Asymptotic representation of Laplace transforms with an application to inverse factorial series

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1. In a recent paper[1] Professor A. C. Aitken directed attention to a little known type of inverse factorial series, which he calls inverse central factorial series. He showed that this type of series is a very powerful means of asymptotic representation of functions of a certain type, and concluded his note with the remark that there is here scope for considerable investigation. As a modest contribution to this investigation, the asymptotic theory, foreshadowed in the last paragraph of Aitken's paper, will be given here: it is hoped to give the convergence theory of inverse central factorial series later.

The main result in the asymptotic theory of inverse central factorial series is

Theorem 3. *If $\phi(z)$ is the Laplace transform of a function $f(t)$ and* $f(t) \sim \Sigma c_n (2 \sinh \tfrac{1}{2} t)^{2n}$ *to N terms as* $t \to +0$,

then
$$\phi(z) \sim \Sigma \frac{(2n)!\, c_n}{(z-n)(z-n+1)\ldots(z+n)} \quad \text{to } N \text{ terms}$$
as $z \to \infty$ *in the sector* $\Delta - \tfrac{1}{2}\pi \leqq \arg z \leqq \tfrac{1}{2}\pi - \Delta$, $\Delta > 0$.

This theorem depends on some general results on asymptotic representation of Laplace transforms which, while not essentially new, are not found in the requisite form in current text-books. For this reason the whole group of results will be given here, although only one of them is required for the proof of theorem 3. The general results apply also to the asymptotic theory of ordinary inverse factorial series.

2. t is a real variable, $p = \rho + i\sigma$ a complex variable. $f(t)$ is said to be $L(p_0)$ if it is integrable over any finite positive interval and if the Laplace transform

$$\phi(p) = \mathcal{L}\{f(t)\} \equiv \int_0^\infty e^{-pt} f(t)\, dt \tag{1}$$

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[1] These *Proceedings* 7, 168-170 (1946).

exists for $p = p_0$ and is absolutely convergent at the origin (but not necessarily absolutely convergent at infinity)[1]. S_Δ is the sector $\Delta - \frac{1}{2}\pi \leqq \arg p \leqq \frac{1}{2}\pi - \Delta$, $\Delta > 0$.

Three functions, $f(t)$, $g(t)$, $h(t)$ will be considered alongside their Laplace transforms $\phi = \mathscr{L}\{f\}$, $\psi = \mathscr{L}\{g\}$, $\chi = \mathscr{L}(h)$. The asymptotic representations of Laplace transforms follow from the two lemmas :

LEMMA 1. *If f, g, h are $L(\rho)$ for every $\rho > 0$, $h(t) > 0$ for $t > 0$, and $\chi(\rho) \to \infty$ as $\rho \to +0$, then*

$$\overline{\lim} \, | \phi(p) - \psi(p) | / \chi(\rho) \leqq \overline{\lim} \, | f(t) - g(t) | / h(t) \tag{2}$$

as $t \to \infty$ and $p \to 0$ in S_Δ.

LEMMA 2. *If f, g, h are $L(\rho)$ for some sufficiently large ρ, $h(t) > 0$ for $t > 0$, and $e^{\kappa\rho}\chi(\rho) \to \infty$, for every $\kappa > 0$, as $\rho \to \infty$, then* (2) *holds as $t \to +0$ and $p \to \infty$ in S_Δ.*

3. Proof of the lemmas.

For any $T > 0$

$$\phi(p) - \psi(p) = \int_0^T + \int_T^\infty e^{-pt} h(t) \frac{f(t) - g(t)}{h(t)} dt = I_1 + I_2, \text{ say.}$$

If the right-hand side of (2) is infinite then there is nothing to prove: if it is finite then under the conditions of lemma 1, I_2 converges absolutely and

$$| I_2 | \leqq \int_T^\infty e^{-pt} h(t) \frac{|f(t) - g(t)|}{h(t)} dt \leqq \chi(\rho) \, U_T, \tag{3}$$

where U_T is the upper bound in $T \leqq t \leqq \infty$ of $|f(t) - g(t)| / h(t)$.

