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ON EQUAL PRODUCTS OF CONSECUTIVE INTEGERS

BY

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Introduction. Using the theory of algebraic numbers, Mordell [1] has shown that the Diophantine equation

$$(1) \qquad n(n+1) = m(m+1)(m+2)$$

possesses only two solutions in positive integers; these are given by $n=2$, $m=1$, and $n=14$, $m=5$. We are interested in positive integer solutions to the generalized equation

$$(2) \qquad n(n+1)\dots(n+k-1) = m(m+1)\dots(m+l-1)$$

and in this paper we prove for several choices of k and l that (2) has no solutions, in other cases the only solutions are given, and numerical evidence for all values of k and l for which $\max(k, l) \le 15$ is also exhibited.

In what follows it is always assumed that $k<l$. In addition to the trivial solution $n=2$, $m=1$ for all $k=l-1$, there is also the infinite class of solutions given by

$$n(n+1)\dots(n+t)(n+t+1)\dots\left(\prod_{i=0}^{t}(n+i)-1\right) = (n+t+1)\dots\left(\prod_{i=0}^{t}(n+i)-1\right)\left(\prod_{i=0}^{t}(n+i)\right)$$

where $k=\prod_{i=0}^{t}(n+i)-n$ and $l=(n+t)\left(\prod_{i=0}^{t-1}(n+i)-1\right)$ for any positive integers n and t, where $n+t\ge 3$. However, with the exception of Theorem 2, all our results are for small specific values of k and l, and we shall ignore the trivial solution $n=2$, $m=1$ for any $k=l-1$.

RESULTS. The simple idea used in the proof of Theorem 1 can be used with some modification to handle several other cases.

THEOREM 1. *Equation* (2) *has no positive integer solutions for* $k=2$, $l=4$.

Proof. Assume that a solution does exist and put $q=m^2+3m$, then

$$(3) \qquad n(n+1) = m(m+1)(m+2)(m+3) = (m^2+3m)(m^2+3m+2) = q(q+2)$$

Now $n^2<n(n+1)<(n+1)^2$ and $q^2<q(q+2)<(q+1)^2$ and since for equation (3) we have $[\sqrt{n(n+1)}]=[\sqrt{q(q+2)}]$, where $[r]$ denotes the integer part of r, it follows that $n=q$ which is clearly impossible, unless $n=q=0$.

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THEOREM 2. *Equation* (2) *has at most a finite number of positive integer solutions for* $l=2k$.

Proof. Assume that a solution does exist, thus

$$n(n+1)\ldots(n+k-1) = m(m+1)\ldots(m+2k-1) \tag{4}$$

The proof consists essentially of obtaining expressions for $[\sqrt[k]{\text{L.S.}}]$ and $[\sqrt[k]{\text{R.S.}}]$ and equating them. Clearly,

$$\text{L.S.} = n^k+\left(\sum_{r_1=1}^{k-1} r_1\right) n^{k-1}+\left(\sum_{r_1=1}^{k-1}\sum_{r_2=1}^{r_1-1} r_1r_2\right) n^{k-2}$$
$$+\left(\sum_{r_1=1}^{k-1}\sum_{r_2=1}^{r_1-1}\sum_{r_3=1}^{r_2-1} r_1r_2r_3\right) n^{k-3}+\cdots+(k-1)!n$$

Now

$$\sum_{r_1=1}^{k-1} r_1 = \frac{k(k-1)}{2},$$

and since

$$1^s+2^s+\cdots+a^s < \int_0^a (x+1)^s\,dx < \frac{(a+1)^{s+1}}{s+1}, \quad \text{for } s > 0,$$

which implies

$$\sum_{r_1=1}^{k-1}\sum_{r_2=1}^{r_1-1}\cdots\sum_{r_t=1}^{r_{t-1}-1} r_1r_2\ldots r_t < \frac{k^{2t}}{2^t t!},$$

then

$$\text{L.S.} < n^k+\frac{k(k-1)}{2}n^{k-1}+\sum_{i=2}^{k-1}\frac{k^{2i}}{2^i i!}n^{k-i}.$$

