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The p-adic monodromy-weight conjecture for p-adically uniformized varieties

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Abstract

A p-adically uniformized variety is a smooth projective variety whose associated rigid analytic space admits a uniformization by Drinfeld's p-adic symmetric domain. For such a variety we prove the monodromy-weight conjecture, which asserts that two independently defined filtrations on the log-crystalline cohomology of the special fiber in fact coincide. The proof proceeds by reducing the conjecture to a combinatorial statement about harmonic cochains on the Bruhat–Tits building, which was verified in our previous work.

Introduction

Let K be a local field of characteristic 0 and residual characteristic p. Denote by $\mathcal{O}_K$ its ring of integers, and by K_0 the maximal unramified subfield of K. Let $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$ be the Frobenius automorphism. Fix a uniformizer π of K.

Let X be a proper scheme over $\mathcal{O}_K$, with semistable reduction. Let $0 \leqslant m \leqslant 2\dim X_K$, and let D be the mth Hyodo–Kato (log-crystalline) cohomology of the special fiber of X (see [Hyo91] and [HK94]). Then D is a finite dimensional vector space over K_0, which comes equipped with a σ-semilinear bijective map Φ, the *Frobenius endomorphism*, and a nilpotent endomorphism N, the *monodromy operator*. The triple (D, Φ, N) depends on the first infinitesimal neighborhood of the special fiber, $X \otimes (\mathcal{O}_K/\pi^2\mathcal{O}_K)$, only. It carries two increasing filtrations. The weight filtration $P.D$ is related to Φ, and the monodromy filtration $M.D$ is related to N. The two endomorphisms satisfy the following commutation relation:

$$N\Phi = p\Phi N. \tag{0.1}$$

Hyodo and Kato constructed an isomorphism (depending on π) between the log-crystalline cohomology of X, and the de Rham cohomology of its generic fiber:

$$\rho_\pi : D \otimes_{K_0} K \simeq H^m_{\mathrm{dR}}(X_K) := \mathbb{H}^m(X_{K,\mathrm{Zar}}, \Omega^\bullet). \tag{0.2}$$

Via this isomorphism, $H^m_{\mathrm{dR}}(X_K)$, with its Hodge filtration, becomes a filtered (Φ, N)-module [Fon94]. (The Hodge filtration will play no role in the present work, but see [IS00] and [AdS02] for its relation to the weight filtration, in our class of p-adically uniformized varieties.)

If X is smooth, D is the usual crystalline cohomology of its special fiber, $N = 0$, and both the monodromy and the weight filtrations are trivial. In general, the *monodromy-weight conjecture* predicts, if X is projective, that the two filtrations coincide, up to a shift in the indices:

$$\text{(MWC)} \quad P_{.+m}D = M.D\,. \tag{0.3}$$

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It is due to Mokrane [Mok93, Conjecture 3.27] who modeled it on a similar conjecture of Deligne in l-adic cohomology [Del71], and on a corresponding theorem of J. Steenbrink and M. Saito in the category of analytic spaces over $\mathbb{C}$. The purpose of this work is to prove the conjecture for a rather special class of varieties, whose special fiber is 'as far as possible' from being smooth.

Theorem 0.1. *The monodromy-weight conjecture holds for p-adically uniformized varieties.*

We call X 'p-adically uniformized', if the associated rigid analytic space X^{an} admits a uniformization as the quotient of Drinfel'd's p-adic symmetric domain of dimension d, denoted $\mathfrak{X}$, by a discrete, cocompact and torsion-free subgroup Γ of $PGL_{d+1}(K)$. By work of Rapoport-Zink and Varshavsky, among these varieties lie some infinite families of unitary Shimura varieties with bad reduction at p.

Although we do not claim to prove it here, Deligne's l-adic monodromy-weight conjecture should follow, for a p-adically uniformized X, along the same lines. One would first have to repeat the developments of [deS00], [AdS02] and [AdS03] in the framework of l-adic cohomology. As pointed out to us by U. Jannsen, the l-adic conjecture also seems to follow (for a p-adically uniformized X) from recent work of M. Saito [Sai00], but our approach is very different and avoids the standard conjectures.[1]

The main tools used in the proof are harmonic analysis and combinatorics on $\mathcal{T}$, the Bruhat–Tits building of $G = PGL_{d+1}(K)$. In our previous work [deS00] we established an isomorphism, based on a rigid-analytic residue map, between the mth de Rham cohomology of $\mathfrak{X}$, and a certain space C^m_{har} of harmonic K-valued m-cochains on $\mathcal{T}$. In [AdS03] we constructed another space $\tilde{C}^m_{\mathrm{har}}$ of m-cochains on $\mathcal{T}$, and a canonical extension of G-modules

$$0 \to C^{m-1}_{\mathrm{har}} \to \tilde{C}^{m-1}_{\mathrm{har}} \xrightarrow{d} C^m_{\mathrm{har}} \to 0, \tag{0.4}$$

where the first map is the inclusion, and the second the coboundary map. The space C^m_{har} is characterized by simple 'harmonicity conditions'. The space $\tilde{C}^m_{\mathrm{har}}$ is characterized by the same conditions, with the exception of '$d = 0$'. For example, if $\mathcal{T}$ is the tree and $m = 1$, $C^0_{\mathrm{har}} = K$, $\tilde{C}^0_{\mathrm{har}}$ are the functions on the vertices annihilated by the combinatorial Laplacian (i.e. having the mean-value property), and C^1_{har} are the alternating functions on the oriented edges, whose values on any $q+1$ edges entering a given vertex sum up to 0.

Let Γ be a discrete, cocompact, torsion-free subgroup of G, and X the scheme over $\mathcal{O}_K$ whose completion, along the special fiber, is the quotient $\Gamma\backslash\widehat{\mathfrak{X}}$, where $\widehat{\mathfrak{X}}$ is the formal scheme underlying $\mathfrak{X}$. This quotient is algebraizable, and even projective, by a theorem of Mumford, Kurihara and Mustafin [Mus78, Theorem 4.1]. Its generic fiber X_K is smooth and projective. Its special fiber X_κ is semistable [Mus78, Theorem 3.1]. It is thus a variety of the type considered above.

The only interesting cohomology of X is in degree $m = d$. In degree $m \neq d$, the cohomology is one-dimensional if m is even, and vanishes if m is odd [SS91, Theorem 5]. The covering spectral sequence [SS91, § 5, Proposition 2] is the spectral sequence

$$E_2^{r,s} = H^r(\Gamma, H^s_{\mathrm{dR}}(\mathfrak{X})) = H^r(\Gamma, C^s_{\mathrm{har}}) \Longrightarrow H^{r+s}_{\mathrm{dR}}(X_K), \tag{0.5}$$

abutting to the cohomology of X_K. The resulting (decreasing) filtration $F^\bullet_\Gamma$ on $H^d_{\mathrm{dR}}(X_K)$, the covering filtration, coincides, up to a change in the indexing, with the weight filtration (inherited from D via the Hyodo–Kato isomorphism). The covering spectral sequence degenerates at E_2 [SS91, § 5], [AdS02], hence the graded pieces $\mathrm{gr}^r_\Gamma H^d_{\mathrm{dR}}(X_K)$ are just the cohomology $H^r(\Gamma, C^{d-r}_{\mathrm{har}})$. The commutation relation between N and Φ implies that N maps $F^r_\Gamma H^d_{\mathrm{dR}}(X_K)$ to $F^{r+1}_\Gamma H^d_{\mathrm{dR}}(X_K)$,

[1] After the completion of this work we learned about the paper by T. Ito [Ito03], in which he proves both the l-adic and p-adic monodromy-weight conjectures for p-adically uniformized varieties, by verifying special cases of the standard conjectures of Hodge type.

and induces therefore a map $\mathrm{gr}_\Gamma N : H^r(\Gamma, C_{\mathrm{har}}^{d-r}) \to H^{r+1}(\Gamma, C_{\mathrm{har}}^{d-r-1})$. To prove the monodromy-weight conjecture for X_K one only has to check that for any $r \leqslant [d/2]$,

$$(\mathrm{gr}_\Gamma N)^{d-2r} : H^r(\Gamma, C_{\mathrm{har}}^{d-r}) \simeq H^{d-r}(\Gamma, C_{\mathrm{har}}^{r}) \tag{0.6}$$

is an isomorphism.

Let ν be the connecting homomorphism in the long exact sequence of Γ-cohomology attached to (0.4):

$$\nu : H^r(\Gamma, C_{\mathrm{har}}^{s}) \to H^{r+1}(\Gamma, C_{\mathrm{har}}^{s-1}) \quad (r+s=d). \tag{0.7}$$

In [AdS03] we showed that for any $r \leqslant [d/2]$, iteration of ν induces an isomorphism

$$\nu^{d-2r} : H^r(\Gamma, C_{\mathrm{har}}^{d-r}) \simeq H^{d-r}(\Gamma, C_{\mathrm{har}}^{r}). \tag{0.8}$$

It remains to prove that the geometrically defined $\mathrm{gr}_\Gamma N$ equals the combinatorially defined ν. This is what we do below.

The following is a brief outline of the paper. In § 1 we recall the logarithmic de Rham–Witt complex $W\omega^\bullet$ on the étale site of the special fiber of X, and how it is used to compute D. We also review, in § 2, a generalization of the work of Hyodo and Kato to the non-proper case, by E. Grosse-Klönne, which allows us to compute, in a similar way, the Hyodo–Kato (= log-rigid) cohomology of the special fiber of $\mathfrak{X}$. In § 3 we introduce the p-adically uniformized varieties, and the calculus of logarithmic forms in de Rham (respectively de Rham–Witt) cohomology of $\mathfrak{X}$ (respectively its special fiber). We have to deal not only with the de Rham–Witt complex itself, but with a certain extension of it

$$0 \to W\omega^\bullet[-1] \to W\tilde{\omega}^\bullet \to W\omega^\bullet \to 0, \tag{0.9}$$

used to define N. The calculus of logarithmic forms is reflected 'fully faithfully' on the level of harmonic cochains on $\mathcal{T}$ via the residue homomorphism. Theorem 0.1 eventually follows, in § 4, from isomorphism (0.8), once we relate the extension (0.4) to (0.9).

