EXHAUSTIVE OPERATORS AND VECTOR MEASURES

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1. Introduction

Let S be a compact Hausdorff space and let $\Phi: C(S)\to E$ be a linear operator defined on the space of real-valued continuous functions on S and taking values in a (real) topological vector space E. Then Φ is called *exhaustive* (7) if given any sequence of functions $f_n \in C(S)$ such that $f_n \geqq 0$ and

$$\sup_s \sum_{n=1}^{\infty} f_n(s) < \infty$$

then $\Phi(f_n)\to 0$. If E is complete then it was shown in (7) that exhaustive maps are precisely those which possess regular integral extensions to the space of bounded Borel functions on S; this is equivalent to possessing a representation

$$\Phi(f) = \int_S f(s)d\mu(s)$$

where μ is a regular countably additive E-valued measure defined on the σ-algebra of Borel subsets of S.

In this paper we seek conditions on E such that every continuous operator $\Phi: C(S)\to E$ (for the norm topology on $C(S)$) is exhaustive. If E is a Banach space then Pelczynski (**14**) has shown that every exhaustive map is weakly compact; then we have from results in (**2**) and (**16**);

Theorem 1.1. *If E is a Banach space containing no copy of c_0, then every bounded $\Phi: C(S)\to E$ is exhaustive.*

Theorem 1.2. *If E is a Banach space containing no copy of l_∞, then if S is σ-Stonian, every bounded $\Phi: C(S)\to E$ is exhaustive.*

These results extend naturally to locally convex spaces, but here we study the general non-locally convex case. We show that Theorem 1.1 does indeed extend to arbitrary topological vector spaces; it seems likely that Theorem 1.2 extends also, but we here only prove special cases. In particular we prove Theorem 1.2 when E is separable (generalising a result due originally to Grothendieck (**6**)).

2. Operators on c_0

We denote by (e_n) the unit vector basis of c_0. If $M\subset \mathbb{N}$ is an infinite subset, then $c_0(M)$ is the subspace of c_0 of all sequences vanishing outside M. Let c_{00} represent the subspace of all sequences which are eventually zero, and let

$$A_n = \{t\in c_{00}\colon \| t \|_\infty \leqq 1 \quad t_1 = t_2 = \dots = t_{n-1} = 0\}. \quad (n \geqq 2)$$

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Now let $\Phi: c_0 \to (E, \tau)$ be a continuous linear operator mapping c_0 into a metrisable topological vector space (E, τ). Let (U_n) be a base of closed balanced τ-neighbourhoods of 0 satisfying $U_{n+1}+U_{n+1} \subset U_n$ for $n \geqq 1$. Define

$$V_n = \bigcap_{m=2}^{\infty} (U_n + \Phi(A_m)).$$

Lemma 2.1. *(V_n) is a base for a metrisable vector topology $\gamma(\Phi)$ on E.*

Proof. Each V_n is balanced since each U_n and $\Phi(A_m)$ is balanced. Since U_n is absorbent, V_n is absorbent. In view of Köthe (**10**, p. 146), it is necessary only to show that $V_{n+1}+V_{n+1} \subset V_n$ for every n, in order to prove that (V_n) defines a vector topology.

Suppose $x, y \in V_{n+1}$; then for any m

$$x = u_1 + \Phi(t),$$

where $u_1 \in U_{n+1}$ and $t \in A_m$. Since $t \in c_{00}$, there exists p such that $t_i = 0$ for $i \geqq p$. Then

$$y = u_2 + \Phi(t'),$$

where $t' \in A_p$. Thus

$$x+y = (u_1+u_2)+\Phi(t+t')$$

and $x+y \in U_n + \Phi(A_m)$. Hence $x+y \in V_n$.

