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ON THE SOLVABILITY AND CONTINUATION TYPE RESULTS FOR NONLINEAR EQUATIONS WITH APPLICATIONS, II

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ABSTRACT. In this paper we continue our study of the solvability of nonlinear equations involving uniform limits of A-proper and pseudo A-proper maps under a new growth condition (1) that we began in [14, 15]. Applications of our results to quasimonotone, ball-condensing pertubations of c-accretive maps and maps of semibounded variation and of type (M) are also given.

Introduction. Let X and Y be normed spaces with an admissible scheme $\Gamma=\{E_n, V_n; F_n, W_n\}$ and $T:X\to Y$ a nonlinear map such that

$$\|Tx\|+(Tx, Kx)/\|Kx\|\to\infty \quad \text{as} \quad \|x\|\to\infty, \tag{1}$$

where $K:K\to Y^*$ is a suitable map with $\|Kx\|\to\infty$ as $\|x\|\to\infty$. Consider the equation

$$T(x)=f \qquad (x\in X, f\in Y) \tag{2}$$

and a sequence of finite dimensional equations associated with (2)

$$W_nTV_n(u)=W_n(f), \qquad (u\in E_n). \tag{3}$$

Unlike the existing (approximation) solvability results for A-proper like maps in the literature (see, e.g. [23, 16, 19 and 20, except Theorem 2.6, cf. Remark 2.7(b)]) we have begun recently the study of Eq. (2) under the new growth condition (1). The first results in that direction were announced in our January 1977 note [14] and later in [15] where we have dealt in detail with the approximation—solvability results for Eq. (2) involving A-proper maps and their applications to elliptic differential equations. Solvability of equations involving monotone and (generalized) pseudo-monotone maps that satisfy condition (1) has been earlier studied by Wille [28], Browder [3], Hess [12], Milojević-Petryshyn [19] etc.

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The purpose of this paper is to establish some solvability type results for Eq. (2) involving a much wider class of the so-called uniform limits of A-proper and pseudo A-proper maps satisfying condition (1). In Section 2 we apply our abstract results in establishing some new solvability results for equations involving quasimonotone maps, ball-condensing perturbations of a-stable maps (and, in particular, of strongly accretive type) and maps of semibounded variation and of type (M). At the end we briefly discuss a continuation theorem for uniform limits of A-proper maps, whose detailed discussion will be given later in [18]. The results of Section 1 are existential extensions of our approximation-solvability results for Eq. (2) involving A-proper maps. The results of this paper are also valid for multivalued maps as stated in [14].

SECTION 1. Let $\{E_n\}$ and $\{F_n\}$ be two sequences of oriented finite dimensional spaces and V_n and W_n continuous linear maps of E_n into X and Y onto F_n, respectively.

DEFINITION 1. A quadruple of sequences $\Gamma=\{E_n, V_n; F_n, W_n\}$ is said to be an *admissible scheme* for (X, Y) if $\dim E_n=\dim F_n$ for each n, V_n is injective, $\operatorname{dist}(x, V_n(E_n))\to 0$ as $n\to\infty$ for each x in X, and $\{W_n\}$ is uniformly bounded.

For various examples of admissible schemes we refer to [19, 20, 23]

DEFINITION 2 ([23]). A map $T:X\to Y$ is said to be *approximation proper* (A-*proper*) with respect to Γ if $T_nW_nTV_n:E_n\to F_n$ is continuous for each n and if $\{V_{n_k}(u_{n_k})\mid u_{n_k}\in E_{n_k}\}$ is any bounded sequence such that $\|T_{n_k}(u_{n_k})-W_{n_k}(f)\|\to 0$ as $k\to\infty$ for some f in Y, then there exists an x in X such that (i) $Tx=f$ and (ii) x belongs to the closure of $\{V_{n_k}(u_{n_k})\}$. T is said to be *pseudo A-proper* w.r.t. Γ if we do not require (ii) in Definition 2.

Many examples of A-proper and pseudo A-proper maps and their uniform limits can be found in [23, 16, 19, 20] (see also Section 2). We just state here some needed ones. The first example is due to Browder [4] when $Y=X^*$ and T is bounded, and in this generality it is a special case of maps of type (KS_+) in [24].

EXAMPLE 1. Let X and Y be reflexive Banach spaces, $K:X\to Y^*$ a linear homeomorphism and $\Psi:X\to R$ weakly upper simicontinuous at 0 with $\Psi(0)=0$. If $T:X\to Y$ is quasibounded, demicontinous and such that

$$(Tx-Ty, K(x-y))\geq c(\|x-y\|)-\Psi(x-y)\qquad (x, y\in X)$$

for some function $c:R^+\to R^+$ with $c(0)=0$ and $c(r)>0$ if $r>0$, then T is A-proper w.r.t. $\Gamma=\{X_n, V_n; Y_n, Q_n\}$ with $X_n\subset X$.