Combining Theorem 0.1 with Tsuji's proof [Tsu99] of Fontaine's conjecture C_{st}, we arrive at the following.

Corollary 0.2. *The Galois representation $H^d_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is not crystalline, unless d is odd and the cohomology vanishes, or d is even and the cohomology is one-dimensional.*

In fact, under the assumptions of the corollary, the weight filtration is non-trivial, hence $N \neq 0$, and this implies that the local Galois representation is not crystalline. Although this is weaker than Theorem 0.1, which may be regarded as a quantitative version of it, we are not aware of an easier proof of the corollary. Compare also with the corresponding theorem by Coleman and Iovita [CI99], asserting that if A_K is an abelian variety and $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Q}_p)$ is crystalline, then A has good reduction. The exceptions in the corollary may occur, at least if Γ has torsion, as was indicated by Mumford.

1. Log-crystalline cohomology and the two filtrations

1.1 Log-crystalline cohomology of proper semistable schemes

In addition to the notation introduced in the Introduction, let $\kappa = \mathcal{O}_K/\pi\mathcal{O}_K = \mathbb{F}_q$ be the residue field, so that K_0 is the field of fractions of $W(\kappa)$, the ring of Witt vectors of κ, and K is a totally ramified extension of K_0.

Let $X \to \operatorname{Spec} \mathcal{O}_K$ be a scheme with semistable reduction. This means that X is regular, flat over $\mathcal{O}_K$, the generic fiber X_K is smooth, and the special fiber $Y = X_\kappa$ is a reduced divisor with normal crossings on X.

Assume that X is proper. Fontaine and Jannsen conjectured that one should attach to it, in a functorial way, and in any degree $0 \leqslant m \leqslant 2\dim X_K$, a filtered (Φ, N)-module $(D, \Phi, N, \mathrm{Fil})$ which will look much simpler than the p-adic étale cohomology $H^m_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$, yet will be sufficiently rich to capture it, as a $\mathrm{Gal}(\bar{K}/K)$ representation, with the help of Fontaine's ring B_{st}. Such a cohomology theory was constructed, in the proper case, by Hyodo and Kato [Hyo91, HK94] (and also by Faltings using a different approach). Kato and Tsuji established the desired relation to p-adic étale cohomology (Fontaine's conjecture C_{st}).

The space D is a finite-dimensional K_0-vector space, endowed with an endomorphism N and a σ-linear bijective map Φ as in the Introduction. Furthermore, there is a comparison isomorphism (the Hyodo–Kato isomorphism) ρ_π between $D \otimes_{K_0} K$ and $H^m_{\mathrm{dR}}(X_K)$, depending on the choice of a uniformizer π. (On the p-adic étale side the choice of π is implicit in the construction of B_{st}.) If π' is another uniformizer, then

$$\rho_{\pi'} = \rho_\pi \circ \exp(\log(\pi'/\pi)N), \tag{1.1}$$

so $N_{\mathrm{dR}} = \rho_\pi \circ N \circ \rho_\pi^{-1}$ is independent of π, and we denote it again by N. In view of the relation

$$N\Phi = p\Phi N, \tag{1.2}$$

we conclude that N is nilpotent.

In the smooth and proper case, $D = H^m_{\mathrm{cris}}(Y/W(\kappa)) \otimes_{W(\kappa)} K_0$, so (D, Φ) depends on the special fiber only, and $N = 0$. In the general case, the triple (D, Φ, N) depends on the first infinitesimal neighborhood of Y.

We shall review, in § 1.4 below, a construction of (D, Φ, N) via the logarithmic de Rham–Witt complex, which is well adapted to the proof of Theorem 0.1. As in crystalline cohomology, Hyodo and Kato defined also an integral version of D. In this paper, we shall ignore questions of torsion (which anyhow should not exist in our examples), and work with K_0-vector spaces.

1.2 The monodromy filtration

The construction of the monodromy filtration uses only the fact that N is nilpotent (see [Del80, Proposition I.6.1]). Define

$$M_r D = \sum_{i-j=r} \ker(N^{i+1}) \cap \mathit{Im}(N^j), \tag{1.3}$$

the 'convolution' of the filtrations by successive kernels and images. This is the unique increasing, exhaustive and separated filtration on D satisfying the following two properties.

i) N maps $M_r D$ to $M_{r-2} D$.

From this we get a map $\mathrm{gr}_M N$ of $\mathrm{gr}^r_M D = M_r D / M_{r-1} D$ to $\mathrm{gr}^{r-2}_M D$.

ii) $(\mathrm{gr}_M N)^r : \mathrm{gr}^r_M D \to \mathrm{gr}^{-r}_M D$ is an isomorphism.

For example, if $N^2 = 0$, then $M_{-2} = 0$, $M_{-1} = \mathit{Im}\, N$, $M_0 = \ker N$, and $M_1 = D$.

1.3 The weight filtration

Recall that a q-Weil number of weight m is an algebraic integer, whose absolute value in any complex embedding is $q^{m/2}$. Let $q = p^f$ be the cardinality of κ, and consider the relative Frobenius automorphism $\phi = \Phi^f$, which now acts linearly on D. If X were smooth, a theorem of Katz and Messing, based on Deligne's proof of the Weil conjectures, says that all the eigenvalues of ϕ on D are q-Weil numbers of weight m. One says that D is pure of weight m. In general, the existence of the weight spectral sequence (for details, see [Mok93], in particular Theorem 3.32) implies that there is a unique increasing filtration $P.D$ on D, preserved by Φ, such that $\mathrm{gr}^r_P D$ is pure of weight r. One expresses this by saying that D is mixed. Once this is proven, one can simply let $P_r D$ be the subspace, which, upon extension of scalars to an algebraic closure of K_0, becomes the sum of the generalized eigenspaces for all the eigenvalues λ satisfying $|\lambda| \leqslant q^{r/2}$. This definition does not say anything about the geometric nature of the filtration, and its relation to vanishing cycles, nor can it be used if we want to treat the integral version of D, but for our purposes it will suffice.

The monodromy-weight conjecture of Mokrane states that, if X is projective,

$$P_{m+r}D = M_r D. \tag{1.4}$$

Since $N\phi = q\phi N$, and ϕ is an isomorphism, N maps $P_r D$ to $P_{r-2}D$. Thus, in order to prove the conjecture, one only has to show that

$$(\mathrm{gr}_P N)^r : \mathrm{gr}_P^{m+r} D \to \mathrm{gr}_P^{m-r} D \tag{1.5}$$

is an isomorphism.

1.4 The logarithmic de Rham–Witt complex

So far we have discussed (D, Φ, N) as a black box. We now extract from [HK94] whatever will be needed about its actual structure. At the heart of it lies an idea of Illusie [Ill79], who constructed, in the smooth and proper case, for each $n \geqslant 0$, a certain complex of sheaves of $W_n(\kappa)$-modules $W_n\Omega^\bullet$ on the étale site of the special fiber, $Y_{\text{ét}}$, and proved that their hypercohomology (in either the étale or the Zariski topology) computes the crystalline cohomology. The complex (more precisely, inverse system of complexes) $W\Omega^\bullet$ employs both the de Rham machinery of differential forms, and the Witt vectors construction, which is used to circumvent the problematics of differential forms in characteristic p. It is therefore called the de Rham–Witt complex. From the Witt construction one gets naturally the Frobenius endomorphism Φ.

Hyodo and Kato proposed to use, in the semistable case, a modified version of this complex, denoted $W\omega^\bullet$. It agrees with $W\Omega^\bullet$ on the smooth locus, but makes room for logarithmic forms at the singularities. They also constructed another complex of sheaves $W\tilde{\omega}^\bullet$ on $Y_{\text{ét}}$, and a short exact sequence of complexes

$$0 \to W\omega^\bullet[-1] \to W\tilde{\omega}^\bullet \to W\omega^\bullet \to 0. \tag{1.6}$$

A fairly quick way to define the Hyodo–Kato cohomology is to put

$$D^m = \mathbb{H}^m(Y_{\text{ét}}, W\omega^\bullet) \otimes_{W(\kappa)} K_0. \tag{1.7}$$

We abuse notation. By $W\omega^\bullet$ we really mean the inverse system $\{W_n\omega^\bullet\}$, and $\mathbb{H}^m(Y_{\text{ét}}, W\omega^\bullet) := \varprojlim \mathbb{H}^m(Y_{\text{ét}}, W_n\omega^\bullet)$, as usual. This definition avoids the log-crystalline site, but see § 2.2 below for the relation with crystalline or rigid cohomologies.

The monodromy operator N is defined to be the connecting homomorphism in the long exact sequence of hypercohomology attached to the short exact sequence (1.6).

To define $W_n\omega^\bullet$, fix any open dense smooth subscheme $u : U \hookrightarrow Y$ in the special fiber of X. The complex $W_n\omega^\bullet$ will be defined as a certain subcomplex of $u_*W_n\Omega^\bullet$. Let

$$Y \xrightarrow{i} X \xleftarrow{j} X_K \tag{1.8}$$

be the inclusion maps. Let $\mathcal{K}_X = j_*(\mathcal{O}_{X_K})$ be the sheaf of functions 'regular in characteristic 0'. The sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ is supported on Y, and its restriction to U is the constant sheaf $\mathbb{Z}$, via the valuation map, because U is smooth.

Illusie [Ill79, § I.3.23] supplies us with a homomorphism of sheaves

$$d\log : \mathcal{O}_U^\times \to W_n\Omega_U^1. \tag{1.9}$$

Combined with the projection $u^{-1}i^{-1}\mathcal{O}_X^\times \to \mathcal{O}_U^\times$, it extends uniquely to a map

$$d\log : u^{-1}i^{-1}\mathcal{K}_X^\times \to W_n\Omega_U^1 \tag{1.10}$$

by specifying $d\log(K^\times) = 0$. This gives a map

$$d\log : i^{-1}\mathcal{K}_X^\times \to u_*W_n\Omega_U^1. \tag{1.11}$$

Define $W_n\omega^\bullet$, to be the subalgebra of $u_*W_n\Omega_U^\bullet$ generated over $W_n(\mathcal{O}_Y)$ by $dW_n(\mathcal{O}_Y)$ and $d\log(i^{-1}\mathcal{K}_X^\times)$ in degree 1.

