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ON SOME PEXIDER-TYPE FUNCTIONAL EQUATIONS CONNECTED WITH THE ABSOLUTE VALUE OF ADDITIVE FUNCTIONS. PART I

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Abstract

We investigate the Pexider-type functional equation

$$\max\{f(x+y), f(x-y)\} = f(y)g(x) + h(x), \quad x, y \in G,$$

where f, g, h are real functions defined on an abelian group G.

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1. Introduction

Our research is inspired by the result obtained by Aczél and presented in [1, Ch. 15]. It concerns the common Pexiderization of two classical Cauchy's functional equations, that is the additive Cauchy's equation,

$$\alpha(x+y) = \alpha(x) + \alpha(y), \tag{1.1}$$

and the multiplicative Cauchy's equation,

$$e(x+y) = e(x)e(y). \tag{1.2}$$

Both equations can be generalized by one equation as follows:

$$f(x+y) = f(x)g(y) + h(y). \tag{1.3}$$

This is just the equation which was solved in [1, Ch. 15]. It turns out that the solutions of (1.3) are one of three types: the trivial solutions with constant function f, solutions connected with either (1.1) or with (1.2). Namely, if $f, g, h : N \to F$, where N is a

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groupoid with a neutral element and F is a field, satisfy (1.3) then either

$$\begin{cases} f(x) = b, \\ g \text{ is an arbitrary function}, \\ h(x) = b(1 - g(x)), \end{cases}$$

or

$$\begin{cases} f(x) = \mathfrak{a}(x) + b, \\ g(x) = 1, \\ h(x) = \mathfrak{a}(x), \end{cases}$$

or

$$\begin{cases} f(x) = c\mathfrak{e}(x) + b, \\ g(x) = \mathfrak{e}(x), \\ h(x) = b(1 - \mathfrak{e}(x)), \end{cases}$$

where b, c are constant, and $\mathfrak{a}, \mathfrak{e} : N \to F$ are solutions of, respectively, (1.1) and (1.2).

This paper is devoted to Pexider generalizations, obtained in an analogous way, from the functional equations

$$\max\{f(x + y), f(x - y)\} = f(x) + f(y) \tag{1.4}$$

and

$$\max\{f(x + y), f(x - y)\} = f(x)f(y). \tag{1.5}$$

Here $f : G \to \mathbb{R}$ is a real function defined on an abelian group $(G, +, 0)$ with neutral element 0. Since the roles of x and y are different in both equations we obtain two Pexider generalizations, that is

$$\max\{f(x + y), f(x - y)\} = f(x)g(y) + h(y) \tag{1.6}$$

and

$$\max\{f(x + y), f(x - y)\} = f(y)g(x) + h(x), \tag{1.7}$$

with $f, g, h : G \to \mathbb{R}$. One can ask, whether here, analogously to [1], we obtain three types of solutions: the trivial one, that connected with (1.4) and that connected with (1.5). As we will see, it turns out that the answer is negative. In this paper we are going to prove the following.

Theorem 1.1. *Let $f, g, h : G \to \mathbb{R}$, where G is an abelian group. Then f, g, h are solutions of (1.7) if and only if they are one of the following forms:*

(1)
$$\begin{cases} f(x) = b, \\ g \textit{ is an arbitrary function}, \\ h(x) = b(1 - g(x)), \end{cases}$$

where $b \in \mathbb{R}$;

(2)
$$\begin{cases} f(x) = c\phi(x) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$$

where $c, b \in \mathbb{R}$, $c > 0$, and $\phi : G \to \mathbb{R}$ is a solution of (1.5);

(3)
$$\begin{cases} f(x) = c\phi(x) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$$

where $c, b \in \mathbb{R}$, $c < 0$, and $\phi : G \to \mathbb{R}$ is a solution of

$$\min\{f(x + y), f(x - y)\} = f(x)f(y); \tag{1.8}$$

(4)
$$\begin{cases} f(x) = \phi(x) + b, \\ g(x) = 1, \\ h(x) = \phi(x), \end{cases}$$

where $b \in \mathbb{R}$, and $\phi : G \to \mathbb{R}$ is a solution of (1.4).