Since

$$1^s+2^s+\cdots+a^s > \int_0^a x^s\,dx = \frac{a^{s+1}}{s+1}, \quad \text{for } s > 0,$$

it follows that

$$\text{L.S.} > n^k+\frac{k(k-1)}{2}n^{k-1}+\sum_{i=2}^{k-1}\frac{(k-i)^{2i}}{2^i i!}n^{k-i}. \tag{5}$$

Now applying the arithmetic geometric mean inequality to the left side of (4), we get

$$\sqrt[k]{\text{L.S.}} < n+\frac{k-1}{2}.$$

On the other hand, from (5) we have for $n>n_0$, $\sqrt[k]{\text{L.S.}}>n+(k-2)/2$. Thus, for $n>n_0$,

$$[\sqrt[k]{\text{L.S.}}] = \left[n+\frac{k-2}{2}\right] = n+\left[\frac{k-2}{2}\right].$$

Writing $u=m(m+2k-1)$, then (4) becomes

$$(6)\qquad \prod_{i=0}^{k-1}(n+i) = \prod_{j=0}^{k-1}(u+j(2k-j-1)).$$

Now

$$\sum_{j=0}^{k-1} j(2k-j-1) = \frac{k(k-1)(2k-1)}{3}$$

and so for (6),

$$(7)\qquad \text{R.S.} = u^k+\frac{k(k-1)(2k-1)}{3}u^{k-1}+\cdots+\prod_{j=0}^{k-1} j(2k-j-1).$$

Again, by the arithmetic–geometric mean inequality,

$$\sqrt[k]{\text{R.S.}} < u+\frac{(k-1)(2k-1)}{3},$$

while by (7), for $n>n_1$,

$$\sqrt[k]{\text{R.S.}} > u+\frac{(k-1)(2k-1)-1}{3}.$$

Thus for $n>n_1$,

$$[\sqrt[k]{\text{R.S.}}] = \left[u+\frac{(k-1)(2k-1)-1}{3}\right] = u+\left[\frac{k(2k-3)}{3}\right].$$

Let $n_2=\max(n_0, n_1)$. Then for $n>n_2$,

$$u+\left[\frac{k(2k-3)}{3}\right] = n+\left[\frac{k-2}{2}\right], \quad\text{or}\quad u = n+\left[\frac{k-2}{2}\right]-\left[\frac{k(2k-3)}{3}\right].$$

Putting this back in (6) we have an equation in n of degree $k-1$, with at most $k-1$ positive integer roots. Thus (4) has at most a finite number of solutions (for each k) in positive integers.

THEOREM 3. *Equation* (2) *has no positive integer solutions for* $k=2$, $l=8$.

Proof. Assume that a solution does exist and put $u=m(m+7)$, then

$$\begin{aligned} n(n+1) &= m(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)(m+7) \\ (8)\qquad &= u(u+6)(u+10)(u+12) \\ &= u^4+28u^3+252u^2+720u. \end{aligned}$$

Now,

$$(u^2+14u+27)^2 = u^4+28u^3+250u^2+756u+729,$$

thus for (8),

$$\text{R.S.} > (u^2+14u+27)^2 \quad\text{when } u > 30, \text{ or } m > 3.$$

Also,

$$(u^2+14u+28)^2 = u^4+28u^3+252u^2+784u+784 > \text{R.S.}$$

Hence for $u>30$, $[\sqrt{\text{R.S.}}]=u^2+14u+27$ and since $[\sqrt{\text{L.S.}}]=n$ it follows that $n=u^2+14u+27$. Substituting this last expression into (8) the equation reduces to $u^2-50u-756=0$ which has no integer solutions. Finally we note that $m=1$, 2, and 3 yield no integer solutions either.