Now

$$\bigcap_{n=1}^{\infty} V_n = \bigcap_{m=2}^{\infty} \bigcap_{n=1}^{\infty} (U_n + \Phi(A_m))$$

$$= \bigcap_{m=2}^{\infty} \overline{\Phi(A_m)} \quad \text{(closure in } \tau)$$

$$\subset \overline{\Phi(A_2)}.$$

However $\bigcap_{n=1}^{\infty} V_n$ is a linear subspace of E, and, as Φ is continuous, $\overline{\Phi(A_2)}$ is bounded. Therefore

$$\bigcap_{n=1}^{\infty} V_n = \{0\},$$

and $\gamma(\Phi)$ is Hausdorff.

Lemma 2.2. *If $\{\Phi(e_n): n \in \mathbb{N}\}$ is not a $\gamma(\Phi)$-precompact set, then for some infinite subset $M \subset \mathbb{N}$, $\Phi: c_0(M) \to (E, \tau)$ is an isomorphism on to its image.*

Proof. We may find $k \in \mathbb{N}$ such that for any $\gamma(\Phi)$-precompact subset S of E, $S + V_k$ does not contain $\{\Phi(e_n): n \in \mathbb{N}\}$. We then select by induction an increasing sequence of integers $p(n)$ such that for every n

$$(\alpha) \quad \Phi(e_{p(n)}) \notin V_k + T_{n-1} \quad n \geqq 1$$

$$(\beta) \quad \Phi(e_{p(n)}) \notin U_{k+1} + T_{n-1} + \Phi(A_{p(n+1)}) \quad n \geqq 1$$

where $T_0 = \{0\}$ and for $n \geqq 1$

$$T_n = \left\{ \sum_{i=1}^{n} a_i \Phi(e_{p(i)}) : |a_i| \leqq 1 \right\}.$$

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Pick $p(1)$ so that (α) holds. Now suppose $p(1)\dots p(r)$ have been selected so that (α) holds for $1 \leqq n \leqq r$ and (β) holds for $1 \leqq n \leqq r-1$. Then by (α)

$$\Phi(e_{p(r)}) \notin V_k + T_{r-1}.$$

Since T_{r-1} is τ-compact and symmetric we may have a finite symmetric subset Σ_{r-1} of T_{r-1} such that

$$T_{r-1} \subset \Sigma_{r-1} + U_{k+1}.$$

Now $\Phi(e_{p(r)}) \notin V_k + \Sigma_{r-1}$, and hence, for each $\sigma \in \Sigma_{r-1}$ there is a $q(\sigma)$ such that

$$\Phi(e_{p(r)}) \notin \sigma + U_k + \Phi(A_{q(\sigma)}).$$

Thus there is a $q = \max(q(\sigma)\colon\ \sigma \in \Sigma_{r-1})$ such that

$$\Phi(e_{p(r)}) \notin \Sigma_{r-1} + U_k + \Phi(A_q).$$

Since $T_{r-1} \subset \Sigma_{r-1} + U_{k+1}$ we conclude that

$$\Phi(e_{p(r)}) \notin T_{r-1} + U_{k+1} + \Phi(A_q).$$

Now pick $p(r+1) > \max(p(r),\ q)$ to satisfy (α), using the fact that T_r is $\gamma(\Phi)$-compact. This completes the inductive construction.

Suppose $(a_i, \dots, a_n)$ is a sequence with $\max_{1 \leqq i \leqq n} |a_i| = |a_j| = 1$. Then

$$a_j \Phi(e_{p(j)}) = \sum_{i=1}^{n} a_i \Phi(e_{p(i)}) - \sum_{i=1}^{j-1} a_i \Phi(e_{p(i)}) - \sum_{=j+1}^{n} a_i \Phi(e_{p(i)})$$

(a summation over the empty set is taken to be zero), and therefore

$$a_j \Phi(e_{p(j)}) \in \sum_{i=1}^{n} a_i \Phi(e_{p(i)}) + T_{j-1} + \Phi(A_{p(j+1)}).$$

