RINGS IN WHICH ALL SUBRINGS ARE IDEALS. I

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In analogy with Hamiltonian groups, an associative ring in which every subring is a two-sided ideal is called a *Hamiltonian ring*, or, more concisely, an H-*ring*. Several attempts at classification of H-rings have been made. H-rings generated by a single element have been studied by M. Šperling (**5**), L. Rédei (**4**), and A. Jones and J. J. Schäffer (**2**). H-rings enjoying additional properties have been characterized by F. Szász (e.g., **6**), and by S.-X. Liu (**3**). A class of closely related rings has been studied by P. A. Freĭdman (**1**). In the present paper and its sequel all H-rings are classified and completely described in terms of their generators and relations.

Among the most natural examples of H-rings are the null rings, in which all products are 0, and the ring $\mathfrak{Z}$ of rational integers together with its subrings and homomorphic images. Result (1.4) shows that a semi-simple H-ring cannot differ much from the subrings and homomorphic images of $\mathfrak{Z}$. One might also expect that a radical H-ring, which by (1.3) must be nil, would not differ much from a null ring or a nilpotent ring of the form $N\mathfrak{Z}/\langle M\rangle$, where the integer N is a divisor of M. We shall in fact find this to be true, although the list of differences which can occur is rather complicated. The complications arise in the classification of the finite, nilpotent H-rings of prime-power order, and so this classification will be done in a separate paper. In § 2 of the present paper other classes of radical H-rings are determined, and in § 3 the extension problem of constructing H-rings from their radicals and semi-simple factor rings is solved. Finally, in § 4, the results are summarized.

1. Preliminaries.

(1.1) *Definitions and special notations.* The *characteristic* of a ring $\mathfrak{R}$ [an element ϕ of a ring] is the least natural number N for which $N\psi = 0$, all $\psi \in \mathfrak{R}$ [for which $N\phi = 0$], provided such an N exists. Otherwise, the characteristic is 0. We write char $\mathfrak{R}$ or char ϕ for the characteristic of $\mathfrak{R}$ or ϕ. The *exponent* of a nilpotent element ϕ is the least integer r for which $\phi^r = 0$. A ring [an H-ring] whose additive group is a p-group is called a p-*ring* [an H-p-*ring*]. The letter p will always denote a prime. Let $\mathfrak{S}$ be an element or a set of elements of a ring. $\langle\mathfrak{S}\rangle$ will denote the subring generated by $\mathfrak{S}$, and $\{\mathfrak{S}\}$ will denote the additive subgroup generated by $\mathfrak{S}$. The word *ideal* will refer only to two-sided ideals; the word *annihilator* to two-sided annihilators. All rings are assumed associative.

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We shall make frequent use of the following known results.

(1.2) Suppose that $\mathfrak{N}$ is a nil ideal in a ring $\mathfrak{R}$. If $\mathfrak{R}/\mathfrak{N}$ contains an idempotent $\phi + \mathfrak{N}$, then $\mathfrak{R}$ contains an idempotent $\epsilon \in \phi + \mathfrak{N}$.

(1.3) (Freĭdman (**1**)). The (Jacobson) radical of an H-ring is nil.

(1.4) (Freĭdman (**1**)). A semi-simple H-ring is isomorphic to a ring direct sum of the form

$$N\mathfrak{Z} \oplus \sum_{p\in\mathfrak{P}} \oplus \mathfrak{F}_p \quad \text{(restricted)},$$

where $\mathfrak{P}$ is a set of primes, $\mathfrak{Z}$ is the ring of rational integers, $N\mathfrak{Z}$ the subring generated by an integer N, $\mathfrak{F}_p$ is the prime field of order p, and each prime p divides N.

(1.5) (Rédei (**4**)). Let ω be a nilpotent element of characteristic 0 in an H-ring. Then ω^2 has non-zero, square-free characteristic, and $\omega^3 = 0$.

The following elementary observations are also helpful.

(1.6) Every subring and every homomorphic image of an H-ring is an H-ring.

(1.7) A ring is an H-ring if and only if every subring generated by a single element is an ideal.

(1.8) A ring $\mathfrak{R}$ with torsion additive group is isomorphic to a restricted ring direct sum of p-rings $\mathfrak{R}_p$ for different primes p. $\mathfrak{R}$ is an H-ring if and only if each $\mathfrak{R}_p$ is an H-p-ring.

