SOME USEFUL MATRIX LEMMAS IN STATISTICAL ESTIMATION THEORY *

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In this note, we present two matrix lemmas (one without proof) which have interesting applications in statistical estimation theory.

LEMMA 1. Let A be a $k \times k$ positive definite matrix. Then for any $k \times 1$ vector c, we have that

$$(1) \qquad (c' A c)(c' A^{-1} c) \geq (c' c)^2 .$$

Proof. Since A is assumed positive definite, the quadratic form $y' A y$ is non-negative for all y. In particular, by setting $y = c + \alpha A^{-1} c$, where α is any scalar; we then have

$$(c + \alpha A^{-1} c)' A (c + \alpha A^{-1} c) \geq 0 \qquad \text{for all } \alpha.$$

This can be written as

$$c' A c + 2\alpha c' c + \alpha^2 c' A^{-1} c \geq 0 \qquad \text{for all } \alpha.$$

Hence, $(c' c)^2 \leq (c' A c)(c' A^{-1} c)$ Q. E. D.

This lemma has an application in statistical estimation theory. Consider sampling from a population with density function $p(x;\theta_1, \theta_2, \ldots, \theta_k)$. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of n independent observations. Further, let

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$T' = (t_1, t_2, \ldots, t_k)$ where $t_i = t_i(X_1, X_2, \ldots, X_n)$ be an unbiased vector for $\theta' = (\theta_1, \theta_2, \ldots, \theta_k)$. We shall denote the covariance matrix of T by V and the covariance matrix of $\left(\frac{\partial \ell np}{\partial \theta_1}, \ldots, \frac{\partial \ell np}{\partial \theta_k}\right)$ by I_θ. We make the usual assumption that I_θ is positive definite. It is well known that the matrix $V - \frac{1}{n} I_\theta^{-1}$ is positive semi-definite (see Kendall and Stuart [4] and Box [2]).

Suppose we wish to find an unbiased estimator of a specific linear combination of θ, say $c'\theta$. One such unbiased estimator is $W = c' T$. This estimator has variance $\sigma_w^2 = c' V c$ and clearly, from the preceeding paragraph, $\sigma_w^2 \geq \frac{1}{n} c' I_\theta^{-1} c$. Applying Lemma 1, we have

$$(2) \qquad \sigma_w^2 = c' V c \geq \frac{1}{n} c' I_\theta^{-1} c \geq \frac{[c' c]^2}{n c' I_\theta c} .$$

This is an interesting result since the extreme right hand quantity in (2) has been claimed in Statistical literature as the greatest lower bound for σ_w^2 (see for example Wilks [6]).

LEMMA 2. Let A be a $k \times k$ matrix. If B is the $(k-2)$-rowed minor obtained from A by deleting the r^{th} and s^{th} rows and the t^{th} and u^{th} columns, then

$$(3) \qquad \begin{vmatrix} \alpha_{rt} & \alpha_{st} \\ \alpha_{ru} & \alpha_{su} \end{vmatrix} = (-1)^{r+t+s+u} |B| \, |A| ,$$

where α_{ij} is the cofactor of the i-jth element in A. For a proof, see Browne [3].

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Again, this lemma has an interesting application in Statistics. As before, let X have density $p(x;\theta)$ and $(X_1, X_2, \ldots, X_n)$ be a random sample of n independent observations on X. Let $d = d(X_1, \ldots, X_n)$ be an unbiased estimator of θ, and $s_1, \ldots, s_k, \ldots$ be a set of linearly independent statistics, where $s_i = s_i(X_1, \ldots, X_n)$, and

$$\operatorname{cov}(d, s_1) = 1, \quad \operatorname{cov}(d, s_j) = 0, \quad j \neq 1 .$$

Further, denote the covariance matrix of $(s_1, \ldots, s_k)$ by A. Let

$$L_k = \frac{\alpha_{11}}{|A|} ;$$

this is known as the k^{th} Bhattacharya bound (see [1] and [5]). Making use of Lemma (2), we now give a new proof that the set of Bhattacharya bounds L_k, $k = 1, 2, \ldots$ is non-decreasing. That is, we wish to show that

$$L_k - L_{k-1} \geq 0 .$$

We note, first of all, that

$$L_k - L_{k-1} = \frac{\alpha_{11}}{|A|} - \frac{|B|}{\alpha_{kk}} = \frac{\alpha_{11}\,\alpha_{kk} - |B|\,|A|}{\alpha_{kk}|A|} ,$$

where B is the $(k-2) \times (k-2)$ matrix obtained by deleting the 1^{st} and k^{th} rows and columns of A. It follows from Lemma (2) that

$$(4) \qquad L_k - L_{k-1} = \frac{\alpha_{1k}\,\alpha_{k1}}{\alpha_{kk}|A|} = \frac{\alpha_{1k}^2}{\alpha_{kk}|A|} \geq 0 ,$$

since α_{kk} and $|A|$ are determinants of covariance matrices of random variables.

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We remark that this result is useful in estimation theory when using Bhattacharya bounds for finding the greatest lower bounds for variances of unbiased estimators of θ. We further note that since $L_k = \sigma_d^2 \rho^2_{d.s_1 \ldots s_k}$, where $\rho^2_{d.s_1 \ldots s_k}$ is the multiple correlation coefficient between d and $s_1 \ldots s_k$ (see Lehmann [5]), we have that

$$L_k - L_{k-1} = \sigma_d^2 (\rho^2_{d.s_1 \ldots s_k} - \rho^2_{d.s_1 \ldots s_{k-1}}) .$$

Hence, from the result given in (4), we have a somewhat simple proof of the fact that the set of multiple correlation coefficients is non-decreasing.

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