In a second paper [7], we solve the functional equation (1.6), but under the assumptions that $G = \mathbb{R}$ and f is continuous.

This first paper is organized as follows. In the second section we recall some information about (1.4), complete the results concerning (1.5), and also solve (1.8), which appears in Theorem 1.1. In the third section we solve (1.7). The last section is devoted to other Pexider equations connected with (1.4) and (1.5), that is

$$\max\{f(x + y), f(x - y)\} = g(x) + h(y), \tag{1.9}$$

and

$$\max\{f(x + y), f(x - y)\} = g(x)h(y). \tag{1.10}$$

At the end of this introduction we would like to mention other papers devoted to equations like (1.3) considered in [1]. These are the papers [3] and [10]. However, in those papers the functions are defined on sets different from those here, or the emphasis is put on the problem of extendability of the solutions.

2. About equations (1.4), (1.5) and (1.8)

Equation (1.4) was considered in a paper [9] and, as the title of this paper indicates, it is a characterization of absolute value of additive functions.

THEOREM 2.1 [9]. *Let G be an abelian group. Function $f : G \to \mathbb{R}$ is a solution of (1.4) if and only if $f(x) = |a(x)|$ for some additive function $a : G \to \mathbb{R}$ (that is, a satisfying (1.1)).*

Later on, rewriting this equation in the form

$$\max\{f(x), f(x + 2y)\} = f(x + y) + f(y),$$

this equation was examined in [4]. This form allowed the authors to avoid subtraction, and, thanks to this, gave them the possibility of defining the function f on a semigroup. See also paper [5].

Under the additional assumption that G is divisible by 6, solutions of (1.5) were determined in [9].

Theorem 2.2 [9]. *Suppose that G is an abelian group divisible by 6. Then $f : G \to \mathbb{R}$ is a solution of (1.5) if and only if $f \equiv 0$ or $f(x) = \exp(|\boldsymbol{a}(x)|)$, with some additive function $\boldsymbol{a} : G \to \mathbb{R}$.*

Remark 2.3. Actually, in the proof of this theorem, it has been shown, without using this additional assumption about divisibility by 6, that the following properties are true.

(1) Either $f(0) = 0$ and then $f \equiv 0$, or $f(0) = 1$.
(2) If $f(0) = 1$ then $|f(x)| \geq 1$ for every $x \in G$.
(3) If $f(x) > 0$ for every $x \in G$ then $f(x) = \exp(|\boldsymbol{a}(x)|)$, $x \in G$, for some additive function $\boldsymbol{a} : G \to \mathbb{R}$. This is a simple corollary from Theorem 2.1.
(4) f is even.

Now we can complete the above theorem in the following way.

Theorem 2.4. *Function $f : G \to \mathbb{R}$ satisfies (1.5) if and only if:*

(1) *$f \equiv 0$; or*
(2) *$f(x) = \exp(|\boldsymbol{a}(x)|)$, $x \in G$, for some additive $\boldsymbol{a} : G \to \mathbb{R}$; or*
(3) *there is a subgroup G_0 of G, with the property*

$$x, y \notin G_0 \Rightarrow (x + y \in G_0 \vee x - y \in G_0), \quad x, y \in G, \tag{2.1}$$

and

$$f(x) = \begin{cases} 1, & x \in G_0, \\ -1, & x \notin G_0. \end{cases}$$

Proof. The 'if' part can easily be checked. For the 'only if' part, we assume that f satisfies (1.5) and will show that it has one of the forms (1)–(3), above. In view of Remark 2.3, we can assume that $f(0) = 1$ and that there exists an $x_0 \in G$ such that $f(x_0) \leq -1$. We will show that, in fact, $f(x_0) = -1$. Assume on the contrary that $f(x_0) < -1$. Since

$$\max\{f(2x_0), f(0)\} = f(x_0)^2 > 1,$$

we have $f(2x_0) = f(x_0)^2$. Moreover,

$$f(x_0) \leq \max\{f(3x_0), f(x_0)\} = f(2x_0)f(x_0) = f(x_0)^3 < f(x_0),$$

which is a contradiction.