If X is compactly embedded in a Banach space Z, then as Ψ we can take $\Psi(x)=\|x\|_Z$, $x\in X$. Hence, all (linear and nonlinear) maps arising, say, in the theory of partial differential equations that satisfy Gårding like inequality in

Example 1 in a space compactly embedded in a bigger space (say L_2) are of A-proper type. As a second example of Ψ we can take $\Psi(x)=(Cx, Kx)$, where $C:X\to Y$ is completely continuous.

Recall that if X is a Banach space and $D\subset X$ bounded, then the *ball-measure of non-compactness* of D is defined by $\chi(D)=\inf\{r>0 \mid D\subset \bigcup_1^n B(x_i, r),\ x_i\in X \text{ and } n>0 \text{ integer}\}$. A map $T:D\subset X\to Y$ is said to be *k-ball-contractive* if $\chi(T(Q))\le k\chi(Q)$ for each $Q\subset D$; it is *ball-condensing* if $\chi(T(Q))<\chi(Q)$ for each $Q\subset D$ with $\chi(Q)\neq 0$. For the theory of these maps, see [22, 26].

EXAMPLE 2 ([16]). Let $E_n\subset X$ and $F_n\subset Y$ with P_n and W_n continuous linear projections of X onto E_n and Y onto F_n, respectively such that $P_n(x)\to x$ and $W_n(y)\to y$ for each x in X and y in Y. If $T:X\to Y$ is continuous, surjective and a-stable, i.e. for some $c>0$.

$$\|T_nx-T_ny\|\ge c\,\|x-y\| \text{ for all } x, y\in E_n,\ n\ge 1,$$

and $F:X\to Y$ is k-ball contractive with $k<c$, or ball condensing if $c=1$, then $T+F$ is *A-proper w.r.t.* $\Gamma_0=\{E_n, P_n; F_n, W_n\}$. In particular, as T one can take a strongly monotone or strongly accretive or strongly K-monotone map.

The importance of this example is that it provides maps that can be treated by the theory of A-proper maps, but not by the other existing ones. The A-properness of $I+A-T$ with A c-monotone and T k-ball contractive, $k-c<1$, was proven in [27].

DEFINITION 3. A map $H:[0,1]\times X\to Y$ is said to be an *A-proper homotopy* on $[0,1]\times X$ w.r.t. Γ if $W_nH:[0,1]\times V_n(E_n)\to F_n$ is continuous and if for all bounded sequences $\{V_{n_k}(u_{n_k}) \mid u_{n_k}\in E_{n_k}\}$ and $\{t_{n_k}\}\subset[0,1]$ such that $\|W_{n_k}H(t_{n_k}, V_{n_k}(u_{n_k}))-W_{n_k}(f)\|\to 0$ as $k\to\infty$ for some f, there are subsequences $t_{n_{k(i)}}\to t_0$ and $V_{n_{k(i)}}(u_{n_{k(i)}})\to x_0$ with $H(t_0, x_0)=f$.

If f in Definition 3 is given in advance, we say that $H(t,x)$ is A-proper at f, while if t_0 is given in advance, $H(t,x)$ is said to be A-proper on X at t_0.

We say that $T:X\to Y$ satisfies condition:

($*$) if $\{x_n\}\subset X$ is bounded whenever $Tx_n\to f$ in Y;

($**$) if $Tx_n\to f$ in Y with $\{x_n\}$ bounded, then $Tx=f$ for some x in X.

THEOREM 1 ([14]). *Let $T:X\to Y$ satisfy condition* (1), *$G:X\to Y$ be bounded and such that $(Gx, Kx)=\|Gx\|\cdot\|Kx\|$ for all $x\in X$, $Gx\neq 0$ for all large $\|x\|$ and $H_\mu(t,x)=tT(x)+\mu G(x)$ an A-proper homotopy on $[0,1]\times X$ w.r.t. Γ for each $\mu>0$ small. Suppose that either one of the following two conditions holds for all large n:*

(i) $\deg(\mu W_nGV_n,\ V_n^{-1}(B(0,r)), 0)\neq 0$ *for all large $r>0$ and small $\mu>0$;*

(ii) *there is $K_n: V_n(E_n)\to F_n^*$ and a linear isomorphism $M_n:E_n\to F_n$ such*

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that

(4) $$(W_n g, K_n x) = (g, Kx) \quad \textit{for all} \quad x \in V_n(E_n), \qquad g \in Y;$$

(5) $$(M_n u, K_n V_n u) > 0 \quad \textit{for all} \quad 0 \neq u \in E_n.$$

Then, if in addition, T either satisfies condition $(**)$ or is pseudo A-proper, T is surjective, i.e. $T(X) = Y$.