To define $W_n\tilde{\omega}^\bullet$, let $W_n\tilde{\Omega}_U^\bullet = W_n\Omega_U^\bullet[\theta]$, where θ is an indeterminate in degree 1, satisfying $\theta^2 = 0$, $d\theta = 0$, and $\theta a = (-1)^q a\theta$ for $a \in W_n\Omega_U^q$. The map $d\log$ lifts to a map

$$d\widetilde{\log} : i^{-1}\mathcal{K}_X^\times \to u_*W_n\tilde{\Omega}_U^1, \tag{1.12}$$

which coincides with $d\log$ on $i^{-1}\mathcal{O}_X^\times$, but on $K^\times$ satisfies

$$d\widetilde{\log}(a) = \mathrm{ord}(a)\theta. \tag{1.13}$$

As before, $W_n\tilde{\omega}^\bullet$ is defined to be the subalgebra of $u_*W_n\tilde{\Omega}_U^\bullet$ generated over $W_n(\mathcal{O}_Y)$ by $dW_n(\mathcal{O}_Y)$ and $d\widetilde{\log}(i^{-1}\mathcal{K}_X^\times)$ in degree 1. The map $W_n\tilde{\omega}^\bullet \to W_n\omega^\bullet$ is 'dividing by θ', and the map $W_n\omega^\bullet[-1] \to W_n\tilde{\omega}^\bullet$ is $a \mapsto \theta \wedge a$. All these constructions are compatible with the transition maps from $W_{n+1}(-)$ to $W_n(-)$. Taking inverse limits over n we get the complexes $W\omega^\bullet$ and $W\tilde{\omega}^\bullet$. The sequence (1.6) is exact.

2. De Rham–Witt cohomology and the Hyodo–Kato isomorphism in the locally proper case

2.1 Logarithmic de Rham–Witt complex in the locally proper case

Let $\widehat{\mathfrak{X}}$ be a p-adic formal scheme over $\mathrm{Spf}\mathcal{O}_K$, and let Y be its special fiber. We assume that:

i) $\widehat{\mathfrak{X}}$ has a strictly semistable reduction (see [Gro02, § 2.1]),

ii) Y is locally proper: its irreducible components are proper, and there exist closed (respectively open) subschemes Z_r (respectively U_r) such that
   1) each Z_r is a finite union of irreducible components
   2) $Z_r \subset U_r \subset Z_{r+1}$ and their union is Y.

These assumptions can probably be relaxed, but they are sufficient for our purposes. We denote by $\mathfrak{X}$ the rigid analytic space which is the generic fiber, in the sense of Raynaud, of $\widehat{\mathfrak{X}}$ (see [Ber96, § 0.2]). We denote by i_r (respectively j_r) the closed (respectively open) embedding of Z_r (respectively U_r) in Y. For any open subscheme U of Y, we denote by

$$]U[_{\mathfrak{X}} = \mathrm{sp}^{-1}(U) \tag{2.1}$$

the rigid subdomain of all the points of $\mathfrak{X}$ whose specialization lies in U. It is called the *tube* of U (see [Ber96, ch. 1]). By our assumption on Y, $\mathfrak{X}$ has a canonical underlying structure of a dagger space $\mathfrak{X}^\dagger$, and we denote by $]U[_{\mathfrak{X}}^\dagger$ the dagger-subdomain of all points specializing to U (see [Gro02], where he preserves the notation $]U[_{\mathfrak{X}}$ for the latter). As sets, $]U[_{\mathfrak{X}}$ and $]U[_{\mathfrak{X}}^\dagger$ coincide, but the structure sheaf on the first consists of convergent (rigid analytic) functions, while the structure sheaf on the second consists of overconvergent functions. We similarly define $]Z[_{\mathfrak{X}}$ and $]Z[_{\mathfrak{X}}^\dagger$ for Z a closed subscheme.

The definition of the sheaves $W\omega^\bullet$ and $W\tilde{\omega}^\bullet$ on $Y_{\text{ét}}$ carries over to the locally proper case (or even more generally). The only modification is that we replace the sheaf $i^{-1}\mathcal{K}_X^\times$ by $\mathcal{K}_{\widehat{\mathfrak{X}}}^\times$, which assigns to any étale neighborhood U in Y, the invertible rigid analytic functions on the tube of U,

$$\mathcal{K}_{\widehat{\mathfrak{X}}}^\times(U) = \mathcal{O}(]U[_{\mathfrak{X}})^\times. \tag{2.2}$$

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A priori this is defined for U a Zariski neighborhood, but it makes sense for étale neighborhoods as well, because the formal scheme 'lifts' to étale covers of the special fiber uniquely. If U is affine, $]U[_{\mathfrak{X}}$ is an affinoid, and $\mathcal{K}^{\times}_{\widehat{\mathfrak{X}}}(U)$ are the units of an affinoid K-algebra. Similarly we replace the sheaf $i^{-1}\mathcal{O}^{\times}_X$ by $\mathcal{O}^{\times}_{\widehat{\mathfrak{X}}}$, the invertible elements in the structure sheaf of $\widehat{\mathfrak{X}}$. As before, the restriction of $\mathcal{K}^{\times}_{\widehat{\mathfrak{X}}}/\mathcal{O}^{\times}_{\widehat{\mathfrak{X}}}$ to an open smooth $U \subset Y$ is the constant sheaf $\mathbb{Z}$. This can be deduced, for example, from the fact that the sup-norm on the affinoid algebra $\mathcal{K}_{\widehat{\mathfrak{X}}}(U)$ (if U is affine and smooth) is multiplicative [BGR84, 6.2.3, Proposition 5]. From here on the construction is identical to the one above. Fix a smooth open dense $u : U \hookrightarrow Y$, and define $d\log : u^{-1}\mathcal{O}^{\times}_{\widehat{\mathfrak{X}}} \to \mathcal{O}^{\times}_U \to W_n\Omega^1_U$ by first reducing to the special fiber, then applying Illusie's map (1.9). Extend to $u^{-1}\mathcal{K}^{\times}_{\widehat{\mathfrak{X}}}$ so that $d\log(K^{\times}) = 0$, and define $W_n\omega^{\bullet}$ in the same way as before. Similarly define $d\widetilde{\log}$ and $W_n\tilde{\omega}^{\bullet}$.

We remark that the complexes $W\omega^{\bullet}$ and $W\tilde{\omega}^{\bullet}$ have a crystalline definition as well [HK94, 4.1], and from that perspective there is no difference between the proper and the locally proper case. All that is needed is the special fiber with its induced log-structure. However, for the 'calculus of logarithmic forms', it will be more convenient to work with the elementary description given above.

2.2 The log-rigid cohomology of Grosse-Klönne

As indicated in the Introduction, E. Grosse-Klönne [Gro02] extended the definition of D^m to certain non-proper situations, including the locally proper set-up described above. He also obtained a generalization of the Hyodo–Kato isomorphism between his D^m and $H^m_{\mathrm{dR}}(\mathfrak{X})$. For (the formal completion along the special fiber of) proper schemes with semistable reduction, his definitions coincide with those of [HK94]. However, his approach uses log-rigid overconvergent cohomology, and not the (logarithmic) de Rham–Witt complex. In this section we make precise the relation between the two in the locally proper case of § 2.1, since we shall eventually be computing with logarithmic de Rham–Witt classes.

We assume familiarity with the various p-adic cohomologies which were invented over the last decade, in order to overcome the limitations of crystalline cohomology. Rigid cohomology was created by Berthelot [Ber96, Ber97], as a substitute for crystalline cohomology, and works well for smooth (but not necessarily proper) schemes. Log-structures on schemes were defined by Kato and Illusie, just so that semistable schemes would be (log) smooth (see [Ill94]). Log-crystalline cohomology [HK94] works well for proper semistable (log) schemes. Log-rigid cohomology is defined in [Gro02], in order to take care of semistable (log) schemes which are not necessarily proper. Convergent cohomology was introduced by Ogus [Ogu90] and log-convergent cohomology is discussed in detail in [Shi02].

We keep our assumptions on $\widehat{\mathfrak{X}}$, and let $\mathfrak{S}^0$ be the formal log-scheme

$$(\mathrm{Spf}W(\kappa), \mathbb{N}, 1 \mapsto 0). \tag{2.3}$$

Its special fiber $\underline{S}_0$ is the standard log point $(\mathrm{Spec}(\kappa), \mathbb{N}, 1 \mapsto 0)$, and $\mathfrak{S}^0$ is its canonical lifting to $\mathrm{Spf}W(\kappa)$ in the sense of [HK94, Definition 3.1]. Let $\underline{\widehat{\mathfrak{X}}}$ be the formal log-scheme given by $\widehat{\mathfrak{X}} \to \mathrm{Spf}\mathcal{O}_K$, endowed with the log-structure defined by the special fiber. Let $\underline{Y}$ be its special fiber. Then $\underline{Y}$ is a fine log-scheme, log-smooth over $\underline{S}_0$, and of Cartier type.

We define $D^m(\underline{Y})$ as $H^m_{\mathrm{rig}}(\underline{Y}/\mathfrak{S}^0)$ (see [Gro02, Lemma 1.4]). Since Y is locally proper, this is equal also to $H^m_{\mathrm{conv}}(\underline{Y}/\mathfrak{S}^0)$ (see [Gro02, 1.5 and Theorem 8.3(ii)]). If Y is *proper*, the two coincide with $H^m_{\mathrm{cris}}(\underline{Y}/\mathfrak{S}^0) \otimes_{W(\kappa)} K_0$ (see [Shi02, Theorem 3.1.1]). Now $H^m_{\mathrm{cris}}(\underline{Y}/\mathfrak{S}^0)$ can be computed in the étale topology, as the cohomology of the logarithmic de Rham–Witt complex (1.7). Hence the definition agrees with that of [HK94], and with (1.7).