Proof. The usual abelian group decomposition, in fact, gives a ring direct sum. Every subring of the direct sum is itself the direct sum of its projections into the direct summands. A subring is an ideal in a direct sum if and only if its projections are ideals in the direct summands.

2. On radical H-rings. To study rings which are close to null it would seem appropriate to consider the two-sided annihilator of the ring. In addition, for our purpose, a somewhat larger subring is useful. We begin with the definition.

(2.1) *Definition.* A subring $\mathfrak{S}$ of a ring $\mathfrak{R}$ *almost annihilates* $\mathfrak{R}$ if, for all $\phi \in \mathfrak{S}$,

(1) $\phi^3 = 0$,

(2) $M\phi^2 = 0$ for some square-free integer M which depends on ϕ,

(3) $\phi\mathfrak{R} \subseteq \{\phi^2\}$ and $\mathfrak{R}\phi \subseteq \{\phi^2\}$.

When $\mathfrak{S} = \mathfrak{R}$, the ring $\mathfrak{R}$ is called *almost-null.*

Three observations are immediate.

(2.2) When subring $\mathfrak{S}$ is a nil H-ring, then (1) of (2.1) follows from (2).

(2.3) All almost-null rings are nilpotent H-rings.

(2.4) Suppose subring $\mathfrak{S}$ almost annihilates ring $\mathfrak{R}$. Choose $\phi \in \mathfrak{S}$. Then ϕ annihilates $\mathfrak{R}$ if and only if $\phi^2 = 0$.

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The importance of almost-null rings is shown by the following two propositions.

(2.5) PROPOSITION. *A nil* p-*ring of characteristic* 0 *is an* H-*ring if and only if it is almost-null.*

(2.6) PROPOSITION. *A nil ring which contains an element of characteristic* 0 *is an* H-*ring if and only if it is almost-null.*

The proofs of these two propositions are similar and will be treated together. We begin with some lemmas.

(2.7) LEMMA. *If* ϕ *is an element of a nil* H-p-*ring, then* $\phi^3 \in \{\phi^2\}$.

Proof. Let k be the minimal integer such that there exists an integer M with $\phi^k = M\phi^{k-1}$. The result is true for $k \leqq 3$ so suppose $k \geqq 4$. Let r be the exponent of ϕ. Then $0 = \phi^{r-k}\phi^k = M\phi^{r-1}$ implies p divides M. From $\phi\phi^2 \in \langle\phi^2\rangle$, it follows that $\phi^3 = F(\phi^2)$ for some polynomial with integral coefficients

$$F(X^2) = F_2X^2 + F_4X^4 + \ldots + F_{2S}X^{2S}.$$

Then, $\phi^{k-1} = \phi^{k-4}\phi^3 = \phi^{k-4}F(\phi^2) = F_2\phi^{k-2} + (F_4M + \ldots + F_{2S}M^{2S-3})\phi^{k-1}$. Since p divides M and char ϕ is a power of p, this equation may be solved for ϕ^{k-1} as an integral multiple of ϕ^{k-2}. This contradicts the minimality of k. Thus $k \leqq 3$ and the result is proved.

(2.8) COROLLARY. *Let* $\mathfrak{R}$ *be a nil* H-p-*ring and let* $\phi \in \mathfrak{R}$, $\xi \in \mathfrak{R}$. *Then there exist integers* U, U', V, V' *such that*

$$\phi\xi = U\phi + V\phi^2 = U'\xi + V'\xi^2.$$

(2.9) LEMMA. *Let* ϕ *and* ω *be elements in a nil* H-*ring. Suppose that* $M = \text{char}\,\phi \neq 0$ *and* M^2 *divides* char ω. *Then* char ϕ^2 *is square-free.*

Proof. Let N be the greatest square-free integer which divides M. (2.8) and (1.8) imply that $\phi(N\phi) = UN\phi + VN^2\phi^2$ for suitable integers U and V. From this it follows that $N\phi^2 \in \{\phi\}$. One of two cases must occur. If $\{\phi\} \cap \{\omega\} = 0$, then $\phi(N\phi + M\omega) \in \langle N\phi + M\omega\rangle$ implies $N\phi^2 = 0$, q.e.d. If $\{\phi\} \cap \{\omega\} \neq 0$, then char $\omega \neq 0$ and by (1.8) it suffices to suppose that $M = p^r$ and char $\omega = p^{2r+s}$ for some prime p and integers $r, s \geqq 0$. Further, there are integers $A \not\equiv 0 \pmod p$ and $t < r$ such that $p^{r+s+t}\omega = Ap^t\phi$. Then

$$\phi(Ap\phi - p^{r+s+1}\omega) \in \langle Ap\phi - p^{r+s+1}\omega\rangle$$

implies $p\phi^2 = 0$. This completes the proof.