Thus $$| \phi(p) - \psi(p) | / \chi(\rho) \leqq | I_1 | / \chi(\rho) + U_T.$$

Since $| I_1 | \leqq \int_0^T | f(t) - g(t) | dt$, it is seen that $| I_1 | / \chi(\rho) \to 0$ as $p \to 0$ in S_Δ, and

$$\overline{\lim} \, | \phi(p) - \psi(p) | / \chi(\rho) \leqq U_T, \quad p \to 0 \text{ in } S_\Delta.$$

Lemma 1 follows on making $T \to \infty$.

With the conditions of lemma 2,

$$| I_1 | \leqq \chi(\rho) \times \text{upper bound of } | f(t) - g(t) | / h(t) \text{ for } 0 \leqq t \leqq T$$

[1] G. Doetsch, *Laplace Transformation* (1937), p. 13.

can be proved similarly to (3). I_2 may not converge absolutely in this case, but it is permissible to integrate by parts, so that

$$I_2 = p \int_T^\infty e^{-pt} \left\{ \int_T^t (f(u) - g(u))\, du \right\} dt.$$

The function which vanishes when $t \leqq T$ and is equal to $f(t) - g(t)$ when $t > T$ is $L(\lambda)$ for sufficiently large $\lambda > 0$ and for such λ there is a constant A such that

$$\left| \int_T^t \{ f(u) - g(u) \}\, du \right| \leqq A\, e^{\lambda t},\ t > T.$$

With $\lambda = \frac{1}{2} \rho$ (ρ sufficiently large),

$$|I_2| \leqq |p| \int_T^\infty e^{-\rho t} A\, e^{\frac{1}{2}\rho t}\, dt = \frac{2\,|p|\,A}{\rho}\, e^{-\frac{1}{2}\rho T} \leqq 2\, A\, e^{-\frac{1}{2}\rho T} \operatorname{cosec} \Delta,$$

and $|I_2| / \chi(\rho) \to 0$ as $p \to \infty$ in S_Δ. Lemma 2 now follows as in the former case by making first $p \to \infty$ and then $T \to +0$.

The more usual form of these lemmas[1] is obtained by taking $g(t) = A\, t^{\alpha-1}$, $h(t) = t^{\alpha-1}$, $\alpha > 0$.

4. By a familiar procedure asymptotic representations and asymptotic expansions can be deduced from (2).

$f(x)$ is said to be asymptotically represented by $g(x)$, $f(x) \sim g(x)$, as $x \to x_0$ if $f(x) / g(x) \to 1$ as $x \to x_0$. x_0 may be a finite point or infinity.

If $h(t)$ is a positive function and A a (possibly complex) constant, the lemmas give conditions under which $f(t) \sim A\, h(t)$ implies $\phi(p) \sim A\, \chi(p)$. It is not necessary to set out in detail the relevant theorems, because they are contained in the theorems on asymptotic expansion (for $N = 0$).

A sequence of functions $g_0(x), g_1(x), g_2(x), \ldots$ is said to be an asymptotic sequence for $x \to x_0$ if $g_{n+1}(x) / g_n(x) \to 0$ as $x \to x_0$. With such an asymptotic sequence, the series $\Sigma\, c_n\, g_n(x)$ is said to represent $f(x)$ asymptotically to N terms as $x \to x_0$ if for every M, $0 \leqq M \leqq N$

$$\{ f(x) - \sum_{n=0}^{M} c_n\, g_n(x) \} / g_M(x) \to 0 \text{ as } x \to x_0.$$

With this definition lemma 1 gives for an asymptotic sequence $g_0(t), g_1(t), \ldots$ for $t \to \infty$ and the sequence of Laplace transforms $\psi_n(p) = \mathcal{L}\{g_n(t)\}$

[1] D. V. Widder, *The Laplace Transform* (1941), Chapter V, § 1.