Hence by (β)

$$\sum_{i=1}^{n} a_i \Phi(e_{p(i)}) \notin U_{k+1}.$$

Let $M = \{p(1), p(2), \dots\}$ and consider $\Phi\colon c_{00} \cap c_0(M) \to (E, \tau)$. If $\Phi(t^{(n)}) \to 0$ and $\| t^{(n)} \|_\infty \geqq \varepsilon > 0$ for all n, then $\Phi(\| t^{(n)} \|_\infty^{-1} t^{(n)}) \to 0$. However

$$\Phi(\| t^{(n)} \|_\infty^{-1} t^{(n)}) \notin U_{k+1}$$

for all n, and so we have a contradiction. Therefore if $\Phi(t^n) \to 0$ then

$$\| t^{(n)} \|_\infty \to 0$$

and Φ is an isomorphism. Clearly Φ is also an isomorphism on the closure of $c_{00} \cap c_0(M)$, i.e. $c_0(M)$.

Note that, since Φ is continuous $\{\sum_{n \in k} \phi(e_n)\colon K \subset \mathbb{N},\ K \text{ finite}\}$ is $\gamma(\Phi)$-bounded. Therefore if $\{\Phi(e_n)\colon n \in \mathbb{N}\}$ is $\gamma(\Phi)$-precompact then $\Phi(e_n) \to 0$ in $\gamma(\Phi)$.

Theorem 2.3. *Suppose (E, τ) is a topological vector space and $\Phi\colon c_0 \to E$ is a continuous linear map; then either*

(i) $\Phi(e_n) \to 0(\tau)$,

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or

(ii) *there is an infinite subset M of $\mathbb{N}$ such that Φ: $c_0(M) \to E$ is an isomorphism onto its image.*

Proof. Suppose neither (i) nor (ii) holds. Then we may find a metrisable topological vector space (F, μ) and a continuous linear map Ψ: $E \to F$ such that (i) does not hold for $\Psi\Phi$. Then (ii) also must fail for $\Psi\Phi$, and so we may reduce consideration to the metrisable case for τ. We may also suppose that (E, τ) is complete. As above, let (U_n) be a base of neighbourhoods for τ.

Now by Lemma 2.2, $\Phi(e_n) \to 0\ \gamma(\Phi)$. Let $\bar{\gamma}$ be the finest vector topology such that $\bar{\gamma} \leqq \tau$ and $\Phi(e_n) \to 0(\bar{\gamma})$ ($\bar{\gamma}$ is given by all τ-continuous F-semi-norms which make $\Phi(e_n)$ a null sequence). Then $\bar{\gamma}$ is Hausdorff since $\gamma(\Phi)$ is Hausdorff. Now let $\bar{\bar{\gamma}}$ be the metrisable topology with a base of neighbourhoods $(\bar{U}_n)$ (closure in $\bar{\gamma}$). Then if $\bar{\gamma} = \bar{\bar{\gamma}}$ the identity map in i: $(E, \bar{\gamma}) \to (E, \tau)$ is almost continuous and therefore by the Closed Graph Theorem (Kelley (9), p. 213), $\bar{\gamma} = \tau$. Since we are assuming (i) to be false we conclude that $\bar{\gamma} < \bar{\bar{\gamma}} \leqq \tau$. Therefore

$$\Phi(e_n) \nrightarrow 0(\bar{\bar{\gamma}})$$

and so by Theorem 3.2 of (8), there is a subsequence $(\Phi(e_n)\colon n \in M)$ which is a regular basic sequence in $(E, \bar{\bar{\gamma}})$. (A sequence is regular if it is bounded away from zero and basic if it forms a basis for its closed linear span in the completion of $(E, \bar{\bar{\gamma}})$.)