Proof of (2.5) *and* (2.6). Let $\mathfrak{P}$ be a nil H-p-ring which contains elements of arbitrarily large characteristic, and let $\mathfrak{R}$ be a nil H-ring which contains an element of characteristic 0. If $\phi \in \mathfrak{P}$, then, by (2.9), $p\phi^2 = 0$. If $\phi \in \mathfrak{R}$, then, by (1.5) or (2.9), ϕ^2 has square-free characteristic. To show that $\mathfrak{P}$ and $\mathfrak{N}$ are almost-null is thus reduced, by (2.2), to showing, for $\phi, \xi \in \mathfrak{P}$ $[\phi, \xi \in \mathfrak{N}]$, that $\phi\xi \in \{\phi^2\} \cap \{\xi^2\}$. We shall prove $\phi\xi \in \{\phi^2\}$. $\phi\xi \in \{\xi^2\}$ follows dually. If char ϕ = char $\xi = 0$, choose integers U and V such that $\phi\xi = U\phi + V\phi^2$. (1.5) implies $\phi\xi^3 = 0$, which implies $U = 0$. Thus, $\phi\xi \in \{\phi^2\}$. If char $\phi \neq 0$, char $\xi = 0$, let $\psi = \phi +$ (char ϕ)(char ξ^2)ξ. Then $\psi\xi \in \langle\psi\rangle$ implies $\phi\xi \in \{\phi^2\}$. If char $\phi = 0$, char $\xi \neq 0$, then $\phi\xi \in \langle\xi\rangle$ implies char $(\phi\xi) \neq 0$ implies $\phi\xi \in \{\phi^2\}$. If, finally, char $\phi \neq 0$, char $\xi \neq 0$, then by (1.8) it suffices to consider the case char $\phi = p^r$, char $\xi = p^s$ for some prime p and integers r and s.

Let $t = \max(r, s) + 1$. If the ring contains an element ω of characteristic 0, let $\psi = \phi + p^t(\text{char } \omega^2)\omega$. Then $\psi\xi \in \langle\psi\rangle$ implies $\phi\xi \in \{\phi^2\}$. Otherwise, choose ω so that char $\omega = p^{2t}$. Then $\phi\xi \in \{\phi^2\}$ follows from $\psi\xi \in \langle\psi\rangle$, where ψ is chosen as follows:

(1) If $\{\omega\} \cap \langle\phi\rangle = 0$, then $\psi = \phi + p^t\omega$.

(2) If $\{\omega\} \cap \{\phi\} \neq 0$, let $p^{2t+k-r}\omega = Ap^k\phi$, where $A \not\equiv 0 \pmod p$ and $k < r$, then $\psi = A\phi - p^{2t-r}\omega$.

(3) If neither (1) nor (2) holds, then there exists integers A and B with $p^{2t-1}\omega = Ap^{r-1}\phi + B\phi^2$. If $A \equiv 0 \pmod p$, then $\psi = \phi + p^t\omega$. If $A \not\equiv 0 \pmod p$, then $\psi = A\phi - p^{2t-r}\omega$.

Thus $\mathfrak{P}$ and $\mathfrak{N}$ are almost-null. An application of (2.3) completes the proof of (2.5) and (2.6).

The remainder of this section is a determination of the structure of almost-null rings. Let $\mathfrak{F}$ be an arbitrary finitely generated subring of such a ring, and let $\mathfrak{N}$ be the annihilator of $\mathfrak{F}$. Then $\mathfrak{F}/\mathfrak{N}$ must be a null ring generated by at most two elements, and all products in $\mathfrak{F}$ must be natural multiples of a fixed element ψ with $M\psi = 0$ for some square-free integer M. This structure is typical of that of an arbitrary almost-null ring, which we now describe in greater detail.