Proof. We shall show that all the hypotheses of Theorem 2.1 in [20] hold with $T_t = T$ for all $t \in [0, 1]$. Let $f \in Y$ be fixed. Then by (1) there exists an $r_f > 0$ and $\gamma > 0$ such that

(6) $$\|Tx - tf\| \geq \gamma \quad \text{for all} \quad \|x\| \geq r_f, \qquad t \in [0, 1],$$

(7) $$\|Tx\| + (Tx, Kx)/\|Kx\| > 0 \quad \text{for all} \quad \|x\| \geq r_f, \qquad Gx \neq 0.$$

Hence, (6) is equivalent to conditions (H1) (and (H2) in Theorem 2.1 of [20], while our assumption on H implies (H3) in this theorem. It remains to show condition (H4) of Theorem 2.1 in [20], that is that for all large n and $\mu > 0$,

$$\deg(W_n T V_n + \mu W_n G V_n, B_n(0, r_f), 0) \neq 0.$$

Consider the mapping $H_\mu(t, x) = tT(x) + \mu G(x)$ for $(t, x) \in [0, 1] \times \bar{B}(0, r_f)$ for μ fixed. If for some $t \in [0, 1]$ and $x \in \partial B$, $H_\mu(t, x) = 0$, then $t \neq 0$ and consequently, $T(x) = -(\mu/t)G(x)$. Hence,

$$\begin{aligned} \|Tx\| + (Tx, Kx)/\|Kx\| &= (\mu/t)\,\|Gx\| - (\mu/t)(Gx, Kx)/\|Kx\| \\ &= (\mu/t)\,\|Gx\| - (\mu/t)\,\|Gx\| = 0, \end{aligned}$$

in contradiction to (7). Thus, $0 \notin H_\mu([0, 1] x \partial B)$.

Next, we shall prove that for all large n,

$$tW_n T V_n(x) + \mu W_n G V_n(x) \neq 0 \quad \text{for} \quad x \in \partial B_n, \qquad t \in [0, 1].$$

If this were not the case, then for all $k \geq 1$ there are $t_{n_k} \in [0, 1]$ and $x_{n_k} \in \partial B_{n_k}$ such that

$$t_{n_k} W_{n_k} T V_{n_k}(x_{n_k}) + \mu W_{n_k} G V_{n_k}(x_{n_k}) = 0.$$

Since the homotopy H_μ is A-proper, it follows that $t_{n_{k(i)}} \to t_0$, $V_{n_{k(i)}}(x_{n_{k(i)}}) \to x_0 \in \partial B$ and $H_\mu(t_0, x_0) = 0$, in contradiction to the above property of H_μ. Now the homotopy theorem for the Brouwer degree implies that for all large n

$$\deg(W_n T V_n + \mu W_n G V_n, B_n, 0) = \deg(\mu W_n G V_n, B_n, 0).$$

Thus, if condition (i) of the theorem holds, hypothesis (H4) of Theorem 2.1 in [20] is satisfied.

Let us now show that (H4) holds when (ii) is satisfied. To that end it is sufficient to show that (i) holds. So, define the new homotopy $U_n : [0, 1] \times \bar{B}_n \to F_n$ by $U_n(t, x) = (1-t)M_n(x) + t\mu W_n G V_n(x)$. If for some n and $t \in [0, 1]$, $x \in$

∂B_n, $U_n(t, x)=0$, we have $t\neq 0$ and for $\alpha=1/t$,

$$\begin{aligned}\alpha(M_nx, K_nV_nx)&=(M_nx, K_nV_nx)-\mu(W_nGV_nx, K_nV_nx)\\&=(M_nx, K_nV_nx)-\mu(GV_nx, K_nV_nx)<(M_nx, K_nV_nx),\end{aligned}$$

which is a contradiction. Hence, $U_n(t, x)\neq 0$ for all $t\in[0, 1]$ and $x\in\partial(B_n)$, and the Brouwer homotopy theorem implies that for all n

$$\deg(\mu W_nGV_n, B_n, 0)=\deg(M_n, B_n, 0)\neq 0.$$

Hence, in either case, (H4) holds and, if T satisfies condition $(**)$, $T(X)=Y$ by Theorem 2.1 in [20].