Further, if $f(y) > 1$ for some $y \in G$, then

$$\max\{f(x_0 + y), f(x_0 - y)\} = f(x_0)f(y) = -f(y) < -1,$$

which is impossible, as we have already shown. Consequently, $f(G) \subset \{1, -1\}$. Put $G_0 = \{x \in G; f(x) = 1\}$. Now it is easy to check that G_0 is a subgroup of G and has the property (2.1). □

Notice that

(1) If G is a group divisible by 6 then there is no subgroup $G_0 \subsetneq G$ with property (2.1).
(2) $2\mathbb{Z}$ and $3\mathbb{Z}$ are subgroups of $\mathbb{Z}$ with property (2.1).

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(3) $\{0\}$ is a subgroup of the cyclic group $\mathbb{Z}_2$ with property (2.1).
(4) $\{0\}$ is a subgroup of the cyclic group $\mathbb{Z}_3$ with property (2.1).

Now let us pass to (1.8), which is strictly related to (1.5).

Theorem 2.5. *Let G be an abelian group and $f : G \to \mathbb{R}$. Then f is a solution of the functional equation (1.8) if and only if it has one of the following forms:*

(1) $f \equiv 0$; *or*
(2) $f(x) = \exp(-|a(x)|)$, $x \in G$, *for some additive* $a : G \to \mathbb{R}$; *or*
(3) *there is a subgroup G_0 of G with the property*

$$x, y \notin G_0 \Rightarrow (x + y \in G_0 \wedge x - y \in G_0), \quad x, y \in G \tag{2.2}$$

and

$$f(x) = \begin{cases} 1, & x \in G_0, \\ -1, & x \notin G_0; \end{cases}$$

or

(4) *there is a subgroup G_0 of G with the property*

$$x, y \notin G_0 \Rightarrow (x + y \notin G_0 \vee x - y \notin G_0), \quad x, y \in G, \tag{2.3}$$

and

$$f(x) = \begin{cases} 1, & x \in G_0, \\ 0, & x \notin G_0. \end{cases}$$

Proof. Assume that $f : G \to \mathbb{R}$ satisfies (1.8). Putting $x = y = 0$ in (1.8) we obtain

$$\min\{f(0), f(0)\} = f(0)f(0),$$

which gives $f(0) = f(0)^2$. Hence, $f(0) = 0$ or $f(0) = 1$. In the first case, if $f(0) = 0$, putting $y = 0$ in (1.8) we get $f(x) = f(x)f(0)$, $x \in G$, which means that $f \equiv 0$.

Now assume that the second case holds, that is $f(0) = 1$. Let us consider the following cases.

(a) $f(x) > 0$ for every $x \in G$.
It is easy to check that the function $1/f : G \to \mathbb{R}$ satisfies (1.5), which, in view of Theorem 2.4 and $1/f > 0$, gives $f(x) = \exp(-|a(x)|)$, $x \in G$, for some additive $a : G \to \mathbb{R}$.

(b) There is $x_0 \in G$ such that $f(x_0) \le 0$.
Putting $x = y$ in (1.8) we get

$$1 = f(0) \ge \min\{f(2x), f(0)\} = f(x)^2,$$

whence $f(x) \in [-1, 1]$, $x \in G$. In particular, $f(x_0) \in [-1, 0]$. If $f(x_0) \in (-1, 0)$ we would have

$$\min\{f(2x_0), f(0)\} = f(x_0)^2 < 1,$$

and hence $f(2x_0) = f(x_0)^2$. However,

$$f(x_0) \ge \min\{f(3x_0), f(x_0)\} = f(2x_0)f(x_0) = f(x_0)^3 > f(x_0),$$

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which is a contradiction. Therefore, $f(x_0) = 0$ or $f(x_0) = -1$. Notice also that the conjunction, $f(x) = 0$ and $f(y) = -1$ for some $x, y \in G$, is impossible. Indeed, in such a case we would obtain

$$-1 \geq \min\{f(y), f(y - 2x)\} = f(y - x)f(x) = 0.$$

Therefore, we have proved that $f(G) \subset \{1, -1\}$ or $f(G) \subset \{1, 0\}$. Notice also that f is even, since

$$\min\{f(y), f(-y)\} = f(0)f(y) = f(y), \qquad y \in G.$$

It is easy to check that if $f(G) \subset \{1, -1\}$, respectively $f(G) \subset \{1, 0\}$, then

$$G_0 := \{x \in G;\ f(x) = 1\}$$

is a subgroup of G with the property (2.2), or respectively (2.3). To finish the proof it is enough to see that every function f described by (1)–(4) is a solution of (1.8). □