THEOREM 1. *Let $g_n(t) > 0$ for $t > 0$ and $L(\rho)$ for every $\rho > 0$, and suppose that $\psi_n(\rho) \to \infty$ as $\rho \to +0$. If $f(t)$ is $L(\rho)$ for every $\rho > 0$ and*

$$f(t) \sim \Sigma c_n g_n(t) \text{ to } N \text{ terms} \tag{4}$$

as $t \to \infty$ then

$$\phi(p) \sim \Sigma c_n \psi_n(p) \text{ to } N \text{ terms} \tag{5}$$

as $p \to 0$ in S_Δ.

Similarly, for an asymptotic sequence $g_0(t)$, $g_1(t)$, ... for $t \to +0$ and the sequence of Laplace transforms $\psi_n = \mathcal{L}\{g_n\}$ lemma 2 yields

THEOREM 2. *Let $g_n(t) > 0$ for $t > 0$ and $L(\rho)$ for some $\rho > 0$, and suppose that $e^{\kappa\rho}\psi_n(\rho) \to \infty$ for every $\kappa > 0$ as $\rho \to \infty$. If $f(t)$ is $L(\rho)$ for some $\rho > 0$ and (4) holds as $t \to +0$ then (5) holds as $p \to \infty$ in S_Δ.*

5. The well-known asymptotic power series for Laplace transforms follow immediately from the above theorems. So does theorem 3, for

$$g_n(t) = (2 \sinh \tfrac{1}{2} t)^{2n}$$

satisfies the conditions of theorem 2 and for this function

$$\psi_n(p) = \int_0^\infty e^{-pt}(e^{\frac{1}{2}t} - e^{-\frac{1}{2}t})^{2n} dt = \int_0^1 u^{p-n-1}(1-u)^{2n} du = \frac{\Gamma(p-n)(2n)!}{\Gamma(p+n+1)}$$

with $u = e^{-t}$.

Many further results follow readily. As examples, two theorems on asymptotic expansions in (ordinary) inverse factorial series will be given.

$g_n(t) = (1 - e^{-t})^n$ satisfies the conditions of theorem 2, and

$$\psi_n(p) = \int_0^\infty e^{-pt}(1 - e^{-t})^n dt = \int_0^1 u^{p-1}(1-u)^n du = \frac{\Gamma(p)\, n!}{\Gamma(p+n+1)}$$

giving

THEOREM 4. *If $\phi(p)$ is the Laplace transform of a function $f(t)$ and*

$$f(t) \sim \Sigma c_n (1 - e^{-t})^n \text{ to } N \text{ terms as } t \to +0,$$

then

$$\phi(p) \sim \Sigma \frac{n!\, c_n}{p(p+1)\ldots(p+n)} \text{ to } N \text{ terms as } p \to \infty \text{ in } S_\Delta.$$

Similarly, $g_n(t) = (e^t - 1)^n$ satisfies the conditions of theorem 2 and

$$\psi_n(p) = \int_0^\infty e^{-pt}(e^t - 1)^n dt = \int_0^1 u^{p-n-1}(1-u)^n du = \frac{\Gamma(p-n)\, n!}{\Gamma(p+1)}.$$

Hence

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THEOREM 5. *If $\phi(p)$ is the Laplace transform of a function $f(t)$ and*

$$f(t) \sim \Sigma\, c_n (e^t - 1)^n \text{ to } N \text{ terms as } t \to +0$$

then

$$\phi(p) \sim \Sigma \frac{n!\, c_n}{p(p-1)\ldots.(p-n)} \text{ to } N \text{ terms as } p \to \infty \text{ in } S_\Delta.$$

N may be infinite in theorem 4, but is essentially finite in theorems 3 and 5. Indeed, one of Aitken's examples shows that a finite inverse central factorial series gives a very good approximation even for values of the variable for which the later terms of the series would become infinite.

The assumption that the functions, to be represented asymptotically by inverse factorial series, are Laplace transforms is not as restrictive as might appear at first, for any function which has an asymptotic inverse factorial expansion to N terms can be represented, to this degree of accuracy, asymptotically by a Laplace transform.

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