(2.10) Proposition. *For a ring* $\mathfrak{R}$ *and a prime* p *define*

$$\mathfrak{R}_p = \langle \phi \in \mathfrak{R} \mid p\phi^2 = 0\rangle.$$

A necessary and sufficient condition that $\mathfrak{R}$ *be almost-null is that* $\mathfrak{R} = \sum_p \mathfrak{R}_p$ *and that each subring* $\mathfrak{R}_p$ *satisfies one of the following conditions.* $\mathfrak{N}_p$ *is the annihilator of* $\mathfrak{R}_p$.

(1) $\mathfrak{R}_p = \mathfrak{N}_p$ *is null.*

(2) $\mathfrak{R}_p = \{\phi\} + \mathfrak{N}_p$, *where* $p\phi \in \mathfrak{N}_p$, $\phi^2 \in \mathfrak{N}_p$, *and* char $\phi^2 = p$.

(3) $\mathfrak{R}_p = \{\phi, \xi\} + \mathfrak{N}_p$, *where* $p\phi, p\xi, \phi^2 \in \mathfrak{N}_p$, char $\phi^2 = p$, *and where there are integers* A, F, *and* F' *such that* $\xi^2 = A\phi^2$, $\phi\xi = F\phi^2$, $\xi\phi = F'\phi^2$, *and for which the congruence*

$$(*) \qquad X^2 + X(F + F') + A \equiv 0 \pmod p$$

has no integer solution X.

Proof. A straightforward verification establishes the sufficiency of the condition. To establish necessity, suppose that $\mathfrak{N}$ is almost-null and define $\mathfrak{N}_p = \langle \phi \in \mathfrak{N} | \, p\phi^2 = 0 \rangle$. Condition (2) of (2.1) implies $\mathfrak{N} = \sum_p \mathfrak{N}_p$, where the (restricted) sum is taken over all primes p. If the subring $\mathfrak{N}_p$ satisfies neither (1) nor (2) of the conclusion, then there exist elements $\phi \in \mathfrak{N}_p$, $\xi \in \mathfrak{N}_p$, with ϕ, ξ linearly independent mod$\langle p\mathfrak{N}_p, \mathfrak{N}_p \rangle$. (2.4) implies $\phi^2 \neq 0$, and $(\xi + X\phi)^2 \neq 0$ for any X; otherwise, $\xi \in \{\phi\} + \mathfrak{N}_p$. We now prove $\{\phi^2\} = \{\xi^2\}$. This follows from (3) of (2.1) if $\phi\,\xi \neq 0$ or $\xi\phi \neq 0$. Otherwise, it follows from the inclusion that

$$\phi(\phi + \xi) \in \{(\phi + \xi)^2\}.$$

Now $(\xi + X\phi)^2 \neq 0$ is equivalent to (*), so if $\mathfrak{N}_p = \{\phi, \xi\} + \mathfrak{N}_p$, then (3) holds. Suppose, on the other hand, that there were some $\psi \in \mathfrak{N}_p$, $\psi \notin \{\phi, \xi\} + \mathfrak{N}_p$. As above, $\{\phi^2\} = \{\xi^2\} = \{\psi^2\}$, and, by (2.4), $(X\phi + Y\xi + Z\psi)^2 = 0$ implies $X \equiv Y \equiv Z \equiv 0 \pmod p$. The existence, however, of a non-zero solution of this equation follows from the well-known fact that every quadratic form in three variables over the field of p elements represents zero. This contradiction completes the proof of (2.10).

3. H-rings which are not nil. The structure of semi-simple H-rings is given in (1.4). In this section we combine this result with the results of § 2 to obtain the structure of general H-rings. To do so we consider three cases. First, we consider those H-rings which contain no elements of characteristic 0, secondly, those whose semi-simple part contains an element of characteristic 0, and thirdly, those which contain a nilpotent element of characteristic 0. By (3.3), the second and third cases are disjoint.

The study of H-rings with no elements of characteristic 0 is reduced, by (1.8), to the study of H-p-rings. By the next result, the problem is reduced to the study of nil H-p-rings. These have been partially described in § 2, and the description will be completed in a sequel to this paper.

(3.1) Proposition. *A ring is an* H-p-*ring if and only if it is isomorphic to a ring* $\mathfrak{R}$ *which satisfies one of the following conditions.*

(1) $\mathfrak{R}$ *is a nil* H-p-*ring.*

(2) $\mathfrak{R} = \mathfrak{F} \oplus \mathfrak{N}$, *where* $\mathfrak{F}$ *is the field of* p *elements and* $\mathfrak{N}$ *is a nil* H-p-*ring.*

(3) $\mathfrak{R} = \mathfrak{Z}/\langle p^n \rangle$ *is the ring of integers modulo* p^n, $n > 1$.

