{F} g_0(y - x)\} dx.$$

That ϕ provides a quantization follows as in Theorem 4.4, the only difference being that the unwanted term in (4.6) is cancelled by the trace in (4.8).

5. Quantization of the sphere. After the Remark in Section 4, one is led to ask what systems do not constitute quantizations. The point is that physically relevant quantizations seem to come from irreducible representations of a Lie group. The difficult step for these quantizations is to show ϕ is well-defined. Once this is accomplished, the fact that ϕ satisfies II and III is a consequence of the structure of Lie algebras and the exponential map. In practice, the factor K in (4.7) is juggled so that requirement V holds [**12**]. In other words, the quantization arising from the left regular representation only suggests how ϕ should be defined—the proof must come from the concrete form of the irreducible representations.

As an illustration, consider the example, where the irreducible representations are well-known, of the real compact semisimple Lie group $SU(2)$ of 2×2 unitary matrices with determinant 1. Let Π: $SU(2) \to SO(3)$ be the usual covering map onto the rotation group of 3-space. The Lie algebra $\mathfrak{G}$ is spanned by three elements

$$e_1 = \tfrac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad e_2 = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad e_3 = \tfrac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

with brackets $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$ and $[e_3, e_1] = e_2$. The negative definite Killing form on $\mathfrak{G}$ provides a natural identification of $\mathfrak{G}$ with $\mathfrak{G}'$. Thus orbits in $\mathfrak{G}'$ become orbits in $\mathfrak{G}$ and the Fourier transform takes functions on $\mathfrak{G}$ into functions on $\mathfrak{G}$. The orbits are spheres centered at the origin and the 2-form ω is invariant under rotations [**4**].

Let us quantize the sphere $\mathscr{O}$ of radius $\frac{1}{2}$. The irreducible representations of $SU(2)$ are indexed by the weight j that takes on all nonnegative integer and half-integer values. Following [**13**], it can be shown that the sphere of radius $j + \frac{1}{2}$ corresponds (in the sense of (5.1)) to the representation of weight j. Let U_λ be the representation corresponding to $(1/\lambda)\mathscr{O}$. Thus λ is restricted to lie in the set $E = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and U_λ is the representation of weight $\frac{1}{2}((1/\lambda)-1)$. For $\mathscr{F}f \in C_0^\infty(\mathfrak{G})$, set

$$T_\lambda f = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathfrak{G}\simeq R^3} \mathscr{F}M_\lambda f(x) U_\lambda(\exp x) \frac{\sin(|x|/2)}{|x|/2}\, dx.$$

where $x = \sum x_i e_i$ and $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then, there is a constant such that

$$(5.1) \qquad c_\lambda \operatorname{Tr}(T_\lambda f) = \int_{\mathscr{O}} f dm.$$

The algebra $\mathscr{A}$ is built from the generators $F_f(\lambda) = T_\lambda f$ and $\phi(F_f) = f_{|\mathscr{O}}$. That this is a quantization follows from

PROPOSITION 5.1. *ϕ is well-defined on the generators F_f.*

Proof. If $\lambda = 1/(2j + 1)$, j an integer, then the representation U_λ acts on the space of spherical harmonics u of degree $j = \frac{1}{2}((1/\lambda) - 1)$ [**15**] by

$$(U_\lambda(\exp x)u)(\xi) = u(\Pi(\exp x)\xi)$$

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where $\Pi(\exp x)\xi$ is the rotation $\Pi(\exp x)$ applied to $\xi \in R^3$. Recall that u is a complex-valued harmonic polynomial on R^3 that is homogeneous of degree j.

Suppose $T_\lambda f$ is the zero operator for all λ. Then, in particular, $((T_\lambda f)u_\lambda)(0, 0, 1) = 0$ where $u_\lambda(\xi_1, \xi_2, \xi_3) = (\xi_3 - i\xi_2)^{\frac{1}{2}((1/\lambda)-1)}$. Therefore

$$0 = \int_{R^3} \mathscr{F}f(x)u_\lambda(\Pi(\exp \lambda x)(0, 0, 1)) \frac{\sin(\lambda|x|/2)}{\lambda|x|/2} dx.$$

As $\lambda \to 0$, by Lemma 5.2

$$0 = \int_{R^3} \mathscr{F}f(x)e^{ix_3/2} dx = f(0, 0, \tfrac{1}{2}).$$

It is clear a similar argument will guarantee that $f(x) = 0$ for all $x \in \mathscr{O}$.