Now if $t \in c_0(M)$ then $\Sigma t_n \Phi(e_n)$ converges in (E, τ) and hence in $(E, \bar{\bar{\gamma}})$. Then as $(\Phi(e_n)\colon n \in M)$ is $\bar{\bar{\gamma}}$-regular it is equivalent to the unit vector basis of c_0. By a result of Arsove and Edwards (1), Φ: $c_0(M) \to G$ is an isomorphism where G is the closed linear span of $\Phi(e_n)$ in $(E, \bar{\bar{\gamma}})$. Then G is also closed in (E, τ) and by the Open Mapping Theorem Φ is also an isomorphism for the topology τ. This contradicts our assumption that (ii) was false.

Theorem 2.4. *Let (E, τ) be a topological vector space containing no copy of c_0 Then any bounded linear map Φ: $c_0 \to E$ takes the unit ball B of c_0 into a precompact subset of E.*

Proof. If $\Phi(B)$ is not precompact, we may find a neighbourhood U of zero in E and a sequence $t^{(n)}$ in $c_{00} \cap B$ such that $\Phi(t^{(n)}) - \Phi(t^{(m)}) \notin U$ for $n \neq m$.

By selecting a subsequence we may suppose $(t^{(n)})$ is co-ordinatewise convergent in l_∞. Thus $t^{(n)} - t^{(n+1)} \to 0$ co-ordinatewise. We may then select a subsequence $s^{(n)}$ of $(t^{(n)} - t^{(n+1)})$ which is disjoint (i.e. if $n \neq m$ $s_k^{(n)} . s_k^{(m)} = 0$ for all k). Define Ψ: $c_0 \to E$ by

$$\Psi(u) = \sum_{i=1}^{\infty} u_i \Psi(s^{(i)}).$$

As $\Psi(s^{(n)}) \notin U$, we may conclude from Theorem 2.3 that E contains a subspace isomorphic to c_0.

If E is complete then the hypotheses of Theorem 2.4 ensure that $\sum_{n=1}^{\infty} \Phi(e_n)$ converges.

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3. Operators on l_∞

Lemma 3.1. *Let E be a separable metrisable topological vector space and suppose $\Phi: l_\infty \to E$ is a continuous operator such that $\Phi(c_0) = 0$. Then there is an infinite subset M of N such that $\Phi(l_\infty(M)) = 0$.*

Here $l_\infty(M) = \{t \in l_\infty : t_i = 0, i \notin M\}$.

Proof. We may assume that E is complete. Let $(M_\alpha : \alpha \in \mathscr{A})$ be an uncountable collection of infinite subsets of $\mathbb{N}$ such that $M_\alpha \cap M_\beta$ is finite for each $\alpha \neq \beta$, see (**19**). Suppose if possible that for each $\alpha \in \mathscr{A}$ there exists $t^{(\alpha)} \in l_\infty(M_\alpha)$ with $\| t^{(\alpha)} \|_\infty = 1$ and $\Phi(t^{(\alpha)}) \neq 0$. Let $\mathscr{A}_k = \{\alpha : \Phi(t^{(\alpha)}) \notin V_k\}$ where (V_k) is a base of neighbourhoods of 0 in E. Then for some k, $\mathscr{A}_k$ is uncountable; however $(\Phi(t^{(\alpha)}) : \alpha \in \mathscr{A}_k)$ is separable and hence there is a sequence (α_n) in $\mathscr{A}_k$ such that

$$\Phi(t^{(\alpha_n)}) \to x_0,$$

where $x_0 \neq 0$. Then for any p

$$\Phi\left(\sum_{n+1}^{n+p} t^{(\alpha_i)}\right) \to p x_0.$$

However since $M_{\alpha_i} \cap M_{\alpha_j}$ is finite if $i \neq j$ and $\Phi(c_0) = 0$ we conclude that

$$\Phi\left(\sum_{n+1}^{n+p} t^{(\alpha_i)}\right) \in \Phi(B),$$

where B is the unit ball of l_∞. Thus $p x_0 \in \overline{\Phi(B)}$ for any p and we have a contradiction.