Proof. Rings which satisfy (1) or (3) are clearly H-p-rings. It is not difficult to show the same for rings which satisfy (2). For the converse, suppose that $\mathfrak{R}$ is an H-p-ring which is not nil. Let $\mathfrak{M}$ be the radical of $\mathfrak{R}$. By (1.4), $\mathfrak{R}/\mathfrak{M}$ is isomorphic to the field of p elements, and so by (1.2), $\mathfrak{R}$ contains an idempotent ϵ. Since $\mathfrak{R}$ is an H-ring, the Peirce decomposition gives

$$\mathfrak{R} = \langle \epsilon \rangle \oplus \mathfrak{N},$$

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where $\mathfrak{N}$ is a two-sided annihilator of ϵ. If $\mathfrak{N}$ were not nil, it would contain an idempotent; but then $\mathfrak{R}/\mathfrak{M}$ would fail to satisfy (1.4). Thus $\mathfrak{N}$ is a nil H-p-ring. Let char $\epsilon = p^n$. If $n = 1$, then condition (2) holds. For $n > 1$ choose any element $\omega \in \mathfrak{N}$. The containment

$$\epsilon(p\epsilon + \omega) \in \langle p\epsilon + \omega\rangle$$

implies $\omega = 0$. Thus $\mathfrak{N} = 0$ and condition (3) holds.

We next determine the structure of an H-ring $\mathfrak{R}$ whose semi-simple part contains an element of characteristic 0. Let $\mathfrak{N}$ be the radical of $\mathfrak{R}$. By (1.4), $\mathfrak{R}/\mathfrak{N}$ is isomorphic to

$$N\mathfrak{Z} \oplus \mathfrak{Z}/\langle M\rangle$$

for some positive integer N and square-free positive integer M which divides N. Thus there is an element $\nu \in \mathfrak{R}$ with $\nu^2 - N\nu = \psi \in \mathfrak{N}$ and char $\nu = 0$. The following must hold.

(3.2) $\nu\psi = \psi\nu = \psi^2 = 0$. $C =$ char $\psi \neq 0$, *is square free, and divides* N.

Proof. Suppose that $C = 0$. Let $K =$ char ψ^2. By (1.5), $K \neq 0$. Then

$$K\nu(2K\nu) \notin \langle 2K\nu\rangle.$$

Thus $C \neq 0$. Next, suppose there is a prime p such that p^2 divides C. Then $\nu(C/p)\nu \notin \langle (C/p)\nu\rangle$. Thus, C is square-free. Now, choose a prime p which does not divide N. The ring $\langle\nu\rangle/\langle p\nu, \psi^2\rangle$ must satisfy (3.1), and this implies $\nu\psi \equiv \psi\nu \equiv 0 \pmod{\langle p\nu, \psi^2\rangle}$. This must hold for infinitely many primes p, and thus $\nu\psi \equiv \psi\nu \equiv 0 \pmod{\langle \psi^2\rangle}$. Then

$$\nu(C\nu + \psi) \in \langle C\nu + \psi\rangle \pmod{\langle\psi^2\rangle}$$

implies C divides N. Finally, (2.7) implies $\psi^3 = 0$, which, with

$$\psi^2 = (\nu^2 - N\nu)\psi = \nu(\nu\psi),$$

implies $\psi^2 = 0$. Thus also $\nu\psi = \psi\nu = 0$. This completes the proof of (3.2).

(3.3) $\mathfrak{N}$ *contains no element of characteristic* 0.

Proof. Suppose that $\omega \in \mathfrak{N}$ and char $\omega = 0$. By (1.5), $K =$ char $\omega^2 \neq 0$. Let $\xi = 2C\nu + K\omega$. Then $\nu\xi \notin \langle\xi\rangle$.

In addition to the subring of characteristic 0 the semi-simple ring $\mathfrak{R}/\mathfrak{N}$ may contain a torsion subring isomorphic to $\mathfrak{Z}/\langle M\rangle$ when M is a square-free integer which divides N. If $M \neq 1$, then, by (1.2), there is an idempotent $\epsilon \in \mathfrak{R}$ with $M\epsilon \equiv \epsilon\nu \equiv \nu\epsilon \equiv 0 \pmod{\mathfrak{N}}$. The following must hold.

