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On the Dispersive Estimate for the Dirichlet Schrödinger Propagator and Applications to Energy Critical NLS

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Abstract. We consider the obstacle problem for the Schrödinger evolution in the exterior of the unit ball with Dirichlet boundary condition. Under radial symmetry we compute explicitly the fundamental solution for the linear Dirichlet Schrödinger propagator $e^{it\Delta_D}$ and give a robust algorithm to prove sharp $L^1 \to L^\infty$ dispersive estimates. We showcase the analysis in dimensions $n = 5, 7$. As an application, we obtain global well-posedness and scattering for defocusing energy-critical NLS on $\Omega = \mathbb{R}^n \setminus \overline{B(0,1)}$ with Dirichlet boundary condition and radial data in these dimensions.

1 Introduction

In this paper, we consider the obstacle problem for the Schrödinger equation in the exterior of the unit ball. Let $\Omega = \mathbb{R}^n \setminus \overline{B(0,1)} = \{x \in \mathbb{R}^n : |x| > 1\}$. We are concerned with the following defocusing energy-critical NLS in Ω:

$$
\begin{gathered}
i\partial_t u + \Delta u = |u|^{\frac{4}{n-2}} u := F(u), \quad (t,x) \in \mathbb{R} \times \Omega, \\
u(t,x)|_{\mathbb{R}\times\partial\Omega} = 0, \quad u(0,x) = u_0(x).
\end{gathered} \tag{1.1}
$$

Here the dimension n is at least 3, and we assume the initial data satisfies $u_0 \in \dot{H}^1_0(\Omega)$. Equation (1.1) has a natural conserved energy

$$
E(u(t)) := \int_\Omega \Big(\frac{1}{2} |\nabla u(t,x)|^2 + \frac{n-2}{2n} |u(t,x)|^{\frac{2n}{n-2}} \Big)\, dx,
$$

and the name "defocusing" corresponds to the "+" sign in the above expression. The special "energy-critical" nonlinearity $F(u) = |u|^{\frac{4}{n-2}} u$ comes from a certain scaling analogy when we consider the general problem

$$
i\partial_t u + \Delta u = |u|^p u
$$

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posed on the whole space $\mathbb{R}\times\mathbb{R}^n$. In that situation, $p=\frac{4}{n-2}$ is the unique exponent such that the scaling transformation

$$\lambda \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

leaves the energy

$$E\big(u(t)\big) = \int_{\mathbb{R}^n} \Big(\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+2}|u(t,x)|^{p+2} \Big)\ dx$$

invariant. Of course, the scaling symmetry breaks down in the obstacle problem (1.1), but we will adopt the same terminologies as in the whole space case.

The energy-critical NLS has a long history and has been intensively studied in the last decade or so. Starting from Bourgain's breakthrough work [6], the large data theory for the Cauchy problem on $\mathbb{R}\times\mathbb{R}^n$, $n\geq 3$ has been successfully worked out in both focusing ($F(u) = -|u|^{\frac{4}{n-2}}u$) and defocusing cases [8, 17, 20]. At the time of this writing, the only unsolved case is the focusing problem in dimensions $d=3,4$ for general nonradial data. For the obstacle problem, the understanding of energy-critical NLS posed on exterior domains is still quite unsatisfactory. Roughly speaking, the main difficulty comes from two aspects. First of all, at the linear level, the $L^1 \to L^\infty$ dispersive estimate for the obstacle problem is difficult to establish in general, and the space-time Strichartz estimates are often more limited than the usual Euclidean case (cf. [4, 5]). Secondly, concerning the nonlinear evolution, the frequency analysis is much more involved, and many technical tools have to be rebuilt for the obstacle case due to lack of translation invariance or scale invariance. For Strichartz estimates of wave and Schrodinger propagators on exterior domains or more general Riemannian manifolds, we refer to [1–3, 11, 12, 18] and references therein.