Condition $(**)$ in Theorem 2.1 [20] was used at the final stage of proof. Let us now show that the theorem remains valid if it is replaced by the pseudo A-properness of T (in our case $T_t=T$ for all t). Condition (6), the boundedness of G and (H4) imply that for all large n and $\mu>0$ fixed independent of n,

$$\deg(W_nTV_n+\mu W_nGV_n, B_n, W_nf)\neq 0$$

Hence, in particular, choosing $\mu_n\to 0$ as $n\to\infty$, we can find $x_n\in B_n$ for all large n such that

$$W_nTV_n(x_n)+\mu_nW_nGV_n(x_n)=W_nf,$$

and consequently, $\|W_nTV_n(x_n)-W_nf\|=\mu_n\|W_nGV_n(x_n)\|\to 0$ as $n\to\infty$. The solvability of $Tx=f$ now follows from the pseudo A-properness of T. $\square$
The following elementary proposition imposes some conditions on T and G that guarantee the A-properness of the homotopy $H_\mu(t, x)$.

Proposition 1 [14]. *If G and $T+\mu G$ are A-proper maps for each $\mu>0$ with T and G bounded, then $H_\mu(t, x)=tT(x)+\mu G(x)$ is an A-proper homotopy on $[0, 1]\times X$.*

Actually, in the above proofs we used the homotopy $H_\mu(t, x)$ only in the sense that $H_\mu(1, x)$ is an A-proper map and that $H_\mu(t, x)$ is A-proper at $0\in Y$ when restricted to $[0, 1]\times(X\setminus B(0, r))$ for some large $r>0$. Thus, only these two properties of $H_\mu(t, x)$ suffice. In view of this and the next proposition we obtain another particular set of condition on T and G for which Theorem 1 holds (cf. [15]).

Proposition 2. *Suppose that G and T are as in Proposition 1 with the boundedness of T replaced by the condition*

(8) *there exist an $R>0$ and $c>0$ such that $(Tx, Kx)\geq -c\|Kx\|$ for all $\|x\|\geq R$.*

Suppose also that K is bounded with $\|Kx\|\to\infty$ as $\|x\|\to\infty$ and that K_n satisfies condition (4) of Theorem 1 with $\{K_n(x_n)\}$ bounded whenever $\{x_n\in V_n(E_n)\}$ is bounded. Then, if $G:X\to Y$ is bounded and $(Gx, Kx)=\|Gx\|\cdot\|Kx\|$, $x\in X$, *the homotopy* $H_\mu(t, x) = tT(x) + \mu G(x)$ *is* A*-proper at* $0 \in Y$ *on* $[0, 1] \times (X \setminus B(0, R_0))$ *for* $R_0 \geq R$.

Proof. Let $\{V_{n_k}(u_{n_k}) \in X \setminus B(0, R_0)\}$ be bounded, $t_k \to t$, $t_k \in [0, 1]$ and $a_k \equiv \|W_{n_k} H_\mu V_{n_k}(u_{n_k})\| \to 0$ as $k \to \infty$ ($\mu > 0$ fixed). If $0 < t \leq 1$, then

$$\|W_{n_k} TV_{n_k}(u_{n_k}) + \frac{\mu}{t} W_{n_k} GV_{n_k}(u_{n_k})\| \leq \frac{a_k}{t_k} + \mu \frac{t_k - t}{t \cdot t_k} \|W_{n_k} GV_{n_k}(u_{n_k})\| \to 0 \quad \text{as} \quad k \to \infty$$

and by the A-properness of $T + \mu G$, a subsequence $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x_0$ with $tTx_0 + \mu Gx_0 = 0$.

If $t = 0$, then using the properties of G and K and condition (4),

$$\begin{aligned} \|GV_{n_k}(u_{n_k})\| &= (W_{n_k} GV_{n_k}(u_{n_k}), K_{n_k} V_{n_k} V_{n_k}(u_{n_k}))\, \|KV_{n_k}(u_{n_k})\|^{-1} \\ &= [(W_{n_k} H_\mu(t_k, V_{n_k}(u_{n_k})), K_{n_k} V_{n_k}(u_{n_k})) - t_k(TV_{n_k}(u_{n_k}), KV_{n_k}(u_{n_k}))](\mu\, \|KV_{n_k} u_{n_k}\|)^{-1} \\ &\leq a_k\, \|K_{n_k} V_{n_k}(u_{n_k})\|\, (\mu\, \|KV_{n_k}(u_{n_k})\|)^{-1} + t_k c \mu^{-1} \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Hence, by the A-properness of G, $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x_0$ with $Gx_0 = H(0, x_0) = 0$. □

Remarks. (1) When T is also A-proper, Theorem 1 was first announced in [14], while details can be found in [15].