(3.4) $M\epsilon = \epsilon\nu = \nu\epsilon = 0$. ϵ *annihilates* $\mathfrak{N}$.

Proof. Let $Q =$ char ϵ. By (3.3), $Q \neq 0$. Suppose for some prime p that p^2 divides Q. Let $\phi = QN\nu + (Q/p)\epsilon$. Then $\epsilon\phi \notin \langle\phi\rangle$. Thus Q is square-free.

It follows that $\langle\epsilon\rangle$ contains no nilpotent elements, so $\langle\epsilon\rangle \cap \mathfrak{N} = 0$. The conclusion follows directly.

To complete the determination of the structure of $\mathfrak{R}$ requires a more careful investigation of the structure of $\mathfrak{N}$. From (3.3) and (1.8) it follows that $\mathfrak{N} = \sum_p \oplus \mathfrak{N}_p$ (restricted) where each subring $\mathfrak{N}_p$ is a nil H-p-ring.

(3.5) $\mathfrak{N}_p = 0$ *unless* p *divides* N.

Proof. For $\mathfrak{N}_p \neq 0$ choose $0 \neq \phi \in \mathfrak{N}_p$, with $\phi^2 = p\phi = 0$. Then

$$\nu(p\nu + \phi) \in \langle p\nu + \phi\rangle$$

implies p divides N.

(3.6) $\mathfrak{N}_p$ *is almost-null.*

Proof. Let $\phi \in \mathfrak{N}_p$. Let $\operatorname{char} \phi = p^s$. Then

$$\phi(p\phi + p^s\nu) \in \langle p\phi + p^s\nu\rangle$$

implies $p\phi^2 = 0$. Let $\xi \in \mathfrak{N}_p$. Let $p^t = \max(\operatorname{char} \phi, \operatorname{char} \xi)$.

$$\phi(\xi + p^t\nu) \in \langle \xi + p^t\nu\rangle$$

implies $\phi\xi \in \{\xi^2\}$. $\phi\xi \in \{\phi^2\}$ is dual. Thus $\mathfrak{N}_p$ is almost-null.

Finally, we investigate the relation between ν and each $\mathfrak{N}_p$ more closely. Let us write $\psi = \sum_p \psi_p$, where $\psi_p \in \mathfrak{N}_p$. Surely, $p\psi_p = 0$, and $\psi_p = 0$ unless p divides C.

Note that no conditions have yet been placed on the choice of the element ν within its coset of $\mathfrak{N}$.

(3.7) *The element* ν *may be chosen in such a way that* ψ_p *and* $\mathfrak{N}_p$ *satisfy one of the following conditions*:

(1) $\psi_p = 0$, $\mathfrak{N}_p$ *annihilates* ν, *and* char $\mathfrak{N}_p$ *divides* N.

(2) $\psi_p \neq 0$, $\mathfrak{N}_p$ *annihilates* $\mathfrak{N}$, *and* char $\mathfrak{N}_p$ *divides* N.

(3) $\psi_p \neq 0$, $\mathfrak{N}_p = \{\phi_p\} + \mathfrak{M}_p$, *where* $\psi_p \in \mathfrak{M}_p$, $p\phi_p \in \mathfrak{M}_p$, $\mathfrak{M}_p$ *annihilates* $\mathfrak{N}$, char $\mathfrak{M}_p$ *divides* N, *and* $\{\psi_p\}$ *contains* ϕ_p^2, $N\phi_p$, $\nu\phi_p$, *and* $\phi_p\nu$. *The equation*

$$(\nu + X\phi_p)^2 = N\nu + \textstyle\sum_{q\neq p} \phi_q$$

has no integer solution X. *Finally, if* $\phi_p^2 = 0$, *then* $p = 2$ *and*

$$N\phi_p = \nu\phi_p = \phi_p\nu = \psi_p.$$

Proof. First suppose that $\psi_p = 0$. Then $\langle\nu\rangle \cap \mathfrak{N}_p = 0$ so $\mathfrak{N}_p$ annihilates ν. Also, for $\xi \in \mathfrak{N}_p$, $\nu(p\nu + \xi) \in \langle p\nu + \xi\rangle$ implies char ξ divides N or else there is an integer A such that $\xi^2 = AN\xi \neq 0$. The latter implies $\xi(A\nu + \xi) \notin \langle A\nu + \xi\rangle$. Thus char $\mathfrak{N}_p$ divides N, and so (1) holds.