In the recent work [16], the authors made a first step and proved the global well-posedness and scattering of (1.1) under the radial assumption in dimension $n=3$. It was first noticed that in the obstacle case, Sobolev spaces defined via the usual Laplacian Δ and the Dirichlet Laplacian Δ_D are not always equivalent unless one works with $L^p(\mathbb{R}^n)$, $1<p<n$. For $p\geq n$, counterexamples can be constructed by modifying the eigenfunctions of the Dirichlet Laplacian. This is the first evidence of the subtle difference between the obstacle case and the whole space case. In a subsequent paper [14], Killip, Visan, and Zhang proved the equivalence between two Sobolev spaces and established general harmonic analysis tools in the general non-radial setting on the exterior domain of a strictly convex obstacle. In a later work, Killip, Visan and Zhang [15] settled energy-critical NLS in the case $n=3$ for general non-radial data outside a strictly convex obstacle. The case $n=4$ is announced in the preprint [9]. The general case in high dimensions still remains open at the time of this writing. The purpose of this work is to extend the analysis of [16] to general dimensions under the radial assumption. In light of the approach in [16], one of the main technical issues in high dimensions is to prove the sharp $L^1\to L^\infty$ dispersive estimate. In dimension $n=3$, the situation is fairly simple, as the radial Dirichlet Laplacian has eigenfunctions of the form

$$\phi_\lambda(r) = \frac{\sin\lambda(r-1)}{r},\ r>1, \tag{1.2}$$

which solve the equation

$$\Delta_D\phi_\lambda + \lambda^2\phi_\lambda = 0.$$

The Dirichlet Schrödinger propagator can then be written as

$$(e^{it\Delta_D})(r,s,t) = \text{Const}\cdot\int_0^\infty \frac{\sin\lambda(r-1)}{r}\frac{\sin\lambda(s-1)}{s}e^{-i\lambda^2 t}\,d\lambda,\qquad r,s>1.$$

Comparing the above expression with the usual radial Schrödinger propagator on $\mathbb{R}\times\mathbb{R}^3$,

$$\left(e^{it\Delta_{rad}}\right)(r,s,t) = \text{Const}\cdot\int_0^\infty \frac{\sin\lambda r}{r}\frac{\sin\lambda s}{s}e^{-i\lambda^2 t}\,d\lambda,\qquad r,s>0,$$

one quickly finds that

$$\begin{aligned}\left|(e^{it\Delta_D})(r,s,t)\right| &= \left|\frac{(r-1)(s-1)}{rs}(e^{it\Delta_{rad}})(r-1,s-1,t)\right|\\ &\lesssim \left|(e^{it\Delta_{rad}})(r-1,s-1,t)\right| \lesssim |t|^{-3/2},\qquad r,s>1,\ t\neq 0.\end{aligned}$$

Here we have invoked the usual dispersive estimate for the radial Schrödinger propagator $e^{it\Delta_{rad}}$.

In dimension $n\geq 4$, the eigenfunctions no longer have the simple form (1.2). They are certain combinations of Bessel functions with suitable normalizations.

The main body of this work is to give a robust construction that computes the sharp dispersive bounds for general dimensions. To showcase the theory, we carry out the explicit computations in dimensions $n=5$ and $n=7$. In some sense, dimension $n=5$ is the first case that requires a qualitatively different computation than $n=3$. On the other hand, dimension $n=7$ is most representative of all dimensions and has some important differences from $n=3,5$. Besides the cumbersome numerology, the general case for $n\geq 4$ requires some careful combinatorics that we plan to address in the future. The results obtained in this work are the following.

Proposition 1.1 (Dispersive estimate) *For $t\neq 0$ and under the radial assumption we have for dimensions $n=5,7$,*

$$|(e^{it\Delta_D})(r,s,t)| \leq C_n|t|^{-\frac{n}{2}},\qquad r,s>1,$$

where C_n is some constant depending on the dimension.

Using Proposition 1.1 and an argument from [6, 16, 19], we obtain the following theorem.

Theorem 1.2 *Let $n=5,7$ and $\Omega=\mathbb{R}^n\setminus\overline{B(0,1)}$. Let $u_0\in\dot H^1_0(\Omega)$ be spherically symmetric. Then there exists a unique solution $u\in C^0_t\dot H^1_0$ to (1.1), and*

$$\|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(\mathbb{R}\times\Omega)} \leq C\left(\|u_0\|_{\dot H^1_0(\Omega)}\right).$$

Moreover, there exist unique radial functions $v_\pm\in\dot H^1_0(\Omega)$ such that

$$\lim_{t\to\pm\infty}\|u(t)-e^{it\Delta_D}v_\pm\|_{\dot H^1_0(\Omega)}=0.$$

Here, Δ_D is the Dirichlet Laplacian, and $e^{it\Delta_D}$ is the free propagator.

Notation

We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some harmless constant $C > 0$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. For any positive number $1 \leq a \leq \infty$, we let $a' = a/(a-1)$ denote the conjugate of a, so that $1/a + 1/a' = 1$.

We will use the notation $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. Of course the implied constants will be clear from the context.