Theorem 3.2. *Let (E, τ) be a separable topological vector space, and let $\Phi: l_\infty \to E$ be a continuous linear operator. Then $\Phi(e_n) \to 0$.*

Proof. Since E may be embedded in a product of separable metrisable spaces, it is sufficient to assume that (E, τ) is metrisable and complete. Now suppose $\Phi(e_n) \nrightarrow 0$ in (E, τ). Then there is an infinite subset M of $\mathbb{N}$ such that $\Phi: c_0(M) \to (E, \tau)$ is an isomorphism onto a closed subspace G of E.

Let $\pi: E \to E/G$ be the quotient map; then $\pi\Phi = 0$ on $c_0(M)$ and by Lemma 3.1 there is an infinite subset M_0 of M such that $\pi\Phi = 0$ on $l_\infty(M_0)$, i.e. $\Phi(l_\infty(M_0)) \subset G$. Now as $G \cong c_0$, we may apply the theorem of Grothendieck (**6**, p. 173), or Rosenthal (**16**) to deduce Φ is weakly compact on $l_\infty(M_0)$ and hence $\sum_{n \in M_0} \Phi(e_n)$ is weakly subseries convergent in G. By the Orlicz-Pettis Theorem $\Phi(e_n) \to 0$ (see e.g. (**5**) p. 318, (**12**) or (**15**)).

It is very possible Theorem 3.2 can be extended to topological vector spaces containing no copy of l_∞. However, here we have only a partial result. The technique of the following theorem is essentially found in Drewnowski (**4**). We identify l_∞ as $C(\beta N)$ and thus we can define exhaustive operators as in the introduction.

Theorem 3.3. *Let $\Phi: l_\infty \to (E, \tau)$ be a continuous linear operator, and suppose there is a Hausdorff vector topology ρ on E such* (i) *$\Phi: l_\infty \to (E, \rho)$ is exhaustive* (ii)

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τ is ρ-polar, i.e. has a base of ρ-closed neighbourhoods of **0**. *Then if $\Phi(e_n)\to 0$ in τ, there is an infinite subset M of $\mathbb{N}$ such that $\Phi\colon l_\infty(M)\to(E,\tau)$ is an isomorphism onto its image.*

Proof. By (8) Proposition 2.1, there is a τ-continuous F-semi-norm η of the form

$$\eta(x)=\sup\,(\lambda(x)\colon\ \lambda\in\Lambda)$$

where Λ is a collection of ρ-continuous F-semi-norms and such that for an infinite subset M_0 of $\mathbb{N}$

$$\eta(\Phi(e_n))\geqq 1\quad n\in M_0.$$

By Theorem 2.3 we may suppose that for some subsequence M_1 of M_0, $\Phi\colon c_0(M_1)\to(\hat{E},\eta_1)$ is an embedding (where $(\hat{E},\eta_1)$ is the Hausdorff quotient of (E,η)). Thus if $t\in c_0(M_1)$, $\|t\|_\infty=1$ then

$$\eta(\Phi(t))\geqq\theta>0.$$

We next select a sequence $(m_k\colon k=1,2,\ldots)$ in $\mathbb{N}$ and a sequence

$$(M_k\colon k=1,2,\ldots)$$

of infinite subsets of $\mathbb{N}$ by induction. First choose $m_1\in M_1$. Next given $(m_1,\ldots,m_k)$ and $(M_1,\ldots,M_k)$ let S_k be a finite subset of

$$L_k=\left\{\sum_{i=1}^{k}t_i\Phi(e_{m_i})\colon\ \max|t_i|=1\right\}$$

such that for $x\in L_k$ there exists $s\in S_k$ with

$$\eta(x-s)\leqq\tfrac{1}{8}\theta.$$

For each $s\in S_k$ pick $\lambda_s\in\Lambda$ such that

$$\lambda_s(s)\geqq\eta(s)-\tfrac{1}{8}\theta.$$

Now let $M_k=\bigcup_{n=1}^{\infty}P_n$ where (P_n) is any sequence of disjoint infinite sets. Since Φ is exhaustive for ρ we may find n_0 such that for $t\in l_\infty(P_{n_0})$, $\|t\|_\infty\leqq 1$ and $s\in S_k$