Notice that

(1) $\mathbb{Q}$ is the subgroup of $\mathbb{R}$ with property (2.3).
(2) $2\mathbb{Z}$ is the subgroup of $\mathbb{Z}$ with property (2.2).

3. Proof of Theorem 1.1

As part ‘if’ of Theorem 1.1 can be easily checked, let us assume that G is an abelian group and $f, g, h : G \to \mathbb{R}$ are solutions of (1.7).

(I) f is constant.

Let $f(x) = b$, $x \in G$. Then (1.7) is of the form $b = bg(x) + h(x)$, $x \in G$, and we see that $h(x) = b(1 - g(x))$, $x \in G$. Hence we get (1) from Theorem 1.1.

(II) f is not constant.

Putting $y = 0$ in (1.7) we get $f(x) = f(0)g(x) + h(x)$, $x \in G$, and hence

$$h(x) = f(x) - f(0)g(x), \qquad x \in G. \tag{3.1}$$

Therefore,

$$\max\{f(x + y), f(x - y)\} = f(y)g(x) + f(x) - f(0)g(x), \qquad x, y \in G. \tag{3.2}$$

Putting $x = 0$ in (3.2) we get

$$\max\{f(y), f(-y)\} = f(y)g(0) + f(0) - f(0)g(0), \qquad y \in G. \tag{3.3}$$

We will show that

$$g(0) = 1 \tag{3.4}$$

Suppose, on the contrary, that $g(0) \neq 1$. Put

$$\mathcal{A} := \{y \in G : f(y) \geq f(-y)\}, \qquad \mathcal{A}' := G \setminus \mathcal{A}.$$

Obviously, $y \in \mathcal{A}' \Rightarrow -y \in \mathcal{A}$. Moreover, if $y \in \mathcal{A}$ then, by (3.3),

$$f(y) = f(y)g(0) + f(0) - f(0)g(0),$$

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which gives

$$(f(y) - f(0))(1 - g(0)) = 0.$$

Thereby, $f(y) = f(0)$ for every $y \in \mathcal{A}$. The set $\mathcal{A}'$ is not empty, since f is not constant. Let $y_0 \in \mathcal{A}'$. We have $f(y_0) < f(-y_0) = f(0)$. From (3.3) with y_0 we get

$$f(0) = f(y_0)g(0) + f(0) - f(0)g(0),$$

whence

$$(f(y_0) - f(0))g(0) = 0.$$

Therefore, $g(0) = 0$. Now, with $x = y_0$ and $y = -y_0$ in (3.2) we obtain

$$\begin{aligned} f(0) \le \max\{f(0), f(2y_0)\} &= f(-y_0)g(y_0) + f(y_0) - f(0)g(y_0) \\ &= f(0)g(y_0) + f(y_0) - f(0)g(y_0) = f(y_0) < f(-y_0) = f(0), \end{aligned}$$

a contradiction. Hence, we proved (3.4).

Notice also that (3.3) implies

$$\max\{f(y), f(-y)\} = f(y) + f(0) - f(0) = f(y), \quad y \in G,$$

whence

$$f(y) = f(-y), \quad y \in G. \tag{3.5}$$

We have, by (3.2) and (3.5),

$$\begin{aligned} f(y)g(x) + f(x) - f(0)g(x) &= \max\{f(x + y), f(x - y)\} \\ &= \max\{f(-x - y), f(y - x)\} \\ &= f(y)g(-x) + f(-x) - f(0)g(-x) \\ &= f(y)g(-x) + f(x) - f(0)g(-x), \end{aligned}$$