(2) Analysing the proof of Theorem 1, we see that condition (1) can be replaced by condition ($*$) for T (which implies (6)) and (7). It is clear that condition (7) is implied by: $\|Tx\| \to \infty$ as $\|x\| \to \infty$. As remarked in [20], the last condition is equivalent to (1) provided condition (8) of Proposition 2 holds. This fact has been used by many authors in the study of monotone like and A-proper maps (cf. [3, 19] and the references there in). In view of this fact, we see that a special case of Theorem 1 (ii), which corresponds to the hypotheses on T and G in Proposition 2, extends Theorem 4 in [24] and is also related to Theorem 2.6 in [20] whose hypotheses imply a stronger K-coercivity condition on $T + \mu G$. Let us also add that Proposition 2 was motivated by Theorem 8 of F. Browder [3] and that conditions (4) and (5) have been used earlier, in a different context, in, for example, [23, 24].

(3) Unlike the results in [20, 24], Theorem 1 (i) gives a new surjectivity result for uniform limits of A-proper with respect to a general admissible scheme Γ maps T; in particular, we do no require that Γ satisfies condition (4). When $Y = X$ or $Y = X^*$, there are natural choices for K, K_n and M_n for which condition (4) holds (cf., e.g., [23]). However, in general, the choice of K, K_n and M_n will depend on a given problem and their existence may impose considerable restriction on T. Illustration of this fact is given in Corollaries 3 and 4 in Section 2.

(4) If $Y = X$, $K = J$ the normalized duality map, $G = I$ and if $T = I - F$, with

$F: X \to X$ 1-set contractive, satisfies conditions (1) and $(**)$, then Theorem 1 (i) (i.e., $T(X) = X$) is still valid (see [15] and [17] for details).

For uniform limits of pseudo A-proper maps we have

THEOREM 2 ([14]). *Let K, M_n and K_n be as in Theorem 1, $Kx = 0$ only if $x = 0$ and $G: X \to Y$ bounded and such that $(Gx, Kx) \geq -c\,\|Kx\|$ for all x and some $c > 0$. Suppose that $T: X \to Y$ satisfies condition (1) and that for each continuous increasing function $\psi: R^+ \to R^+$ with*

$$\psi(\|Kx\|) + \frac{(Tx, Kx)}{\|Kx\|} \to \infty \quad \text{as} \quad \|x\| \to \infty \tag{9}$$

there exists a map $F: X \to Y$ such that $(Fx, Kx) = \|Fx\|\,\|Kx\|$, $\|Fx\| = \psi(\|Kx\|)$ for $x \in X$ and $T + F + \mu G$ is pseudo A-proper for each $\mu > 0$. Then, if $T + F$ satisfies condition $(**)$ or is pseudo A-proper, $T(X) = Y$.

Proof. For $f \in Y$, define $T_f x = Tx - f$, $x \in X$. Since T_f satisfies the same conditions as T, it is sufficient to show that $Tx = 0$ is solvable. Let $r > 0$ be fixed and observe that if $\|Tx\| + (Tx, Kx)/\|Kx\| \leq r$, then $\|x\| \leq R$ for some $R > 0$. Let $\psi: R^+ \to R^+$ be continuous and such that $\psi(\|Kx\|) = 0$ for $\|x\| \leq R$ and condition (9) holds. For F that corresponds to this ψ we get that $T + F + \mu G$ is K-coercive, i.e. $(Tx + Fx + \mu Gx, Kx)/\|Kx\| \to \infty$ as $\|x\| \to \infty$. Consequently, for each $\mu \to 0$ there exists [23] $x_\mu \in X$ such that $Tx_\mu + Fx_\mu + \mu Gx_\mu = 0$. Since $T + F$ is also K-coercive and $(Tx_\mu + Fx_\mu, Kx_\mu)/\|Kx_\mu\| \leq \mu c$, we get that $\{x_\mu\}$ is bounded with $Tx_\mu + Fx_\mu \to 0$ as $\mu \to 0$. Thus, $Tx_0 + Fx_0 = 0$ for some x_0 in X if $T + F$ satisfies $(**)$. If $x_0 = 0$, then $\|Tx_0\| = \|Fx_0\| = \psi(\|Kx_0\|) = 0$ and so $Tx_0 = 0$. If $x_0 \neq 0$, $(Tx_0 + Fx_0, Kx_0)/\|Kx_0\| = \|Tx_0\| + (Tx_0, Kx_0)/\|Kx_0\| = 0$ and so $\|Tx_0\| = \|Fx_0\| = \psi(\|Kx_0\|) = 0$, i.e. $Tx_0 = 0$.