From now on assume that $\psi_p \neq 0$. Let $\xi \in \mathfrak{N}_p$ be arbitrary and let $p^t = \operatorname{char} \xi$. We shall show that either ξ annihilates $\mathfrak{N}$ and p^t divides N or ξ acts like the element ϕ_p of condition (3).

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It is always true that $\nu\xi \in \{\psi_p\}$, $\xi\nu \in \{\psi_p\}$, and $p\xi$, ψ_p, and ξ^2 annihilate $\mathfrak{R}$. Moreover, the inclusion $\nu(N\nu + \xi) \in \langle N\nu + \xi\rangle$ implies $\nu\xi - N\xi \in \{\xi^2\}$. Dually, $\xi\nu - N\xi \in \{\xi^2\}$. First, suppose that $\{\psi_p\} \cap \langle\xi\rangle = 0$. Then $\nu\xi = \xi\nu = 0$ and $\nu(\nu + \xi) \in \langle \nu + \xi\rangle$ implies p^t divides N and $\xi^2 = 0$. (2.4) completes the proof that ξ annihilates $\mathfrak{R}$.

Next, suppose that $\{\psi_p\} \cap \langle\xi\rangle \neq 0$ and $\xi^2 \in \{\xi\}$. If $\xi^2 = 0$, then $\nu\xi = \xi\nu = N\xi$. If $N\xi = 0$, then ξ annihilates $\mathfrak{R}$. If $N\xi \neq 0$, then $p = 2$ and $N\xi = \psi_p$, or else

$$(\nu + X\xi)^2 = N\nu + \textstyle\sum_{q\neq p} \psi_q$$

has a solution. If $\xi^2 \neq 0$, then $\{\psi_p\} \cap \langle\xi\rangle \neq 0$ implies $\{\psi_p\}$ contains ξ^2 and $N\xi$.

Lastly, suppose that $\{\psi_p\} \cap \langle\xi\rangle \neq 0$ and $\xi^2 \notin \{\xi\}$. Let $\nu\xi = H\psi_p$. Then

$$\nu(H\nu - \xi) \in \langle H\nu - \xi\rangle \quad \text{and} \quad \nu(\nu + \xi) \in \langle \nu + \xi\rangle$$

imply $N\xi = 0$ and $\xi^2 \in \{\psi_p\}$.

We have now shown that if all ξ annihilate $\mathfrak{R}$, then (2) holds, while if not, when we set $\phi_p = \xi$ for some ξ which does not annihilate $\mathfrak{R}$, then to show that (3) holds we need only verify that $\mathfrak{N}_p = \{\phi_p\} + \mathfrak{M}_p$, where $\mathfrak{M}_p$ annihilates $\mathfrak{R}$, and that we can assume that

$$(\nu + X\phi_p)^2 = N\nu + \textstyle\sum_{q\neq p} \psi_q$$

has no solution. But if there is a solution X, then, if ν is replaced by $\nu + X\phi_p$, $\psi_p = 0$, and so this choice of ν makes $\mathfrak{N}_p$, ψ_q satisfy (1). Finally, let $\mathfrak{M}_p$ be the annihilator of $\mathfrak{R}$ contained in $\mathfrak{N}_p$. If there is an element $\xi \in \mathfrak{N}_p$, $\xi \notin \langle\phi_p, \mathfrak{M}_p\rangle$, then it is possible to find integers X, Y such that

$$(\nu + X\phi_p + Y\xi)^2 = N\nu + \textstyle\sum_{q\neq p} \psi_q,$$

and so, by changing ν, $\psi_p = 0$ and (1) holds. This completes the proof of (3.7).

A straightforward verification shows that a ring which satisfies (3.2)–(3.7) is an H-ring. We can therefore summarize as follows.

(3.8) Proposition. *A ring which contains a non-nilpotent element of characteristic* 0 *is an* H-*ring if and only if it is isomorphic to a ring* $\mathfrak{R}$ *which satisfies* (3.2)–(3.7).

We conclude this section with a characterization of H-rings which contain a nilpotent element of characteristic 0. The result follows.