$$\lambda_s(\Phi(t))\leqq\tfrac{1}{8}\theta.$$

Let $M_{k+1}=P_{n_0}$ and then choose $m_{k+1}\in M_{k+1}$. This constructs a set

$$M=(m_1,m_2,\ldots,m_k,\ldots)$$

such that $(m_{k+1},m_{k+2},\ldots)\subset M_{k+1}$ for all k.

Now suppose $t\in l_\infty(M)$ with $\|t\|_\infty=1$. For $\varepsilon>0$ there exists k such that $|t_{m_k}|>1-\varepsilon$; thus there exists δ with $|\delta|<\varepsilon$ and such that if $t'=t+\delta e_{m_k}$ then $\|t'\|=|t'_{m_k}|=1$. Then let $t''=\sum_{i=1}^{k}t'_{m_i}e_{m_i}$ and choose $s\in S_k$ such that

$$\eta(\Phi(t'')-s)\leqq\theta\tfrac{1}{8}.$$

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Then

$$\begin{aligned}\eta(\Phi(t')) &\geqq \lambda_s(\Phi(t'))\\ &\geqq \lambda_s(\Phi(t''))-\tfrac{1}{8}\theta\\ &\geqq \lambda_s(s)-\tfrac{1}{4}\theta\\ &\geqq \eta(s)-\tfrac{3}{8}\theta\\ &\geqq \tfrac{5}{8}\theta.\end{aligned}$$

Hence

$$\eta(\Phi(t)) \geqq \tfrac{5}{8}\theta-\eta(\varepsilon\Phi(e_{m_k})),$$

and therefore, as $\varepsilon>0$ is arbitrary and $(\Phi(e_n): n \in \mathbb{N})$ is bounded,

$$\eta(\Phi(t)) \geqq \tfrac{5}{8}\theta.$$

It follows easily that Φ is an isomorphism on $l_\infty(M)$.

4. Applications

In this section we collect together the main results of the paper, which are deductions from the more technical results of Sections 2 and 3.

Theorem 4.1. *Let E be a topological vector space containing no copy of c_0; then every continuous linear operator $\Phi: C(S)\to E$, where S is compact Hausdorff, is exhaustive (and can therefore be represented in the form*

$$\Phi(f) = \int_S f(s)d\mu,$$

where μ is a regular countably additive E-valued vector measure defined on the Borel sets of S).

Proof. Let (f_n) be any sequence of positive functions in $C(S)$ such that

$$\sup_s \sum_{n=1}^{\infty} f_n(s)<\infty.$$

Then we can define $\Psi: c_0\to E$ by

$$\Psi(t) = \Phi\left(\sum_{n=1}^{\infty} t_n f_n\right)$$

$\left(\sum_{n=1}^{\infty} t_n f_n \text{ converges in the norm topology of } C(S)\right)$. By Theorem 2.3 $\Psi(e_n)\to 0$ i.e. $\Phi(f_n)\to 0$ and so Φ is exhaustive.

Theorem 4.2. *Suppose S is a σ-Stonian compact Hausdorff space and that E is a separable topological vector space. Then any continuous linear operator $\Phi: C(S)\to E$ is exhaustive.*

Proof. Suppose $f_n \in C(S), f_n \geqq 0$ and

$$\sup_s \sum_{n=1}^{\infty} f_n(s)<\infty.$$

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Then since $C(S)$ is σ-order-complete we can define for $t \in l_\infty$ and $t \geqq 0$ the order-sum $o-\sum_{n=1}^{\infty} t_n f_n = \sup_n \sum_{k=1}^{n} t_k f_k$. We can extend this definition to a linear map Γ: $l_\infty \to C(S)$ and Γ is continuous. Now let $\Psi = \Gamma\Phi$ and apply Theorem 3.2.