Next, let $T + F$ be pseudo A-proper. Since it is K-coercive, $Tx_0 + Fx_0 = 0$ for some x_0 in X and, as before, we get that $Tx_0 = 0$. □

REMARK. Condition (9) holds if, e.g. T is bounded or $(Tx, Kx) \geq -c_1\,\|Kx\|$ for some $x \in X$ and some $c_1 > 0$.

SECTION 2. We now state briefly some special cases of the abstract results in Section 1.

If $K: X \to Y^*$, according to Brezis [1], $T: X \to Y$ is a map of type (KM) if $x_n \rightharpoonup x$ in X, $Tx_n \rightharpoonup f$ in Y and $\limsup(Tx_n, K(x_n - x)) \leq 0$ imply that $Tx = f$.

A map $T: X \to Y$ is K-quasimonotone if $x_n \rightharpoonup x$ in X, then $\limsup(Tx_n, K(x_n - x)) \geq 0$. This class was introduced independently by Calvert-Webb [6] and Hess [11] for $K = I$, $Y = X^*$ and later studied by many authors (see, e.g. [10], [24], [20]). It is known that under suitable conditions pseudo-monotone maps [13] are of type (M) and/or are generalized pseudo-monotone [1, 5]. Under suitable conditions on K and T, it has been shown in [24, 20] that

K-quasimonotone, pseudo K-monotone and generalized pseudo K-monotone maps are uniform limits of A-proper maps. In addition to many examples of pseudo A-proper maps (see, e.g. [23]) we have the following new useful result.

PROPOSITION 3. *Let X and Y be reflexive and $\Gamma=\{E_n, V_n; F_n, W_n\}$ with $E_n \subset X$, $F_n \subset Y$ and V_n the inclusion map. Let $K: X \to Y^*$ be bounded, weakly continuous at* 0, *uniformly continuous on closed balls in X, α-positively homogeneous, $Kx \neq 0$ for $x \neq 0$, $\overline{R(K)}=Y^*$ and K_n as in Proposition* 2. *Let $T: X \to Y$ be K-quasibounded (i.e., $\{Tx_n\}$ is bounded whenever $\{x_n\}$ and $\{(Tx_n, Kx_n)\}$ are bounded), demicontinuous and of type (KM) and $G: X \to Y$ bounded, weakly continuous and of type (KS) (i.e., if $x_n \rightharpoonup x$ and* $\lim(Tx_n, K(x_n-x))=0$, *then $x_n \to x$). Then $T_\mu=T+\mu G$ is pseudo A-proper w.r.t. Γ for each $\mu \geq 0$.*

Proof. Let $\mu \geq 0$ be fixed and $\{x_{n_k} \in X_{n_k}\}$ bounded and such that $\|W_{n_k}(Tx_{n_k}+\mu Gx_{n_k})-W_{n_k}(g)\| \to 0$ as $k \to \infty$. Since

$$(Tx_{n_k}+\mu Gx_{n_k}, K(x_{n_k}))=(W_{n_k}(Tx_{n_k}+\mu Gx_{n_k})-W_{n_k}(g), K_{n_k}(x_{n_k}) + (W_{n_k}(g), K_{n_k}(x_{n_k})),$$

by condition (4), the sequence $\{(Tx_{n_k}+\mu Gx_{n_k}, Kx_{n_k})\}$ is bounded and consequently, $\{Tx_{n_k}+\mu Gx_{n_k}\}$ is bounded by the K-quasiboundedness of T. By the reflexivity of X, we may assume that $x_{n_k} \rightharpoonup x_0$. Then, as in [19]

$$(Tx_{n_k}+\mu Gx_{n_k}, K(x_{n_k}-x_0)) \to 0 \quad \text{as} \quad k \to \infty.$$

Next, by the boundedness of G, we may assume that $Tx_{n_k} \rightharpoonup y_0$ and $Gx_{n_k} \rightharpoonup G(x_0)$ and either $\limsup(Gx_{n_k}, K(x_{n_k}-x_0)) \leq 0$ or $\limsup(Gx_{n_k}, K(x_{n_k}-x_0))>0$.

Suppose first that $\limsup(Gx_{n_k}, K(x_{n_k}-x_0)) \leq 0$ and passing to a subsequence, we may assume that $a=\lim(Gx_{n_k}, K(x_{n_k}-x_0)) \leq 0$ with $a \neq -\infty$ by the boundedness of G and K. Hence, by property (KS) of G, $x_{n_k} \to x_0$ and

$$\limsup(Tx_{n_k}, K(x_{n_k}-x_0))=\lim(Tx_{n_k}+\mu Gx_{n_k}, K(x_{n_k}-x_0)) - \mu \lim(Gx_{n_k}, K(x_{n_k}-x_0))=0$$

by the boundedness of $\{Gx_{n_k}\}$ and continuity of K. By property (KM) of T, $Tx_0=y_0$. To show that $Tx_0+\mu Gx_0=g$, let $v \in X$ be arbitrary and $v_n \in X_n$ such that $v_n \to v$. Then

$$\begin{aligned}(Tx_0+\mu Gx_0-g, Kv)&=\lim(Tx_{n_k}+\mu Gx_{n_k}-g, K(v_{n_k}))\\ &=\lim(W_{n_k}(Tx_{n_k}+\mu Gx_{n_k})-W_{n_k}(g), K_{n_k}(v_{n_k}))=0.\end{aligned}$$