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Please note that the essential ingredient in Shiho's comparison theorem [Shi02, 3.1.1] is for $\underline{Y}/\mathfrak{S}^0$ to be log-smooth of finite type. As we assumed Y to be locally proper, if it is of finite type, it must be proper. But we shall have the occasion below to use the comparison theorem between log-convergent and log-crystalline cohomology in a non-proper situation, in fact when both cohomologies are pathological! We remark that already in the classically smooth case, when the comparison theorem between crystalline and convergent cohomologies becomes Ogus' theorem [Ogu90, Theorem 0.7.7], properness is not used in the proof. It was included by Ogus in the hypothesis simply because these cohomologies are known to be pathological otherwise. Nevertheless, they are canonically isomorphic.

It follows that in the proper case the cohomology has a canonical integral structure. In general, in itself, log-rigid (or log-convergent) cohomology is a $\mathbb{Q}$-cohomology, and does not come equipped with a canonical integral structure. For Y not of finite type, when the cohomology might be infinite-dimensional, it is not at all clear that such an integral structure exists. For the Drinfel'd symmetric domain, this will turn out to be the case at the end, thanks to explicit computations with logarithmic forms. But we do not know if the existence of a dense subspace of 'bounded cohomology' is a general phenomenon. These remarks explain, why, in part ii of Theorem 2.1 below, we have to substitute a rather awkward-looking space for $\mathbb{H}^m(Y_{\text{ét}}, W\omega^\bullet)\,\widehat{\otimes}_{W(\kappa)}\,K_0$, which eventually will turn out to be the same. *A priori* we do not know that the latter is non-zero!

Theorem 2.1.

i) [Gro02] *There is an isomorphism, depending on* π,

$$\rho_\pi : D^m(\underline{Y}) \otimes_{K_0} K \simeq H^m_{\mathrm{dR}}(\mathfrak{X}). \tag{2.4}$$

ii) *There is a canonical isomorphism*

$$D^m(\underline{Y}) \simeq \varprojlim \{\mathbb{H}^m(Z_{r,\acute{e}t}, i_r^* W\omega^\bullet) \otimes_{W(\kappa)} K_0\}. \tag{2.5}$$

iii) *Consider the edge-homomorphism in the first spectral sequence of hypercohomology (see also Lemma 3.2)*

$$H^1(W\omega^\bullet(Y)) \to \mathbb{H}^1(Y_{\acute{e}t}, W\omega^\bullet), \tag{2.6}$$

followed by the map to $D^1(Y)$ obtained from part ii, and the comparison isomorphism from part i with $H^1_{\mathrm{dR}}(\mathfrak{X})$. Let $f \in \mathcal{O}(\mathfrak{X})^\times$ be a nowhere vanishing rigid analytic function on $\mathfrak{X}$. The function f defines a global section of the sheaf $\mathcal{K}^\times_{\widehat{\mathfrak{X}}}$, hence a closed 1-form $d\log f \in W\omega^1(Y)$. Then the cohomology class of this element corresponds, under the above map, to $[df/f] \in H^1_{\mathrm{dR}}(\mathfrak{X})$.

Proof. The Hyodo–Kato isomorphism in the non-necessarily proper case is the isomorphism (see [Gro02, Theorem 3.4, Corollary 3.7], and the discussion in § 3.8)

$$H^m_{\mathrm{rig}}(\underline{Y}/\mathfrak{S}^0) \otimes_{K_0} K \simeq H^m_{\mathrm{dR}}(\mathfrak{X}). \tag{2.7}$$

This gives part i.

On the other hand, we have the logarithmic de Rham–Witt complex $W\omega^\bullet$, described in § 2.1. Denote by $\underline{S}_n$ the log-scheme

$$\underline{S}_n = (\operatorname{Spec} W_n(\kappa), \mathbb{N}, 1 \mapsto 0) \tag{2.8}$$

(denoted by $(W_n, W_n(L))$ in [HK94]) and by $u_{\underline{Y}/\underline{S}_n}$ the canonical morphism of topoi

$$u_{\underline{Y}/\underline{S}_n} : (\underline{Y}/\underline{S}_n)^\sim_{\mathrm{crys}} \to Y^\sim_{\acute{e}t}. \tag{2.9}$$

Then, in the derived category $D(Y_{\text{ét}})$,

$$W_n\omega^\bullet_Y \simeq Ru_{\underline{Y}/\underline{S}_n*}(\mathcal{O}_{Y/W_n(\kappa)}) \tag{2.10}$$

(see [HK94, Theorem 4.19 and (4.20)]). Thus

$$\begin{aligned} R\Gamma_{\text{ét}}(Y_{\text{ét}}, W_n\omega^\bullet_Y) &\simeq R\Gamma_{\text{ét}} \circ Ru_{\underline{Y}/\underline{S}_n*}(\mathcal{O}_{Y/W_n(\kappa)}) \\ &\simeq R\Gamma_{\text{cris}}((\underline{Y}/\underline{S}_n)_{\text{crys}}, \mathcal{O}_{Y/W_n(\kappa)}), \end{aligned} \tag{2.11}$$

and, taking inverse limits,

$$H^m_{\text{cris}}(\underline{Y}/\mathfrak{S}^0) \simeq \mathbb{H}^m(Y_{\text{ét}}, W\omega^\bullet_Y). \tag{2.12}$$

Similarly, for every r,

$$H^m_{\text{cris}}(\underline{U}_r/\mathfrak{S}^0) \simeq \mathbb{H}^m(U_{r,\text{ét}}, j_r^* W\omega^\bullet_Y). \tag{2.13}$$

We shall need this canonical isomorphism in the 'open' case, despite the fact that crystalline (or log-crystalline) cohomology is then 'bad'. See the discussion below.

By [SS91, p. 64],

$$H^m_{\text{dR}}(\mathfrak{X}) = \varprojlim H^m_{\text{dR}}(\,]Z_r[^\dagger_{\mathfrak{X}}). \tag{2.14}$$

By [Gro02, Theorem 0.1 (= Theorem 3.1)],

$$H^m_{\text{dR}}(\,]Z_r[^\dagger_{\mathfrak{X}}) \simeq H^m_{\text{rig}}(\underline{Z}_r/\mathfrak{S}^0) \otimes_{K_0} K. \tag{2.15}$$

In fact, in part (a) of that theorem, it is proven that

$$H^m_{\text{dR}}(\,]M[^\dagger_{\mathfrak{X}}) \simeq H^m_{\text{rig}}(\underline{M}/\mathfrak{S}^0) \otimes_{K_0} K \tag{2.16}$$

for the intersection M of any number of irreducible components of Y, endowed with the induced log-structure. Then (2.15) is deduced from (2.16) by means of the Čech spectral sequence for the closed covering of Z_r by its irreducible components [Gro02, § 3.3 (1)], in the same way that Grosse-Klönne deduces part (b) of his theorem for the full space.

Remark. Recall that Z_r is the union of a finite number of irreducible components of Y, and that the log structure on $\underline{Z}_r$ is the one induced from that of Y. At points of Z_r at which Y is defined (locally étale in $\widehat{\mathfrak{X}}$) by an equation of the form $t_1 \ldots t_k = \pi$, this log structure is the reduction modulo π of the standard log-structure given by the chart

$$\mathbb{N}^k \to \mathcal{O}_{\widehat{\mathfrak{X}}} : (m_i) \mapsto \prod t_i^{m_i}, \tag{2.17}$$

even if the point is smooth on Z_r. Thus $\underline{Z}_r$ may not be log-smooth over $\underline{S}_0$ at such a point. The same remark applies to the intersections M figuring in (2.16).

Now

$$H^m_{\text{rig}}(\underline{Z}_r/\mathfrak{S}^0) \simeq H^m_{\text{conv}}(\underline{Z}_r/\mathfrak{S}^0) \tag{2.18}$$

because Z_r is proper, by the same argument involving the Čech spectral sequence. For the irreducible components of Z_r, which are also those of Y, and their intersections M, this is proved in [Gro02, Proposition 8.6]. For $\underline{Z}_r$ itself, apply the Čech spectral sequence for the closed covering by the irreducible components [Gro02, § 3.3 (1)]. Notice that [Gro02, Theorem 8.3(ii)] cannot be applied directly because $\underline{Z}_r$ is not a semistable log-scheme.

Next, there are functorial restriction maps

$$H^m_{\text{conv}}(\underline{Z}_r/\mathfrak{S}^0) \leftarrow H^m_{\text{conv}}(\underline{U}_r/\mathfrak{S}^0) \leftarrow H^m_{\text{conv}}(\underline{Z}_{r+1}/\mathfrak{S}^0). \tag{2.19}$$

When we take the inverse limit we get a canonical isomorphism

$$\varprojlim H^m_{\text{conv}}(\underline{Z}_r/\mathfrak{S}^0) \simeq \varprojlim H^m_{\text{conv}}(\underline{U}_r/\mathfrak{S}^0). \tag{2.20}$$

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Remark. It is well known that the $H^m_{\mathrm{conv}}(\underline{U}_r/\mathfrak{S}^0)$ are pathological, and even infinite-dimensional. However, in the inverse limit all the bad classes die out, and only the good ones survive. The reader should keep in mind the picture of the Néron model of $\mathbb{G}_m$. When its special fiber, which is a string of projective lines, labeled by $\mathbb{Z}$, is exhausted by open subschemes, their tubular neighborhoods are closed annuli of the shape

$$\{z \mid p^{-r} \leqslant |z| \leqslant p^r\}. \tag{2.21}$$

The convergent cohomology of the U_r contains then the good class of dz/z, but also all the 'bad' classes of Taylor–McLaurin expansions with zero residue, which converge up to the boundary of the annulus, but whose integral does not converge up to the boundary. Nevertheless, these bad classes are killed by the map from the rth annulus to the $(r-1)$th. In the limit, only the class of dz/z survives.

The reason we have passed to the U_r is that, unlike the Z_r, when they are given the induced log-structure, they become log-smooth over the base. They are also of finite type, hence Shiho's comparison theorem [Shi02, Theorem 3.1.1], allows us to write

$$H^m_{\mathrm{conv}}(\underline{U}_r/\mathfrak{S}^0) \simeq H^m_{\mathrm{cris}}(\underline{U}_r/\mathfrak{S}^0) \otimes_{W(\kappa)} K_0. \tag{2.22}$$

We may now invoke (2.13) to pass to logarithmic de Rham–Witt cohomology, and the isomorphism

$$\varprojlim\{\mathbb{H}^m(U_{r,\text{ét}}, j_r^* W\omega^\bullet_Y) \otimes_{W(\kappa)} K_0\} \simeq \varprojlim\{\mathbb{H}^m(Z_{r,\text{ét}}, i_r^* W\omega^\bullet_Y) \otimes_{W(\kappa)} K_0\} \tag{2.23}$$

resulting from the restriction maps (as we did in convergent cohomology).