whence

$$(f(y) - f(0))(g(x) - g(-x)) = 0, \quad x, y \in G.$$

This, together with the assumption that f is not constant, shows that g is even. Further, by (3.1), also h is even. Moreover, by (3.2) and (3.5),

$$\begin{aligned} f(x)g(y) + f(y) - f(0)g(y) &= \max\{f(y + x), f(y - x)\} \\ &= \max\{f(x + y), f(x - y)\} = f(y)g(x) + f(x) - f(0)g(x), \end{aligned}$$

thereby,

$$(g(x) - 1)(f(y) - f(0)) = (g(y) - 1)(f(x) - f(0)), \quad x, y \in G.$$

Let $y_0 \in G$ be such that $f(y_0) \neq f(0)$. Put

$$\alpha := \frac{g(y_0) - 1}{f(y_0) - f(0)}.$$

Hence

$$g(x) - 1 = \alpha(f(x) - f(0)).$$

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Let us define

$$\tilde{g}(x) := g(x) - 1, \quad x \in G,$$
$$\tilde{f}(x) := f(x) - f(0), \quad x \in G.$$

We have $\tilde{g} = \alpha \tilde{f}$ and $\tilde{f}(0) = 0$. Moreover, by (3.2),

$$\begin{aligned} \max\{\tilde{f}(x+y), \tilde{f}(x-y)\} &= \max\{f(x+y), f(x-y)\} - f(0) \\ &= f(y)g(x) + f(x) - f(0)g(x) - f(0) \\ &= \tilde{f}(y)g(x) + \tilde{f}(x) = \tilde{g}(x)\tilde{f}(y) + \tilde{f}(x) + \tilde{f}(y) \\ &= \alpha\tilde{f}(x)\tilde{f}(y) + \tilde{f}(x) + \tilde{f}(y), \quad x, y \in G. \end{aligned} \tag{3.6}$$

Let us consider three cases.

• $\alpha = 0$.

Then $\tilde{g}(x) = 0$, for $x \in G$, which means that $g(x) = 1$, $x \in G$. By (3.6) we get

$$\max\{\tilde{f}(x+y), \tilde{f}(x-y)\} = \tilde{f}(x) + \tilde{f}(y), \quad x, y \in G,$$

which means that $\tilde{f}$ satisfies (1.4). Hence, $\tilde{f}(x) = |\boldsymbol{a}(x)|$, $x \in G$, for some additive $\boldsymbol{a} : G \to \mathbb{R}$. Finally, using (3.1), we see that f, g, h are as in point (4) from Theorem 1.1:

$$\begin{cases} f(x) = |\boldsymbol{a}(x)| + f(0), \\ g(x) = 1, \\ h(x) = |\boldsymbol{a}(x)|. \end{cases}$$

• $\alpha > 0$.

Put $\bar{f}(x) := \alpha\tilde{f}(x)$, $x \in G$. Multiplying (3.6) by α we get

$$\begin{aligned} \max\{\bar{f}(x+y), \bar{f}(x-y)\} &= \alpha\bar{f}(x)\bar{f}(y) + \bar{f}(x) + \bar{f}(y) \\ &= (\bar{f}(x) + 1)(\bar{f}(y) + 1) - 1, \quad x, y \in G. \end{aligned}$$

Therefore, with $\phi(x) := \bar{f}(x) + 1$, $x \in G$, we get

$$\max\{\phi(x+y), \phi(x-y)\} = \phi(x)\phi(y), \quad x, y \in G,$$

which means that ϕ satisfies (1.5). We have that f, g, h are as in point (2) from Theorem 1.1:

$$\begin{cases} f(x) = \dfrac{1}{\alpha}\phi(x) + f(0) - \dfrac{1}{\alpha}, \\ g(x) = \phi(x), \\ h(x) = \Big(f(0) - \dfrac{1}{\alpha}\Big)(1 - \phi(x)). \end{cases}$$

• $\alpha < 0$.