Thus, since $R(K)$ is dense in Y^* and $(Tx_0+\mu Gx_0-g, \omega)=0$ for each $\omega \in R(K)$, it follows that $Tx_0+\mu Gx_0=g$.

Next, suppose that $\limsup(Gx_{n_k}, K(x_{n_k}-x_0))>0$. Passing to a subsequence, we may assume that $\lim(Gx_{n_k}, K(x_{n_k}-x_0))>0$ and consequently,

$$\limsup(Tx_{n_k}, K(x_{n_k}-x_0))=\limsup(Tx_{n_k}+\mu Gx_{n_k}, K(x_{n_k}-x_0)) - \mu \lim(Gx_{n_k}, K(x_{n_k}-x_0))<0.$$

Again, by property (KM), $Tx_0 = y_0$ and $Tx_0 + \mu Gx_0 = g$ as before. Hence, $T+\mu G$ is pseudo A-proper. $\square$

In view of the results of Kadec and Asplund, in the rest of the paper (except Theorem 4) we may assume without loss of generality that X and X^* are locally uniformly convex. Let $T: X \to X^*$ be such that for some $R>0$,

$$\|Tx\| + \frac{(Tx, x)}{\|x\|} > 0 \quad \text{for all} \quad \|x\| \geq R. \tag{10}$$

Let $\psi: R^+ \to R^+$ be a continuous increasing function such that $\psi(t)=0$ for $t \leq R$ and

$$\psi(\|x\|) + \frac{(Tx, x)}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty. \tag{11}$$

Let $J_\psi: X \to 2^{X^*}$ be the duality mapping corresponding to this ψ, i.e., $J_\psi(0)=0$ for $t \leq R$ and

$$J_\psi(x) = \{w \in X^* \mid (w, x) = \|w\| \cdot \|x\|, \|w\| = \psi(\|x\|)\}.$$

PROPOSITION 4. *Let X be reflexive and separable and $T: X \to X^*$ quasibounded, demicontinuous, of type (M) and satisfy condition* (10). *Then, if J_ψ is weakly continuous, $T+J_\psi$ is peudo A-proper w.r.t.* $\Gamma_0 = \{X_n, V_n; X_n^*, V_n^*\}$.

Proof. Let $\{x_{n_k} \in X_{n_k}\}$ be bounded and such that for some $g \in X^*$

$$\|V_{n_k}^*(T(x_{n_k}) + J_\psi(x_{n_k})) - V_{n_k}^*(g)\| \to 0 \quad \text{as} \quad k \to \infty.$$

Going to a subsequence if necessary, we may assume that either $\|x_{n_k}\| \leq R$ for all k or $\|x_{n_k}\| > R$ for all k. If the first case happens, then $T(x_{n_k}) + J_\psi(x_{n_k}) = T(x_{n_k})$ and by Proposition 3 ($\mu=0$), we get $x_{n_k} \rightharpoonup x_0 \in \bar{B}(0, R)$ with $Tx_0 = g$, i.e., $Tx_0 + J_\psi x_0 = g$.

Next, suppose that $\|x_{n_k}\| > R$ for all k. Then, since J_ψ restricted to $X \setminus \bar{B}(0, R)$ is weakly continuous and of type (S), we have $x_{n_k} \rightharpoonup x_0 \in X$ and $T_{x_0} + J_\psi x_0 = g$ as in Proposition 3 ($\mu = 1$). Hence, $T + J_\psi$ is pseudo A-proper on X. $\square$

From our abstract results we can obtain surjectivity results for various special classes of mappings. We illustrate this by the following few new surjectivity results.