LEMMA 5.2. *With the above notation, let* $w_\lambda(x) = u_\lambda(\Pi(\exp \lambda x)(0, 0, 1))$. *Then* w_λ *is a sequence of analytic functions of* x *that converges uniformly for compact subsets as* $\lambda \to 0$ *to the analytic function* $e^{ix_3/2}$.

Proof. $w_\lambda(x) = u_\lambda((I + \lambda A + (1/2!)(\lambda A)^2 + \ldots)(0, 0, 1))$ where I is the identity matrix on R^3 and

$$A = \begin{bmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}$$

Therefore $w_\lambda(x) = (1 + i\lambda x_3 + \lambda^2(\dagger))^{\frac{1}{2}((1/\lambda)-1)}$ and $\dagger$ is an analytic function that is uniformly bounded in λ on compact subsets of x. The binomial expansion implies

$$\lim_{\lambda \to 0} w_\lambda(x) = 1 + ix_3/2 + \frac{1}{2!}(ix_3/2)^2 + \ldots = e^{ix_3/2}.$$

6. Exponential Lie groups—the "*ax* + *b*" group. Exponential Lie groups are characterized by the condition that the exponential map is a diffeomorphism onto the group. The nilpotent groups of Section 3 are exponential. There is again a method of constructing irreducible representations from orbits [**12**; **14**]. A quantization of the orbit results if the factor K in (4.7) is taken to be the function K_0 of [**12**]. Unfortunately, the operators are not of trace class—especially if the orbit is not closed.

Rather than carry out the details of the quantization for arbitrary exponential Lie groups, let us explore more fully a simple example. Take for $\mathscr{G}$ the "$ax + b$" group of affine transformations on the line [**12**]. The Lie algebra $\mathfrak{G}$ is spanned by $\{e_1, e_2\}$ with $[e_1, e_2] = e_2$. Exponential coordinates used globally on $\mathscr{G}$ yield the following expression for $\exp^{-1}(\exp(x, y)\exp(z, w))$ in terms of the chosen basis for $\mathfrak{G}$,

$$(x + z, ((x + z)/(e^{x+z} - 1))((e^x - 1)y/x + e^x w(e^z - 1)/z)).$$

The orbits in $\mathfrak{G}'$, relative to the dual basis $\{e_1', e_2'\}$ are of the following three types:

i) points on the line spanned by e_1',

ii) the (upper) half-plane with positive second coordinate, and

iii) the (lower) half-plane with negative second coordinate.

The invariant measure of ii) is $dm = (1/l_2)dl_1dl_2$.

Let us quantize the upper half-plane $\mathcal{O}$. The set E is all positive real numbers. The representations U_λ corresponding to $(1/\lambda)\mathcal{O}$ is induced by the character $\chi_\lambda(\exp(se_2)) = e^{is/\lambda}$. Explicitly, for $u \in L^2(R)$, U_λ is given by

$$(6.1)\quad (U_\lambda(\exp(x, y)u))(t) = e^{i(e^t(e^x-1)y/\lambda x)}u(t + x).$$

For this orbit, K is identically 1, so set for $\mathscr{F}f \in C_0^\infty(\mathfrak{G})$

$$(6.2)\quad T_\lambda f = \frac{1}{2\pi}\int_{\mathfrak{G}\simeq R^2} \mathscr{F}M_\lambda f(x, y)U_\lambda(\exp(x, y))dxdy.$$

As usual, $\mathscr{A}$ is generated by $F_f(\lambda) = T_\lambda f$ and $\phi(F_f) = f_{|\mathcal{O}}$. Combine (6.1) and (6.2), then substitute $x - t$ for x, to obtain

$$((T_\lambda f)u)(t) = \frac{1}{2\pi}\int_{R^2} \mathscr{F}M_\lambda f(x, y)e^{i(e^t(e^x-1)y/\lambda x)}\, u(t + x)dxdy$$

$$= \frac{1}{\sqrt{2\pi}}\int_R \mathscr{F}_1M_\lambda f(x - t, (e^x - e^t)/\lambda(x - t))u(x)dx.$$