(3.9) Proposition. *A ring which contains a nilpotent element of characteristic* 0 *is an* H-*ring if and only if it is isomorphic to a ring* $\mathfrak{R}$,

$$\mathfrak{R} = \mathfrak{N} \oplus \textstyle\sum_{p\in\mathfrak{P}} \oplus \mathfrak{F}_p \quad \text{(restricted)},$$

where $\mathfrak{N}$ *is almost-null,* $\mathfrak{P}$ *is a set of primes, and* $\mathfrak{F}_p$ *is the field of* p *elements.*

Proof. A straightforward verification shows that a ring of the form $\mathfrak{N} \oplus \sum_{p\in\mathfrak{P}} \oplus \mathfrak{F}_p$ is an H-ring. For the converse let $\mathfrak{R}$ be an H-ring which

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contains a nilpotent element ω with char $\omega = 0$. Let $\mathfrak{N}$ be the radical of $\mathfrak{R}$. By (2.6), $\mathfrak{N}$ is almost null. By (3.3) and (1.4), $\mathfrak{R}/\mathfrak{N}$ is isomorphic to $\sum_{p \in \mathfrak{P}} \oplus \mathfrak{F}_p$, where $\mathfrak{P}$ is a set of primes and $\mathfrak{F}_p$ is the field of p elements. $\mathfrak{F}_p$ is generated by an idempotent, which, by (1.2), may be lifted to an idempotent $\epsilon_p \in \mathfrak{R}$ with $p\epsilon_p \in \mathfrak{N}$. Let $C = \text{char } \epsilon_p$. Since $p\epsilon_p$ is nilpotent and $\epsilon_p{}^2 = \epsilon_p$, $C = p^r$ for some integer r. Let $D = \text{char } \omega^2$. By (1.5), $D \neq 0$. Then the inclusion

$$\epsilon(p\epsilon + p^r D\omega) \in \langle p\epsilon + p^r D\omega \rangle$$

implies $r = 1$. Thus $\langle \epsilon_p \rangle$ is isomorphic to $\mathfrak{F}_p$, $\langle \epsilon_p \rangle \cap \mathfrak{N} = 0$, and $\mathfrak{R}$ is then isomorphic to

$$\mathfrak{N} \oplus \textstyle\sum_{p \in \mathfrak{P}} \oplus \mathfrak{F}_p.$$

4. Summary. A class of rings, called *almost-null*, is of fundamental importance in the determination of rings in which every subring is a two-sided ideal. The specific structure of almost-null rings is given in (2.10). Such a ring is nilpotent with cube 0; the square of the ring, locally, consists of the natural multiples of a fixed element; and the ring over its annihilator, locally, is at most a two-generator null ring. If a radical ring contains elements of sufficiently "large" characteristic, then it is an H-ring if and only if it is almost-null. B "larger" than A means A divides B. More specifically:

Suppose that $\mathfrak{R}$ is a radical ring in which, for every $\phi \in \mathfrak{R}$ which does not annihilate $\mathfrak{R}$ and for which char $\phi \neq 0$, there exists some $\omega \in \mathfrak{R}$ for which (char $\phi)^3$ divides char ω. Then $\mathfrak{R}$ is an H-ring if and only if $\mathfrak{R}$ is almost-null.

The structure of radical H-rings which contain no elements of "large" characteristic is more complicated and will be given in a separate paper. The idea of "almost-null", however, may also be used to describe concisely those algebras in which all subalgebras are ideals (Liu (**3**)):

An associative algebra over a field k has every subalgebra an ideal if and only if it is almost-null, or is isomorphic to the direct sum of an almost-null algebra and the field k.

Almost-null rings are also important in the description of H-rings which contain elements of characteristic 0. A ring which contains a nilpotent element of characteristic 0 is isomorphic to the direct sum of an almost-null ring and a ring which is, locally, isomorphic to the rational integers modulo a square-free integer (see (3.9)). If the semi-simple part of an H-ring contains an element of characteristic 0, then its radical is almost-null, with the more special structure given in (3.7). See (3.2)–(3.8). Finally, the determination of H-rings with no elements of characteristic 0 is reduced (by (1.8), (3.1), and (2.5)) to the study of nil H-rings of prime-power characteristic. Such rings, in fact, must be the direct sum of a finite nilpotent ring with at most four generators and a null ring. The proof is too lengthy to be included in this paper.

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For a more complete bibliography, see **6.**

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