Putting it all together we deduce the desired isomorphism

$$H^m_{\mathrm{dR}}(\mathfrak{X}) \simeq \varprojlim\{\mathbb{H}^m(Z_{r,\text{ét}}, i_r^* W\omega^\bullet_Y)_K\}. \tag{2.24}$$

Part iii is proved by following the definitions and we omit it. $\square$

With regard to part iii we remark that the open dense smooth subscheme U used in the construction of $W\omega^\bullet$ will in general not be connected. On any of its irreducible components U_i we shall have to normalize f differently, dividing by a power of the uniformizer π, so that the resulting function is regular and invertible on the restriction of $\widehat{\mathfrak{X}}$ to U_i. This normalization does not change $d\log f$, but the power by which we divide, 'the multiplicity of U_i in the divisor of f', is recorded in $d\widetilde{\log} f$, as the coefficient of θ.

One should also bear in mind that the association between logarithmic classes in de Rham–Witt cohomology of the special fiber, and de Rham cohomology of the rigid analytic space, is well defined only on the level of cohomology, and not at the level of differential forms. For example, if $U \subset Y$ is an open subset such that $f_U = f|_{]U[}$ satisfies $|f_U - 1| < 1$, then $0 = d\log f_U \in W\omega^1(U)$, since the image of f_U in $\mathcal{O}^\times_Y(U)$ is already the constant 1. On the tube $]U[_{\mathfrak{X}}$, on the other hand, df_U/f_U will be only exact.

3. *p*-adically uniformized varieties

3.1 The de Rham cohomology of $\mathfrak{X}$ and of $X_{\Gamma,K}$

Let V_K be a $(d+1)$-dimensional vector space over K. Denote by H_a, for $a \in \mathbb{P}(V_K)$, the corresponding hyperplane in $\mathbb{P}(V^*)$, and let

$$\mathfrak{X} = \mathbb{P}(V^*) - \bigcup_{a \in \mathbb{P}(V_K)} H_a. \tag{3.1}$$

This is a rigid analytic space on which $G = PGL(V_K)$ acts, called Drinfeld's p-adic symmetric domain of dimension d. Fix a discrete cocompact subgroup Γ of G. The quotient space $\Gamma\backslash\mathfrak{X}$ is the rigid analytic space associated to a (unique) projective and smooth variety $X_{\Gamma,K}$ over K (Mumford, Mustafin). Now $\mathfrak{X}$ has an underlying structure of a formal scheme $\widehat{\mathfrak{X}}$, and $X_{\Gamma,K}$ has a uniquely determined model X_Γ over $\mathcal{O}_K$ which is proper, with strictly semistable reduction, and whose formal completion along the special fiber is $\Gamma\backslash\widehat{\mathfrak{X}}$. Since Γ is fixed, from now on we omit the subscript Γ from the notation.

We review some results concerning the rigid analytic de Rham cohomology of $\mathfrak{X}$ and $X_K = X_{\Gamma,K}$ (see [SS91], [deS00], [AdS02], [IS00] and [Gro02]). Let $\mathcal{T}$ be the Bruhat–Tits building of G, and let $C^r(\mathcal{T}, A)$, for any ring A, be the group of alternating r-cochains on $\mathcal{T}$ with values in A. Let $C^r_{\text{har}} \subset C^r(\mathcal{T}, K)$ be the group of harmonic r-cochains, as defined in [deS00, § 3.1] (see also [AdS03, § 2.5]). If ω is a closed rigid analytic r-form on $\mathfrak{X}$, then we defined in [deS00, Definition 7.1] the *residue* of ω along any oriented r-simplex σ of $\mathcal{T}$, denoted $\operatorname{res}_\sigma \omega$. Fixing ω, and putting

$$c_\omega(\sigma) = \operatorname{res}_\sigma \omega, \tag{3.2}$$

we obtained an r-cochain which we proved was harmonic [deS00, Theorem 7.7]. The main result [deS00, Theorem 8.2] is that this yields an isomorphism

$$H^r_{\text{dR}}(\mathfrak{X}) \simeq C^r_{\text{har}}. \tag{3.3}$$

An immediate consequence is that $H^r_{\text{dR}}(\mathfrak{X})$ has a distinguished subspace, of bounded cohomology classes, corresponding to the bounded harmonic cochains, C^r_{bhar}. It is dense in the topology of uniform convergence on affinoid subdomains, but is complete in the Banach supremum norm transported from C^r_{bhar} via (3.3). The integral cohomology is, by definition, the part mapping to $C^r_{\text{har}}(\mathcal{O}_K) \subset C^r_{\text{bhar}}$. All these spaces are stable under G.

Let $\mathcal{A} = \mathbb{P}(V_K)$ be the set of K-rational hyperplanes in V^*, viewed as a compact space in the p-adic topology. To

$$S = (a_0, \dots, a_r) \in \mathcal{A}^{r+1} \tag{3.4}$$

we associate the differential form

$$\omega_S = \sum_{i=0}^{r} (-1)^i da_0/a_0 \wedge \cdots \wedge \widehat{da_i/a_i} \wedge \cdots \wedge da_r/a_r. \tag{3.5}$$

Notice that da/a is independent of the representative of a in V_K, but is a 1-form on the pre-image of $\mathfrak{X}$ in V^* only. Nevertheless, ω_S descends to $\mathfrak{X}$ because it is equal to $\bigwedge_{i=1}^r d\log(a_i/a_0)$. The harmonic cochain $c_S = c_{\omega(S)}$ is particularly nice. It is given by a combinatorial pairing ([deS00, § 2.3 and Corollary 7.6])

$$c_S(\sigma) = (\sigma, S). \tag{3.6}$$

Let σ be represented by the lattice-flag

$$\sigma = [L_0 \supset L_1 \supset \cdots \supset L_r \supset \pi L_0] \tag{3.7}$$

(the lattices determined up to a common homothety, and the inclusions are strict). Normalize the elements of S to lie in $L_0 - \pi L_0$. Then $(\sigma, S) = \operatorname{sgn}(\alpha)$ if there exists a permutation α such that $a_{\alpha(i)} \in L_i - L_{i+1}$, and is 0 otherwise.

More generally, let μ be any bounded K-valued distribution on $\mathcal{A}^{r+1}$ (a bounded, finitely additive K-valued function on the compact-open subsets of this profinite space). Such a μ is called a measure. The space of measures is denoted by $M(\mathcal{A}^{r+1})$. Then

$$\omega(\mu) = \int \omega_S \, d\mu(S) \tag{3.8}$$

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is well defined on any affinoid subdomain of $\mathfrak{X}$, hence on all of $\mathfrak{X}$ (see [IS00]). We call such an ω a logarithmic r-form and denote the space of them all by $\Omega^r_{\log}(\mathfrak{X})$. Logarithmic forms are closed. The corresponding harmonic cochain is given by

$$c_\mu(\sigma) = c_{\omega(\mu)}(\sigma) = \int (\sigma, S)\, d\mu(S). \tag{3.9}$$

An important observation of Iovita and Spiess is that the bounded cohomology is precisely the image of the map

$$M(\mathcal{A}^{r+1}) \to H^r_{\mathrm{dR}}(\mathfrak{X}), \quad \mu \mapsto [\omega(\mu)] \tag{3.10}$$

and that the logarithmic forms represent the bounded cohomology classes one-to-one, namely, if $[\omega(\mu)] = 0$, then already $\omega(\mu) = 0$ (see [AdS02, Theorem 1.2]). It is not true that $\omega(\mu) = 0$ implies $\mu = 0$. The kernel of this map is the space $M(\mathcal{A}^{r+1})_{\deg}$ of degenerate measures [AdS02, 1.4]. Among other things, we prove in [AdS02] that the Banach quotient norm of the class of μ modulo $M(\mathcal{A}^{r+1})_{\deg}$ is equal to the supremum norm of c_μ.

There is a spectral sequence

$$E_2^{r,s} = H^r(\Gamma, C^s_{\mathrm{har}}) = H^r(\Gamma, H^s_{\mathrm{dR}}(\mathfrak{X})) \Longrightarrow H^\bullet_{\mathrm{dR}}(X_K) \tag{3.11}$$

which degenerates at E_2 (the covering spectral sequence, see [SS91, § 5, Proposition 2]). It therefore induces a filtration $F^\bullet_\Gamma$ on the cohomology of X_K, whose graded pieces are

$$\mathrm{gr}^r_\Gamma\, H^d_{\mathrm{dR}}(X_K) = H^r(\Gamma, C^{d-r}_{\mathrm{har}}). \tag{3.12}$$

One may substitute the bounded harmonic cochains (or bounded cohomology) C^s_{bhar} for C^s_{har} and get the same spectral sequence (see [AdS02, Lemma 3.4]).

The results of § 2 apply to $\mathfrak{X}$. If we denote by Y its special fiber, then the log-rigid cohomology $D^m(Y)$ is defined, and carries a Frobenius endomorphism. Theorem 2.1 holds. The relative Frobenius automorphism ϕ (of degree q) acts linearly. It was computed by Grosse-Klönne in [Gro02, Corollary 5.6], and was found to act like q^m on D^m. This is not surprising if we note that, on the logarithmic de Rham–Witt 1-forms $d\log f$, the absolute Frobenius acts by multiplication by p. It will eventually turn out that these logarithmic classes generate the entire cohomology.