Theorem 4.3. *Suppose (E, τ) is an F-space containing no copy of l_∞ and Φ: $C(S) \to E$ is a continuous linear operator which is exhaustive for a weaker Hausdorff vector topology ρ on E. Then Φ is exhaustive for τ.*

Proof. Let γ be the largest vector topology on E such that $\gamma \leqq \tau$ and Φ is γ-exhaustive. Let $\bar{\gamma}$ be the topology with a base of τ-neighbourhoods consisting of the γ-closures of τ-neighbourhoods of 0.

Suppose now for some $f_n \geqq 0$ with $\sup_s \Sigma f_n(s) < \infty$ that $\Phi(f_n) \to 0(\bar{\gamma})$. Then we may form the map Ψ: $l_\infty \to E$ as in Theorem 4.2 and by Theorem 3.3 there is infinite subset M of $\mathbb{N}$ such that Ψ: $l_\infty(M) \to (E, \bar{\gamma})$ is an embedding. Hence Ψ: $l_\infty(M) \to (E, \tau)$ is an embedding and this contradicts the hypotheses of the theorem. Hence $\Phi(f_n) \to 0(\bar{\gamma})$ and so Φ is $\bar{\gamma}$-exhaustive. However, $\bar{\gamma} \leqq \tau$ and therefore $\bar{\gamma} \leqq \gamma$; thus the identity I: $(E, \gamma) \to (E, \tau)$ is almost continuous and by the Closed Graph Theorem (Kelley (9), p. 213) is also continuous, i.e. $\gamma = \tau$ and Φ is τ-exhaustive.

Remark. If E has the property that the continuous linear operators with separable range separate points then a topology ρ can also be found to satisfy the conditions of Theorem 4.3.

Next we mention two other applications. Our first result generalises a theorem of Diestel (3).

Theorem 4.4. *Let E be a separable locally bounded F-space, and let $\mathscr{S}$ be a σ-algebra of subsets of a set S. Let μ: $\mathscr{S} \to E$ be a bounded (finitely-additive) measure. Then μ is exhaustive.*

Note. A measure μ is called exhaustive or strongly bounded if for any sequence (S_n) of disjoint sets $\mu(S_n) \to 0$.

Proof. Since E is locally bounded, the topology may be given by a p-norm $\|.\|$ where $0 < p \leqq 1$. Let

$$\sup_{S \in \mathscr{S}} \| \mu(S) \| = \theta.$$

We use a technique due to Robertson (**15**); μ may be extended to a linear map Φ_0 on the simple functions $\Sigma(\mathscr{S})$ on $\mathscr{S}$. Then $\Sigma(\mathscr{S})$ is a normed space under

$$\| f \| = \sup_{s \in S} | f(s) |;$$

suppose $f \in \Sigma(\mathscr{S})$ and $\| f \|_\infty \leqq 1$. Then

$$\Phi_0(f) = \sum_{=1}^{\infty} 2^{-i}(\mu(S_i) - \mu(T_i))$$

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where S_i, $T_i \in \mathscr{S}$ (only finitely many of S_i, T_i are distinct). Then

$$\| \Phi_0(f) \| \leqq 2\theta \sum_{i=1}^{\infty} 2^{-ip} \leqq \frac{\theta 2^{1+p}}{1-2^{-p}}.$$

Thus Φ_0: $\Sigma(\mathscr{S}) \to E$ is continuous and extends to a continuous operator Φ: $B(\mathscr{S}) \to E$ where $B(\mathscr{S})$ in the space of bounded $\mathscr{S}$-measurable functions on S. Using the techniques of Theorem 4.2 it follows that if (S_n) is a disjoint sequence in $\mathscr{S}$, $\mu(S_n) = \Phi(\chi_{S_n}) \to 0$ where χ_{S_n} is the characteristic of S_n. (Alternatively $B(\mathscr{S})$ is isometrically isomorphic to $C(T)$ where T is σ-Stonian.)