In the last integral, $\mathscr{F}_1 f$ is the partial Fourier transform $\mathscr{F}_1 f(a, b) = (1/\sqrt{2\pi})\int f(z, b)e^{-iaz}dz$. Hence, $T_\lambda f$ is the operator on $L^2(R)$ with continuous kernel

$$(6.3)\quad (1/\sqrt{2\pi})\mathscr{F}_1M_\lambda f(x - t, (e^x - e^t)/\lambda(x - t))$$

as a function of x and t. Since $(e^x - e^t)/\lambda(x - t)$ is always positive, (6.3) is identically zero if and only if $\mathscr{F}_1 f(x, r) = 0$ for all $x \in R$ and all $r > 0$. That is, $T_\lambda f$ being the zero operator is equivalent to f being zero on all of $\mathcal{O}$. Thus, not only is ϕ well-defined, but for each λ there is a 1-1 correspondence between operators in A_λ and functions on the upper half-plane. It is interesting to note that, if $T_\lambda f$ is a positive definite operator of trace class, then (6.3) implies

$$\mathrm{Tr}\,(T_\lambda f) = (1/\sqrt{2\pi})\int_R \mathscr{F}_1M_\lambda f(0, e^t/\lambda)dt$$

$$= \frac{1}{2\pi}\int_{R^2} M_\lambda f(z, e^t/\lambda)dtdz = \frac{1}{2\pi\lambda}\int_{\mathcal{O}} f(l_1, l_2)(1/l_2)dl_1dl_2.$$

Requirement V fails simply because the algebras A_λ were generated by the wrong function space. It can be shown that a special quantization will result if A_λ is chosen properly.

If coordinates (p, q) are introduced on $\mathcal{O}$, this classical mechanical system could be a model for a particle whose position coordinate is restricted. A deeper analysis of the correspondence principle is possible in this case just as in the 1-1 correspondence for the case of the Heisenberg group [**3**; **6**]. One initial feature is that operators (not necessarily bounded) may be defined for more general functions. As an elementary example, the observables $f(p, q) = p$ and $g(p, q) = q$ become the essentially self-adjoint operators $-i\lambda d/dx$ and e^x on $L^2(R)$ respectively.

References

1. R. Abraham, *Foundations of mechanics* (Benjamin, New York, 1967).

2. R. F. V. Anderson, *The Weyl functional calculus*, J. Functional Analysis *4* (1969), 240–269.

3. ——— *The multiplicative Weyl functional calculus*, J. Functional Analysis *9* (1972), 423–440.

4. F. A. Berezin, *General concept of quantization*, Comm. Math. Phys. *40* (1975), 153–174.

5. R. J. Blattner, *Quantization and representation theory*, in Harmonic Analysis on Homogeneous Spaces, Proceedings of Symposia in Pure Mathematics, Vol. XXVI (American Mathematical Society, Providence, R.I., 1973).

6. R. Cressman, *An evolution equation in phase space and the Weyl correspondence*, J. Functional Analysis *22* (1976), 405–419.

7. A. Grossmann, G. Loupias, and E. M. Stein, *An algebra of pseudodifferential operators and quantum mechanics on phase space*, Ann. Inst. Fourier (Grenoble) *18* (1968), 343–368.

8. S. Helgason, *Differential geometry and symmetric spaces* (Academic Press, New York, 1962).

9. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk. *17* (1962), 57–110; Russian Math. Surveys *17* (1962), 53–104.

10. B. Kostant, *Quantization and unitary representations*, in Lectures in Modern Analysis and Application III, Lecture Notes in Mathematics, Vol. 170 (Springer-Verlag, Heidelberg, 1970).

11. G. W. Mackay, *The mathematical foundations of quantum mechanics* (Benjamin, New York, 1963).

12. C. C. Moore, *Representations of solvable and nilpotent groups and harmonic analysis on nil and solvmanifolds*, in Harmonic Analysis on Homogeneous Spaces, Proceedings of Symposia in Pure Mathematics, Vol. XXVI (American Mathematical Society, Providence, R.I., 1973).

13. L. Pukansky, *On the characters and the Plancherel formula of nilpotent groups*, J. Functional Analysis *1* (1967), 255–280.

14. ——— *On the unitary representations of exponential groups*, J. Functional Analysis *2* (1968), 73–113.

15. H. Weyl, *Theory of groups and quantum mechanics* (Dover, New York, 1950).

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