Put $\bar{f}(x) := -\alpha\tilde{f}(x)$, $x \in G$. Multiplying (3.6) by $-\alpha$ we get

$$\begin{aligned} \max\{\bar{f}(x+y), \bar{f}(x-y)\} &= -\alpha\bar{f}(x)\bar{f}(y) + \bar{f}(x) + \bar{f}(y) \\ &= -(\bar{f}(x) - 1)(\bar{f}(y) - 1) + 1, \quad x, y \in G. \end{aligned}$$

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Therefore, with $\bar{\phi}(x) := \bar{f}(x) - 1$, $x \in G$, we get

$$\max\{\bar{\phi}(x+y), \bar{\phi}(x-y)\} = -\bar{\phi}(x)\bar{\phi}(y), \qquad x, y \in G,$$

which means that $\phi := -\bar{\phi}$ satisfies (1.8). We have that f, g, h are as in point (3) from Theorem 1.1:

$$\begin{cases} f(x) = \dfrac{1}{\alpha}\phi(x) + f(0) - \dfrac{1}{\alpha}, \\ g(x) = \phi(x), \\ h(x) = \Big(f(0) - \dfrac{1}{\alpha}\Big)(1 - \phi(x)). \end{cases}$$

The proof is finished. $\square$

Let us finish this section with the remark from the above proof, which we will need in [7].

REMARK 3.1. If $f, g, h : G \to \mathbb{R}$ satisfy the functional equation (1.7), then f is even, moreover if f is not constant then also g and h are even and $g(0) = 1$.

4. Other Pexider equations

We recall a result about the functional equation (1.9) from [8].

THEOREM 4.1 [8, Satz 2]. *Let G be an abelian group and $f, g, h : G \to \mathbb{R}$. Then f, g and h are solutions of the equation*

$$\max\{f(x+y), f(x-y)\} = g(x) + h(y)$$

if and only if

$$\begin{cases} f(x) = \boldsymbol{a}(x) + c_1 + c_2 \\ g(x) = \boldsymbol{a}(x) + c_1 \\ h(x) = |\boldsymbol{a}(x)| + c_2, \end{cases}$$

or

$$\begin{cases} f(x) = |\boldsymbol{a}(x) + c_0| + c_1 + c_2 \\ g(x) = |\boldsymbol{a}(x) + c_0| + c_1 \\ h(x) = |\boldsymbol{a}(x)| + c_2, \end{cases}$$

with some $\boldsymbol{a} : G \to \mathbb{R}$ satisfying (1.1) and c_0, c_1, c_2 constants.

As a corollary from [7] we get the result concerning the multiplicative version of the above equation.

THEOREM 4.2. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$, f continuous. Then*

$$\max\{f(x+y), f(x-y)\} = g(x)h(y) \tag{1.10}$$

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if and only if one of the following holds:

$$\begin{cases} f \equiv 0 \\ g \equiv 0 \\ h \textit{ is arbitrary}; \end{cases}$$

or

$$\begin{cases} f \equiv 0 \\ h \equiv 0 \\ g \textit{ is arbitrary}; \end{cases}$$

or

$$\begin{cases} f(x) = bce^{a|x-x_0|} \\ g(x) = be^{a|x-x_0|} \\ h(x) = ce^{a|x|}, \end{cases}$$

where $a, b, c, x_0 \in \mathbb{R}$, $abc > 0$; *or*

$$\begin{cases} f(x) = bce^{ax} \\ g(x) = be^{ax} \\ h(x) = ce^{\operatorname{sgn}(bc)|ax|}, \end{cases}$$

where $a, b, c \in \mathbb{R}$.

Proof. One can check that functions f, g, h, described by the formulas above, satisfy (1.10).

Now assume that f, g, h are solutions of (1.10). First notice that if $f \equiv 0$, then either $g \equiv 0$ or $h \equiv 0$, moreover, if f is constant, then g and h are constant and $f = gh$. Now, consider the case, that f is not constant. With $y = 0$ in (1.10), we get $f(x) = g(x)h(0)$, $x \in \mathbb{R}$. Therefore $h(0) \neq 0$. Put $\tilde{h} := h/h(0)$. We get that $f, \tilde{h}, 0$ are solutions of (1.6), and from [7, Theorem 3.1] we obtain the formulas for f, g and h. This ends the proof. □

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