In [Gro02, Theorem 6.3], this is used to prove that the relative Frobenius ϕ respects the covering filtration on the de Rham cohomology and in fact acts like q^s on the graded pieces $H^r(\Gamma, C^s_{\mathrm{har}})$. It follows that the weight filtration and the covering filtration are essentially identical, up to a change of the indexing:

$$F^r_\Gamma H^d_{\mathrm{dR}}(X_K) = P_{2d-2r} D^d(Y) \otimes_{K_0} K. \tag{3.13}$$

3.2 The extensions $\tilde{C}^s_{\mathrm{har}}$

We now review and elaborate on some results from [AdS03]. We recall the extension of G-modules

$$0 \to C^{s-1}_{\mathrm{har}}(A) \to \tilde{C}^{s-1}_{\mathrm{har}}(A) \xrightarrow{d} C^s_{\mathrm{har}}(A) \to 0 \tag{3.14}$$

constructed in [AdS03, § 3] (A any ring). If $A = K$ we drop it from the notation.

Let $\widetilde{\mathcal{A}} = (V_K \backslash \{0\})/\mathcal{O}_K^\times$. This is a locally compact space fibered over $\mathcal{A}$, whose fibers are principal homogeneous spaces for $\mathbb{Z} = K^\times/\mathcal{O}_K^\times$. Any lattice L in V_K determines a splitting $\mathcal{A} \to \widetilde{\mathcal{A}}$, namely $a \mapsto a_L$ where a_L lies in $L - \pi L$. Via this splitting, $\widetilde{\mathcal{A}}$ is identified with $\mathcal{A} \times \mathbb{Z}$. A measure on $\widetilde{\mathcal{A}}^{s+1}$ is, as before, a bounded finitely additive K-valued function on the compact-open subsets of $\widetilde{\mathcal{A}}^{s+1}$. We say that a measure μ is compactly supported if there exists a compact-open subset C such that $\mu(U) = \mu(U \cap C)$ for every compact-open U. Let $M_c(\widetilde{\mathcal{A}}^{s+1})$ be the space of compactly supported measures. We can integrate against such a μ any continuous K-valued function on $\widetilde{\mathcal{A}}^{s+1}$. We denote by μ_* the push-down of μ to a measure on $\mathcal{A}^{s+1}$.

Let $\tilde{S} = (\tilde{a}_0, \dots, \tilde{a}_s) \in \widetilde{\mathcal{A}}^{s+1}$. The element $\tilde{c}_{\tilde{S}} \in \tilde{C}^{s-1}_{\mathrm{har}}$ is defined by

$$\tilde{c}_{\tilde{S}}(\tau) = -\sum_{j=0}^{s} (-1)^j \operatorname{ord}_{L_0}(\tilde{a}_j)(\tau, S_j), \tag{3.15}$$

where L_0 is the leading vertex of $\tau \in \widehat{\mathcal{T}}_{s-1}$ and S_j is the projection to $\mathcal{A}^s$ of the sequence obtained from $\tilde{S}$ by deleting $\tilde{a}_j$. We proved in [AdS03, § 3] that this is indeed an element of $\tilde{C}^{s-1}_{\mathrm{har}}$, and [AdS03, Lemma 3.2] that

$$d(\tilde{c}_{\tilde{S}}) = c_S. \tag{3.16}$$

More generally, we may define $\tilde{c}_\mu$ for any $\mu \in M_c(\widetilde{\mathcal{A}}^{s+1})$ and $d(\tilde{c}_\mu) = c_{\mu_*}$.

3.3 Special logarithmic forms in the de Rham–Witt cohomology

Let $S \in \mathcal{A}^{s+1}$ be as before. Then $[\omega_S] \in H^s_{\mathrm{dR}}(\mathfrak{X})$ is the image, under the comparison isomorphism ρ_π, of a logarithmic class in de Rham–Witt cohomology, which we denote $[W\omega_S] \in D^s(Y)$. In fact, a_i/a_0 represents an element of $\mathcal{K}^\times_{\hat{\mathfrak{X}}}(Y) = \mathcal{O}(\mathfrak{X})^\times$, uniquely determined up to $K^\times$, so $d\log(a_i/a_0) \in W\omega^1(Y)$. Then

$$W\omega_S = \bigwedge_{i=1}^{s} d\log(a_i/a_0) \tag{3.17}$$

is a closed s-form in $W\omega^s(Y)$. Its class in $H^s(W\omega^\bullet(Y))$ maps, under the edge-homomorphism

$$H^s(W\omega^\bullet(Y)) \to \mathbb{H}^s(Y_{\text{ét}}, W\omega^\bullet) \subset D^s(Y), \tag{3.18}$$

to a cohomology class which we denote by $[W\omega_S]$ (see Lemma 4.2 below). In view of part iii of Theorem 2.1, it maps under ρ_π to $[\omega_S] \in H^s_{\mathrm{dR}}(\mathfrak{X})$.

Similarly, if $\tilde{S} \in \widetilde{\mathcal{A}}^{s+1}$, we define

$$[W\tilde{\omega}_{\tilde{S}}] = \left[\bigwedge_{i=1}^{s} d\,\widetilde{\log}(\tilde{a}_i/\tilde{a}_0)\right] \in \mathbb{H}^s(Y_{\text{ét}}, W\tilde{\omega}^\bullet). \tag{3.19}$$

The element $W\tilde{\omega}_{\tilde{S}}$ is a closed global section in $W\tilde{\omega}^s(Y)$. We wish to compute it, using the definition of the complex of sheaves $W\tilde{\omega}^\bullet$ as a subcomplex of $u_*W\Omega^\bullet[\theta]$ for an appropriate smooth open dense subscheme $u : U \hookrightarrow Y$ (see § 2.1). To this end we have to specify U.

Let $\mathrm{red} : \mathfrak{X} \to |\mathcal{T}|$ be the reduction map from the rigid analytic space to the real realization of $\mathcal{T}$, and $\mathrm{sp} : \mathfrak{X} \to Y(\bar{\kappa})$ the specialization map to the special fiber. We take

$$U = \mathrm{sp}(\mathrm{red}^{-1}(\mathcal{T}_0)), \tag{3.20}$$

the set of specializations of points of $\mathfrak{X}$ reducing to the vertices of $\mathcal{T}$. Here we have assumed some familiarity with the structure of $\mathfrak{X}$ and the associated formal scheme. The irreducible components $\bar{Y}_v$ of Y are labeled by the vertices of $\mathcal{T}$. Let $v = [L]$ be such a vertex, representing the homothety class of the lattice L. One has $\bar{Y}_v = \mathrm{sp}(\mathrm{red}^{-1}(\mathrm{St}(v)))$ where $\mathrm{St}(v)$ is the open star of v in $|\mathcal{T}|$. It is a proper smooth variety over κ which is obtained by successive blow-ups of κ-rational linear subspaces in projective spaces. Its core is $Y_v = \mathrm{sp}(\mathrm{red}^{-1}(v))$, which is just the complement in $\mathbb{P}_v = \mathbb{P}((L/\pi L)^*)$ of all the κ-rational hyperplanes. Then Y_v is also the intersection of $\bar{Y}_v$ with the smooth locus of Y. The union

$$U = \coprod_{v \in \mathcal{T}_0} Y_v \tag{3.21}$$

is disjoint. It is in fact the whole smooth locus of Y, and is easily seen to be open and dense.

To give $W\tilde{\omega}_{\tilde{S}}$ as a global section of $u_*W\tilde{\Omega}^s(Y)$ we simply have to specify

$$W\tilde{\omega}_{\tilde{S},v} \in W\tilde{\Omega}^s(Y_v) \tag{3.22}$$

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independently for each $v=[L]$. (Of course, not every such a collection is allowable.) But $\tilde a_i/\tilde a_0$ has to be divided precisely by $\pi^{\mathrm{ord}_L(\tilde a_i/\tilde a_0)}$ to make it an invertible element in $\mathcal{O}_{\widehat{\mathfrak{X}}}(Y_v)$. Denote by a the image of $\tilde a$ in $\mathcal{A}$, and by S that of $\tilde S$. From the definition of $d\widetilde{\log}$, and from $\theta^2=0$, we get

$$\begin{aligned} W\tilde\omega_{\tilde S,v} &= \bigwedge_{i=1}^{s}[d\log(a_i/a_0)+\theta(\mathrm{ord}_L(\tilde a_i)-\mathrm{ord}_L(\tilde a_0))] \\ &= W\omega_S - \theta\wedge\sum_{i=1}^{s}(-1)^i(\mathrm{ord}_L(\tilde a_i)-\mathrm{ord}_L(\tilde a_0))W\omega_{S_i}. \\ &= W\omega_S - \theta\wedge\sum_{i=0}^{s}(-1)^i\,\mathrm{ord}_L(\tilde a_i)W\omega_{S_i}. \end{aligned} \tag{3.23}$$

The last equality uses the fact that $\sum_{i=0}^{s}(-1)^iW\omega_{S_i}=0$. If we use the abbreviation

$$W\eta_{\tilde S,v} = -\sum_{i=0}^{s}(-1)^i\,\mathrm{ord}_L(\tilde a_i)W\omega_{S_i},$$

then

$$W\tilde\omega_{\tilde S,v} = W\omega_S + \theta\wedge W\eta_{\tilde S,v}. \tag{3.24}$$

We denote by $\eta_{\tilde S,v}$ the corresponding element of $\Omega^{s-1}_{\log}(\mathfrak{X})$, namely

$$\eta_{\tilde S,v} = -\sum_{i=0}^{s}(-1)^i\,\mathrm{ord}_L(\tilde a_i)\omega_{S_i}. \tag{3.25}$$

Proposition 3.1.

i) *If v is a vertex of $\tau\in\widehat{\mathcal{T}}_{s-1}$ then*

$$\tilde c_{\tilde S}(\tau) = \mathrm{res}_\tau(\eta_{\tilde S,v}). \tag{3.26}$$

ii) *If $\sigma\in\widehat{\mathcal{T}}_s$, $\sigma=(v_0,\dots,v_s)$ and $\sigma^{(i)}=(v_0,\dots,\widehat{v}_i,\dots,v_s)$ is its ith face, and if $v^{(i)}=v_j$ $(j\neq i)$ denotes any vertex of $\sigma^{(i)}$, then*

$$\mathrm{res}_\sigma\,\omega_S = \sum_{i=0}^{s}(-1)^i\,\mathrm{res}_{\sigma^{(i)}}\,\eta_{\tilde S,v^{(i)}}. \tag{3.27}$$

iii) *If σ is as in part ii,*

$$\mathrm{res}_\sigma\,\omega_S = \mathrm{res}_{\sigma^{(0)}}\,\eta_{\tilde S,v_1} - \mathrm{res}_{\sigma^{(0)}}\,\eta_{\tilde S,v_0}. \tag{3.28}$$

Remark. If $d=1$, $s=1$, and $\sigma=[L_0\supset L_1\supset\pi L_0]$ (a one-dimensional annulus) then ii says that

$$\mathrm{res}_\sigma\,d\log f = \mathrm{ord}_{L_1}(f)-\mathrm{ord}_{L_0}(f). \tag{3.29}$$

Proof. Part i follows, for v the leading vertex of τ, from a comparison of (3.15) with (3.25). If we apply a cyclic permutation to τ, both $\tilde c_{\tilde S}(\tau)$ and the residue along τ change by the sign of the permutation, but v is replaced by another vertex, hence the formula remains valid with any vertex of τ.