Clearly the preceding theorem generalises to semi-convex topological vector spaces (i.e. spaces which can be embedded in a product of locally bounded spaces).

A C-series is a sequence x_n in a topological vector space such that $\Sigma t_n x_n$ converges whenever $t_n \to 0$. If E is a space such that every C-series converges, then E is called a C-space (Schwartz (**17**), Thomas (**18**)). Clearly Theorem 2.3 yields

Theorem 4.4. *A complete topological vector space is a C-space if and only if it contains no subspace isomorphic to* c_0.

A topological vector space is said to have property (O) (Orlicz (**13**), Labuda (**10**)) if every series Σx_n in E such that the set $\left\{ \sum_{n \in \Delta} x_n \colon \Delta \subset \mathrm{N}, \Delta \text{ finite} \right\}$ is bounded, is also convergent. Again by Theorem 2.3 and a similar argument to Theorem 4.3 we conclude

Theorem 4.5. *A complete semi-convex topological vector space E has property* (O) *if and only if E contains no subspace isomorphic to* c_0.

REFERENCES

(1) M. G. Arsove and R. E. Edwards, Generalised bases in topological linear spaces, *Studia Math.* **19** (1960), 95-113.

(2) C. Bessaga and A. Pełczyński, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **17** (1958), 151-164.

(3) J. Diestel, Applications of bases and weak compactness to vector measures and vectorial integration, *Rev. Roum. Math. Pures et Appl.* **18** (1973), 211-224.

(4) L. Drewnowski, Another note on Kalton's theorem (to appear).

(5) N. Dunford and J. T. Schwartz, *Linear Operators*, Vol. I (Interscience, New York, 1958).

(6) A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type C(K), *Canad. J. Math.* **5** (1953), 129-173.

(7) N. J. Kalton, Topologies on Riesz groups and applications to measure theory, *Proc. London Math. Soc.* (3) **18** (1974), 253-273.

https://doi.org/10.1017/S0013091500015571 Published online by Cambridge University Press

(8) N. J. KALTON, Basic sequences in F-spaces and their applications, *Proc. Edinburgh Math. Soc.* **19** (1974), 151-167.

(9) J. L. KELLEY, *General Topology* (van Nostrand, New York, 1955).

(10) G. KÖTHE, *Topological Vector Spaces* (Springer, Berlin, 1969).

(11) I. LABUDA, Sur quelques théorèmes du type d'Orlicz-Pettis I, *Bull. Acad. Polon. Sci.* **21** (1973), 127-132.

(12) C. W. MCARTHUR, On a theorem of Orlicz and Pettis, *Pacific J. Math.* **22** (1967), 297-302.

(13) W. ORLICZ, Absolute continuity of set functions with respect to a finitely subadditive measure, *Prace Mat.* **14** (1970), 101-118.

(14) A. PELCZYŃSKI, Banach spaces on which every unconditionally converging operator is weakly compact, *Bull. Acad. Polon. Sci.* **10** (1962), 641-648.

(15) A. P. ROBERTSON, On unconditional convergence in topological vector spaces, *Proc. Roy. Soc. Edinburgh, Sect.* A **68** (1969), 145-157.

(16) H. P. ROSENTHAL, On relatively disjoint families of measures, with some applications to Banach space theory. *Studia Math.* **37** (1971), 13-36.

(17) L. SCHWARTZ, Un théorème de la convergence dans les espaces L^p, $0 \leqq p < \infty$, *C.R. Acad. Sci. Paris, Ser.* A **268** (1969), 704-706.

(18) E. THOMAS, Sur le théorème d'Orlicz et un problème de M. Laurent Schwartz, *C.R. Acad. Sci. Paris, Ser.* A. **267** (1968), 7-10.

(19) R. J. WHITLEY, Projecting m onto c_0, *Amer. Math. Monthly* **73** (1966), 285-286.

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