THEOREM 4. *Equation* (2) *has just one positive integer solution for* $k=3$, $l=6$.

Proof. Assume that a solution does exist and put $u=m(m+5)$, then

$$
\begin{aligned}
n(n+1)(n+2) &= m(m+1)(m+2)(m+3)(m+4)(m+5)\\
&= u(u+4)(u+6) \qquad (9)\\
&= u^3+10u^2+24u
\end{aligned}
$$

Now,

$$(u+3)^3 = u^3+9u^2+27u+27$$

thus for (9)

$$\text{R.S.} > (u+3)^3 \quad \text{when } u > 7, \text{ or } m > 1.$$

Also,

$$(u+4)^3 = u^3+12u^2+48u+64 > \text{R.S.}$$

Hence for $u>7$, $[\sqrt[3]{\text{R.S.}}]=u+3$.

Now, $n^3<\text{L.S.}<(n+1)^3$ and so $[\sqrt[3]{\text{L.S.}}]=n$. For $u>7$ it follows that $n=u+3$, and substituting this expression into (9) the equation reduces to $2u+15=0$ which has no integer solutions. We note that $m=1$ yields $8\cdot 9\cdot 10=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6$, the only integer solution.

Similarly we can prove:

THEOREM 5. *Equation* (2) *has no positive integer solutions for* $k=4$, $l=8$.

THEOREM 6. *Equation* (2) *has no positive integer solutions for* $k=5$, $l=10$.

Since further progress in this direction is limited, we return to the case where $k=2$ for the next two results.

THEOREM 7. *Equation* (2) *has no positive integer solutions for* $k=2$, $l=6$.

THEOREM 8. *Equation* (2) *has no positive integer solutions for* $k=2$, $l=12$.

Both of these theorems are proved by first taking congruences and then proceeding along the above lines: we shall omit the details here.

In addition to the trivial solutions and the infinite class of solutions noted in the Introduction, the following solutions have been found by computer. These results represent an exhaustive search for all possible solutions in the range defined by $\max(k, l)\le 10$ where $\max(n, m)\le 10{,}000$, and $\max(k, l)\le 15$ where $\max(n, m)\le 1{,}000$.

$k=2, l=3$	$14\cdot 15 = 5\cdot 6\cdot 7$
$k=3, l=4$	$4\cdot 5\cdot 6 = 2\cdot 3\cdot 4\cdot 5$
	$55\cdot 56\cdot 57 = 19\cdot 20\cdot 21\cdot 22$

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$k = 3, l = 5$	$4\cdot5\cdot6 = 1\cdot2\cdot3\cdot4\cdot5$
	$8\cdot9\cdot10 = 2\cdot3\cdot4\cdot5\cdot6$
$k = 3, l = 6$	$8\cdot9\cdot10 = 1\cdot2\cdot3\cdot4\cdot5\cdot6$
$k = 4, l = 6$	$7\cdot8\cdot9\cdot10 = 2\cdot3\cdot4\cdot5\cdot6\cdot7$
$k = 4, l = 7$	$7\cdot8\cdot9\cdot10 = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7$
	$63\cdot64\cdot65\cdot66 = 8\cdot9\cdot10\cdot11\cdot12\cdot13\cdot14$
$k = 8, l = 9$	$5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12 = 3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11$
	$8\cdot9\cdot10\cdot11\cdot12\cdot13\cdot14\cdot15 = 5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13.$

We finish with a conjecture for which some supporting evidence has been provided in the results above.

CONJECTURE. Equation (2) has no positive integer solutions for $k=2$, $l\geq 4$, i.e. the product of two consecutive positive integers is never equal to the product of four or more consecutive positive integers.

REFERENCE

1. L. J. Mordell, *On the integer solutions of* $y(y+1)=x(x+1)(x+2)$, Pacific J. Math. **13B** (1963), 1347–1351.

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