Part ii follows from (3.16) since

$$\mathrm{res}_\sigma(\omega_S) = c_S(\sigma) = d\tilde c_{\tilde S}(\sigma). \tag{3.30}$$

Part iii follows from part ii and the fact that $\sum_{i=0}^{s}(-1)^i\,\mathrm{res}_{\sigma^{(i)}}\,\eta_{\tilde S,v_0}=0$. □

All that we did in this section with $W\omega_S$, $W\tilde\omega_{\tilde S}$ and $W\eta_{\tilde S,v}$ can be 'integrated' to give $W\omega(\mu_*)$, $W\tilde\omega(\mu)$ and $W\eta(\mu)_v$, if $\mu\in M_c(\widetilde{\mathcal{A}}^{s+1})$. We follow the construction of [I-S], write $\lambda=\mu_*$, and describe, as an example, $W\omega(\lambda)$. It is enough to give $W\omega(\lambda)|_V$, its restriction to any open subscheme of finite

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type $V \subset Y$. Such a V intersects only finitely many Y_v, and therefore $d\log(a_i/a_0)|_V \in W\omega^1(V)$ is locally constant in a_i and a_0. The integral

$$W\omega(\lambda)|_V = \int_{\mathcal{A}^{s+1}} W\omega_S|_V \, d\lambda(S) \tag{3.31}$$

makes sense as a finite linear combination of the previously defined $W\omega_S$.

Thus $W\tilde{\omega}(\mu)$ is specified by giving a collection of $W\tilde{\omega}(\mu)_v \in W\tilde{\Omega}^s(Y_v)$ and

$$W\tilde{\omega}(\mu)_v = W\omega(\mu_*) + \theta \wedge W\eta(\mu)_v. \tag{3.32}$$

Notice that the differential forms

$$\eta(\mu)_v = -\int_{\widetilde{\mathcal{A}}^{s+1}} \sum_{i=0}^{s} (-1)^i \operatorname{ord}_L(\tilde{a}_i)\omega_{S_i} \, d\mu(\tilde{S}) \tag{3.33}$$

($\tilde{S} = (\tilde{a}_0, \ldots, \tilde{a}_s)$ and $v = [L]$ as usual) are global logarithmic forms, although the corresponding de Rham–Witt forms $W\eta(\mu)_v$ are regarded as sections of $W\Omega^{s-1}$ over Y_v only.

For the proof of the Main Theorem we shall need some technical lemmas on these special logarithmic forms.

LEMMA 3.2. *If S is linearly dependent, then $W\omega_S = 0$. If S_L (representatives of S in $L - \pi L$, for a lattice L) are linearly dependent modulo πL, then under the map*

$$W\omega^s(Y) \subset u_* W\Omega^s_U(Y) = \prod_{v \in \mathcal{T}_0} W\Omega^s(Y_v) \tag{3.34}$$

the image of $W\omega_S$ in $W\Omega^s(Y_v)$, for $v = [L]$, vanishes.

Proof. It is enough, of course, to prove the second claim, because if S is a linearly dependent sequence, S_L is linearly dependent modulo πL for every L. To compute $W\omega_S|Y_v$ we have to consider the representatives $a_{i,L}$ in $L - \pi L$, reduce them modulo πL to get $\alpha_i \in \mathcal{O}_U(Y_v)^\times$, and then use Illusie's map (1.9). If there is a linear dependence we may assume that it is

$$\alpha_0 + \cdots + \alpha_t = 0 \tag{3.35}$$

for some $t \leqslant s$, hence $1 + x_1 + \cdots + x_t = 0$ where $x_i = \alpha_i/\alpha_0$, and clearly $\bigwedge_{i=1}^s d\log(x_i) = 0$. ☐

LEMMA 3.3. *Suppose that $\tilde{S} = (\tilde{a}_0, \ldots, \tilde{a}_{s+1}) \in \widetilde{\mathcal{A}}^{s+2}$ and $\tilde{S}_i$ is obtained by deleting the ith vector. Let*

$$\mu = \sum_{i=0}^{s+1} (-1)^i \delta_{\tilde{S}_i}. \tag{3.36}$$

Then

i) $W\tilde{\omega}(\mu) = 0$,

ii) $\tilde{c}_\mu = 0$.

Proof. i) The measure μ_* is degenerate, hence $W\omega(\mu_*) = 0$. It remains to show that, for every $v = [L]$, $W\eta(\mu)_v = 0$. We compute

$$\begin{aligned} W\eta(\mu)_v &= -\sum_{i=0}^{s+1} (-1)^i \left\{ \sum_{j=0}^{i-1} (-1)^j \operatorname{ord}_L(\tilde{a}_j) W\omega_{S_{ji}} + \sum_{j=i+1}^{s+1} (-1)^{j-1} \operatorname{ord}_L(\tilde{a}_j) W\omega_{S_{ij}} \right\} \\ &= \sum_{j=0}^{s+1} (-1)^j \operatorname{ord}_L(\tilde{a}_j) \left\{ \sum_{i<j} (-1)^i W\omega_{S_{ij}} + \sum_{j<i} (-1)^{i-1} W\omega_{S_{ji}} \right\} = 0, \end{aligned} \tag{3.37}$$

since for every $T = S_j$ of length $s+1$, $\sum_{i=0}^{s} (-1)^i W\omega_{T_i} = 0$.

ii) As in part i we compute $\eta(\mu)_v = 0$ for all v, hence $\tilde{c}_\mu = 0$ (by Proposition 3.1). ☐

For the next lemma, let $S = (a_0, \ldots, a_s)$ be linearly dependent. We may assume that $a_0, \ldots, a_t$ is a minimal linearly dependent subset, and choose representatives $\tilde{S}$ so that

$$\tilde{a}_0 + \cdots + \tilde{a}_t = 0. \tag{3.38}$$

LEMMA 3.4. *Under the above assumptions:*

i) $W\tilde{\omega}_{\tilde{S}} = 0$,

ii) $\tilde{c}_{\tilde{S}} = 0$.

Proof. i) We already know that $W\omega_S = 0$ (see Lemma 3.2). It remains to check that, for any $v = [L]$, $W\eta_{\tilde{S},v} = 0$. Normalize L so that $\tilde{a}_i$, for $i \leqslant t$, all lie in L, but not all of them lie in πL. Rearrange the indices so that $\tilde{a}_i \in L - \pi L$ for $i \leqslant r$ and $\tilde{a}_i \in \pi L$ for $r < i \leqslant t$. Then $\{\tilde{a}_0, \ldots, \tilde{a}_r\}$ are linearly dependent modulo πL, hence $W\omega_{S_i}|Y_v = 0$ if $i > r$. If $0 \leqslant i \leqslant r$, on the other hand, $\mathrm{ord}_L(\tilde{a}_i) = 0$. It follows from (3.23) that $W\tilde{\omega}_{\tilde{S},v} = 0$, as desired.

ii) Although, as in the previous lemma, we could deduce part ii from part i, let us prove it directly. Let $\tau \in \widehat{\mathcal{T}}_{s-1}$ and let $v = [L]$ be the leading vertex of τ. Normalize L and rearrange the indices as in part i. Recall that

$$\tilde{c}_{\tilde{S}}(\tau) = -\sum_{i=0}^{s} (-1)^i \, \mathrm{ord}_L(\tilde{a}_i)(\tau, S_i). \tag{3.39}$$

Since $\{\tilde{a}_0, \ldots, \tilde{a}_r\}$ are linearly dependent modulo πL, $(\tau, S_i) = 0$ for $i > r$. For $i \leqslant r$, $\mathrm{ord}_L(\tilde{a}_i) = 0$. It follows that $\tilde{c}_{\tilde{S}}(\tau) = 0$. $\square$

4. The Main Theorem

Let

$$\nu : H^r(\Gamma, C^{d-r}_{\mathrm{har}}) \to H^{r+1}(\Gamma, C^{d-r-1}_{\mathrm{har}}) \tag{4.1}$$

be the connecting homomorphism in group cohomology coming from the extension (0.4). It follows immediately from [AdS03, Theorem 4.3] that, for any $r \leqslant [d/2]$, iteration of ν yields an isomorphism

$$\nu^{d-2r} : H^r(\Gamma, C^{d-r}_{\mathrm{har}}) \simeq H^{d-r}(\Gamma, C^{r}_{\mathrm{har}}). \tag{4.2}$$

Because of the relation between the covering filtration and the weight filtration (3.13), and because of the characterization given in § 1.2 of the monodromy filtration, the following theorem is all that we need to conclude the proof of Theorem 0.1.