COROLLARY 1. *Let X be separable and reflexive and let $T: X \to X^*$ be quasibounded, demicontinuous and quasimonotone. Suppose that T satisfies condition* $(**)$ *and either condition* (1) *or conditions* $(*)$ *and* (7) *for some $r>0$. Then, if $H_\mu(t, x) = tT(x) + \mu J(x)$ is an A-proper homotopy at $0 \in Y$ on $[0,1] \times (X \setminus B(0, r))$ for some large $r>0$ (which is so if, e.g. T is bounded or $(Tx, x) \geq -c\|x\|$ for all $\|x\| \geq r$), T is surjective.*

COROLLARY 2. *Let X be separable and reflexive and $T: X \to X^*$*

quasibounded, demicontinuous and quasimonotone and satisfy conditions (1), (11) *and* $(**)$. *Then T is surjective.*

Proof. Since J_Ψ is maximal montone, $T+J_\Psi$ is demiclosed quasibounded, quasimonotone and satisfies condition $(**)$. Moreover, if J is the normalized duality mapping, then $T+J_\Psi+\mu J$ is A-proper w.r.t. Γ_0 (cf. [20, 16]). The conclusion now follows from Theorem 3. $\square$

For our next corollary we need

DEFINITION 4. Let $\|\cdot\|_1$ be a norm on X compact relative to the norm $\|\cdot\|$ on X and $K:X\to Y^*$. Then $T:X\to Y$ is said to be of *semi-bounded variation*, if for each $R>0$ and $\|x\|\le R$, $\|y\|\le R$

$$(Tx-Ty, K(x-y))\ge -c(R, \|x-y\|_1),$$

where $c(R,\phi)\ge 0$ is a continuous function in R and ϕ such that $c(R, t\phi)/t\to 0$ as $t\to 0$ for fixed R and ϕ.

Such maps have been studied by Browder [2], Dubinsky [8], Milojević-Petryshyn [20].

Assume that X and Y are separable Hilbert spaces and $K:X\to Y$ a linear bijection. Let $\{X_n\}\subset X$ be a sequence of finite dimensional subspaces such that $\operatorname{dist}(x, X_n)\to 0$ for each x in X. Set $Y_n=K(X_n)$. Then $\operatorname{dist}(y, Y_n)\to 0$ for each y in Y and, if $P_n:X\to X_n$ and $Q_n:Y\to Y_n$ are the orthogonal projections, the scheme $\Gamma_0=\{X_n, P_n; Y_n, Q_n\}$ is projectionally complete. We have

COROLLARY 3. *If $A:X\to Y$ is of semibounded variation, $T:X\to Y$ continuous and compact and $A+T$ satisfies condition* (1) *and either condition* (8) *or A is bounded, then $(A+T)(X)=Y$.*

Proof. For each $R>0$ and $\mu>0$, the map $A+\mu K$ is A-proper w.r.t. Γ_0 on $\bar{B}(0,R)$ by Example 1 and clearly so is K. Since T is compact, $A+T+\mu K$ is also A-proper. Since T is completely continuous and A is strongly demiclosed ([20]), $A+T$ satisfies condition $(**)$. Taking $G=K$ and $K_n=M_n=K\mid X:X_n\to Y_n$, we see that the conclusion follows from Theorem 1. $\square$

Using similar arguments one can establish the following continuation theorem, valid also in the multivalued case, whose detailed discussion will be given in [18] (compare also with Theorem 2.1 in [20]).

THEOREM 3. *Let $H:[0,1]\times X\to Y$ and $G:X\to Y$ be bounded and such that for each $\mu>0$ the homotopy $H_\mu(t,x)=H(t,x)+\mu G(x)$ is A-proper at $0\in Y$ when restricted to $[0,1]\times(X\setminus B(0,R_0))$ for some large $R_0>0$. Suppose that $H_\mu(1,\cdot)$ is A-proper w.r.t. Γ for each $\mu>0$, $H(1,\cdot)$ satisfies condition $(*)$ and that $K:X\to Y^*$ is such that $Kx\ne 0$ for $\|x\|\ge R$ with $H_\mu(t,x)\ne 0$ for all $\|x\|\ge R$, $t\in[0,1]$ and small $\mu>0$. Then, if either $(H(0,x), Kx)\ge 0$ and $(Gx, Kx)\ge 0$ for $\|x\|\ge R$ and there are M_n and K_n satisfying conditions* (4) *and* (5) *of Theorem* 1,

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or there exists an $n_0 \geq 1$ *such that for each* $n \geq n_0$, $\mu > 0$, $\deg(W_n H(0, \cdot) V_n + \mu W_n G V_n, V_n^{-1}(B(0, r)), 0) \neq 0$ *for all large* $r > 0$, *the equation* $H(1, x) = f$ *is solvable for each* f *in* Y *provided also* $H(1, \cdot)$ *satisfies condition* $(**)$.

Note added in proof. The boundedness condition in Proposition 1 and the corresponding result in [15] can be weakened as shown in the author's *Theory of A-proper and pseudo A-closed mappings*, Habilitation Memoir, UFMG, Belo Horizonte, Brazil, 1980, 1–195.

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