THEOREM 4.1. *Let $\mathrm{gr}_\Gamma N : \mathrm{gr}^r_\Gamma H^d_{\mathrm{dR}}(X_K) \to \mathrm{gr}^{r+1}_\Gamma H^d_{\mathrm{dR}}(X_K)$ be the map induced from the monodromy operator N of $H^d_{\mathrm{dR}}(X_K)$, by passing to graded pieces in the covering filtration. Then*

$$\mathrm{gr}_\Gamma N = \nu. \tag{4.3}$$

Proof. We begin by noting that not only $H^m_{\mathrm{dR}}(\mathfrak{X})$, but also $\mathbb{H}^m(Y_{\text{ét}}, W\omega^\bullet)$, admits an action of Γ (or even G). Indeed, Γ acts on Y, and the quotient map $Y \to Y_\Gamma$ is étale, since we assumed Γ to be torsion-free. The category of abelian sheaves on $Y_{\Gamma,\text{ét}}$ is canonically equivalent, under the restriction functor, to the category of Γ-equivariant abelian sheaves on $Y_{\text{ét}}$. From general principles we get the desired Γ-action on $\mathbb{H}^m(Y_{\text{ét}}, W_n\omega^\bullet)$, as well as a Hochschild–Serre spectral sequence

$$H^r(\Gamma, \mathbb{H}^s(Y_{\text{ét}}, W_n\omega^\bullet)) \Rightarrow \mathbb{H}^{r+s}(Y_{\Gamma,\text{ét}}, W_n\omega^\bullet). \tag{4.4}$$

Remark. For the existence of a Hochschild–Serre spectral sequence one only has to verify that, if I is an injective abelian sheaf on $Y_{\Gamma,\text{ét}}$, then $H^0(Y_{\text{ét}}, I)$ is an acyclic Γ-module. This is proved precisely as in [Mil80, Theorem III.2.20]. Note that the discussion in Example 2.6 of that work applies to our situation, although Γ is infinite. The group Γ is discrete, and Γ-cohomology is ordinary group cohomology, without any reference to continuity.

LEMMA 4.2. *There are Γ-isomorphisms*

$$\rho_k : \mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet) \otimes_{W(\kappa)} K \simeq C^k_{\text{bhar}}(K) \tag{4.5}$$

sending the class $[W\omega_S]$ to c_S.

Proof. Let Z_r be as in § 2.1. Define ρ_k to be the compositum of the three maps

$$\mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet) \otimes_{W(\kappa)} K \hookrightarrow \varprojlim\{\mathbb{H}^k(Z_{r,\text{ét}}, i_r^* W\omega^\bullet) \otimes_{W(\kappa)} K\}, \tag{4.6}$$

$$\varprojlim\{\mathbb{H}^k(Z_{r,\text{ét}}, i_r^* W\omega^\bullet) \otimes_{W(\kappa)} K\} \simeq H^k_{\text{dR}}(\mathfrak{X}), \tag{4.7}$$

$$H^k_{\text{dR}}(\mathfrak{X}) \simeq C^k_{\text{har}}(K). \tag{4.8}$$

Here the first map is obvious, the second comes from Theorem 2.1, parts i and ii, and the third is the canonical isomorphism constructed in [deS00]. The edge homomorphism

$$H^k(W\omega^\bullet(Y)) \to \mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet) \tag{4.9}$$

of the (first) spectral sequence of hypercohomology

$$E_1^{pq} = H^q(Y_{\text{ét}}, W\omega^p) \Rightarrow \mathbb{H}^{p+q}(Y_{\text{ét}}, W\omega^\bullet) \tag{4.10}$$

(note that $E_2^{p0} = H^p(W\omega^\bullet(Y))$), allows us to talk about the logarithmic classes $[W\omega(\lambda)]$ in $\mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet) \otimes_{W(\kappa)} K$, for any K-valued measure λ on $\mathcal{A}^{k+1}$. This class maps to $c_\lambda \in C^k_{\text{bhar}}$. As it is clear that $\mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet)$ gets mapped to $C^k_{\text{har}}(\mathcal{O}_K)$, and as every bounded harmonic cochain is of the form c_λ, we conclude that the image of $\mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet) \otimes_{W(\kappa)} K$ is precisely $C^k_{\text{bhar}}(K)$, and that every element in it is of the form $[W\omega(\lambda)]$. $\square$

We now return to the proof of the theorem. Consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r, dashed] & H^{k-1}(W\omega^\bullet(Y)) \arrow[r] \arrow[d] & H^k(W\tilde{\omega}^\bullet(Y)) \arrow[r] \arrow[d] & H^k(W\omega^\bullet(Y)) \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r, dashed] & \mathbb{H}^{k-1}(Y_{\text{ét}}, W\omega^\bullet)_K \arrow[r] \arrow[d, "\cong\rho_{k-1}"] & \mathbb{H}^k(Y_{\text{ét}}, W\tilde{\omega}^\bullet)_K \arrow[r] \arrow[d, "\tilde{\rho}_k"] & \mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet)_K \arrow[r, dashed] \arrow[d, "\cong\rho_k"] & 0 \\
0 \arrow[r] & C^{k-1}_{\text{bhar}}(K) \arrow[r] & \tilde{C}^{k-1}_{\text{bhar}}(K) \arrow[r] & C^k_{\text{bhar}}(K) \arrow[r] & 0
\end{tikzcd} \tag{4.11}$$

Here the middle row is part of the long exact sequence in hypercohomology coming from (1.6), and the maps in the top row are the obvious ones. The arrows from the top to the middle row are supplied by the edge-homomorphisms of the first spectral sequence of hypercohomology. The map $\tilde{\rho}_k$ will be constructed soon.

Step I. We first note that the middle row is exact (namely, the 'monodromy operator' on $D^k(Y)$ vanishes). The results of [AdS02] show that every $c \in C^k_{\text{bhar}}$ is of the form c_{μ_*} for some $\mu_* \in M(\mathcal{A}^{k+1})$, which is unique up to $M(\mathcal{A}^{k+1})_{\text{deg}}$. The fact that ρ_k is an isomorphism implies that every element of $\mathbb{H}^k(Y_{\text{ét}}, W\omega^\bullet)_K$ is of the form $[W\omega(\mu_*)]$. Since μ_* can be lifted to a measure μ on $\widetilde{\mathcal{A}}^{k+1}$, $W\omega(\mu_*)$ is the image of $W\tilde{\omega}(\mu)$, and we obtain exactness on the right. The same exactness on the right, for index $k-1$ instead of k, proves exactness on the left in the diagram.

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Step II. We now define the map $\tilde{\rho}_k$. From Step I we deduce that every element of $\mathbb{H}^k(Y_{\text{ét}}, W\tilde{\omega}^\bullet)_K$ is of the form $[W\tilde{\omega}(\mu)]$, although μ is not unique. We put

$$\tilde{\rho}_k([W\tilde{\omega}(\mu)]) = \tilde{c}_\mu \tag{4.12}$$

and proceed to show that this is well defined. We have to check that, if $[W\tilde{\omega}(\mu)] = 0$, then also $\tilde{c}_\mu = 0$. Before we begin recall (Proposition 3.1, part i) that, if v is the leading vertex of τ,

$$\tilde{c}_\mu(\tau) = \operatorname{res}_\tau \eta(\mu)_v. \tag{4.13}$$

Suppose that $[W\tilde{\omega}(\mu)] = 0$. Then $[W\omega(\mu_*)] = 0$ too, hence $[\omega(\mu_*)] = 0$, so $\omega(\mu_*) = 0$, and μ_* is degenerate. Now every degenerate measure is a linear combination (integral) of

a) λ_*, for $\lambda = \sum_{i=0}^{k+1} (-1)^i \delta_{\tilde{S}_i}$, and

b) λ_* that are supported on $(\mathcal{A}^{k+1})_0$, the locus of linearly dependent vectors in $\mathcal{A}^{k+1}$.

If λ is of type a), both $W\tilde{\omega}(\lambda) = 0$ and $\tilde{c}_\lambda = 0$, by Lemma 3.3. If λ_* is of type b), Lemma 3.4 shows that we can lift it to a compactly supported measure λ on $\widetilde{\mathcal{A}}^{k+1}$ so that the same conclusion holds. In either case, we can modify μ without affecting $W\tilde{\omega}(\mu)$ or $\tilde{c}_\mu$, and make sure that μ_* is not only degenerate, but vanishes identically.

We claim that, if $\mu_* = 0$, then $\eta(\mu)_v$ is independent of v. For that, it is enough to treat the following special case. Pick $S = (a_1, \dots, a_k) \in \mathcal{A}^k$, and consider any lifting $\tilde{S}$ to $\widetilde{\mathcal{A}}^k$, and any $\tilde{a}_0$. Let

$$\mu = \delta_{\tilde{a}_0, \tilde{S}} - \delta_{\pi\tilde{a}_0, \tilde{S}}, \tag{4.14}$$

where δ is the Dirac measure. Then it follows from the definitions that

$$\eta(\mu)_v = \omega_S, \tag{4.15}$$

independently of v, as claimed.

We conclude that, if $\mu_* = 0$, then $W\tilde{\omega}(\mu) = \theta \wedge W\eta(\mu)$. If in addition $[W\tilde{\omega}(\mu)] = 0$, then $[W\eta(\mu)] = 0$, so $[\eta(\mu)] = 0$ and $\eta(\mu) = 0$; hence

$$\tilde{c}_\mu(\tau) = \operatorname{res}_\tau \eta(\mu) = 0. \tag{4.16}$$

This proves that $\tilde{\rho}_k$ is well defined.

Step III. It is clear that $\tilde{\rho}_k$ is G-equivariant. It also makes the diagram commutative. For the square on the right, use

$$\rho_k([W\omega(\mu_*)]) = c_{\mu_*} = d(\tilde{c}_\mu) = d\tilde{\rho}_k([W\tilde{\omega}(\mu)]). \tag{4.17}$$

For the square on the left, use the same computation from Step II. Let μ be as in (4.14). Then

$$W\tilde{\omega}(\mu) = \theta \wedge W\omega_S; \tag{4.18}$$

hence

$$\tilde{\rho}_k([\theta \wedge W\omega_S]) = \tilde{\rho}_k([W\tilde{\omega}(\mu)]) = \tilde{c}_\mu = c_S, \tag{4.19}$$

the last equality being a consequence of (3.15). This proves the commutativity of the left square.

Step IV. When we take Γ-cohomology of the middle row in the diagram, the connecting homomorphism

$$H^r(\Gamma, \mathbb{H}^s(Y_{\text{ét}}, W\omega^\bullet)_K) \to H^{r+1}(\Gamma, \mathbb{H}^{s-1}(Y_{\text{ét}}, W\omega^\bullet)_K) \tag{4.20}$$

is precisely $\operatorname{gr}^r_\Gamma N$ where N is the Hyodo–Kato monodromy operator. In the bottom row, however, we get the map ν of [AdS03]. This completes the proof of Theorem 4.1. □

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