Let $I \subset \mathbb{R}$ be a time interval. We write $L_t^q L_x^r(I \times \Omega)$ to denote the Banach space endowed with the norm

$$\|u\|_{L_t^q L_x^r(I\times\Omega)} := \left(\int_I \left(\int_\Omega |u(t,x)|^r \, dx \right)^{q/r} dt \right)^{1/q},$$

with the usual modifications when q or r are equal to infinity. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$. We shall write $u \in L_{t,loc}^q L_x^r(I \times \Omega)$ if $u \in L_t^q L_x^r(J \times \Omega)$ for any compact $J \subset I$.

The rest of this paper is organized as follows. In Sections 2 and 3, we prove the dispersive estimate for dimensions $n = 5$ and 7 respectively. In Section 4, we recall some basic facts about the Littlewood–Paley operators, Bernstein inequalities, and L^p-based Sobolev spaces on exterior domains. In Section 5, we complete the nonlinear analysis and finish the proof of Theorem 1.2.

2 Fundamental Solution and the Dispersive Estimate for $n = 5$

In dimension $n = 5$, we consider the radial eigenfunctions of the Dirichlet Laplacian on the exterior domain $\overline{B(0,1)}^c$:

$$\Delta\phi_\lambda + \lambda^2\phi_\lambda = 0, \quad \lambda > 0.$$

Using the Sommerfeld radiation condition, $\phi_\lambda(r)$ is given by

$$\phi_\lambda(r) = \frac{1}{r^2}\left(\frac{\sin(\lambda(r-1)+a)}{\lambda r} - \cos(\lambda(r-1)+a) \right).$$

Here $a \in [0, \pi/2)$ satisfies

$$\sin a = \frac{\lambda}{\sqrt{1+\lambda^2}}, \quad \cos a = \frac{1}{\sqrt{1+\lambda^2}}. \tag{2.1}$$

To show that $\phi_\lambda(r)$ constitute a complete basis, we first prove the following resolution of identity.

Lemma 2.1

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi_\lambda(r)\phi_\lambda(s)\, d\lambda = \frac{1}{r^4}\delta(r-s), \quad r, s > 1.$$

Proof Using the elementary trigonometric identities, we write

$$\phi_\lambda(r)\phi_\lambda(s) = \frac{1}{2r^2s^2}\sum_{j=1}^{4} I_j,$$

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where

$$I_1(r,s) = \frac{1}{rs}\frac{\cos(\lambda(r-s)) - \cos(\lambda(r+s-2)+2a)}{\lambda^2},$$
$$I_2(r,s) = -\frac{1}{s}\frac{-\sin(\lambda(r-s)) + \sin(\lambda(r+s-2)+2a)}{\lambda},$$
$$I_3(r,s) = -\frac{1}{r}\frac{\sin(\lambda(r-s)) + \sin(\lambda(r+s-2)+2a)}{\lambda},$$
$$I_4(r,s) = \cos\big(\lambda(r-s)\big) + \cos\big(\lambda(r+s-2)+2a\big).$$

Observe that

$$-sI_2 = \partial_r(rsI_1), \quad -rI_3 = \partial_s(rsI_1), \quad I_4 = \partial_{rs}(rsI_1).$$

Now define

$$F_1(r,s) = \int_{-\infty}^{\infty} \frac{\cos(\lambda(r-s)) - \cos(\lambda(r+s-2)+2a)}{\lambda^2}\, d\lambda, \tag{2.2}$$

and we can write

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi_\lambda(r)\phi_\lambda(s)\, d\lambda = \frac{1}{2\pi r^2 s^2}\Big(\frac{1}{rs}F_1 - \frac{1}{s}\partial_r F_1 - \frac{1}{r}\partial_s F_1 + \partial_{rs}F_1\Big). \tag{2.3}$$

We first compute (2.2). Indeed, using (2.1) we expand (2.2) as

$$\begin{aligned} F_1(r,s) &= \int_{-\infty}^{\infty} \frac{\cos(\lambda(r-s)) - \cos(\lambda(r+s-2))\frac{1-\lambda^2}{1+\lambda^2} + \sin(\lambda(r+s-2))\frac{2\lambda}{1+\lambda^2}}{\lambda^2}\, d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\cos(\lambda(r-s)) - 1}{\lambda^2}\, d\lambda + \int_{-\infty}^{\infty} \frac{1-\cos(\lambda(r+s-2))}{\lambda^2}\, d\lambda \\ &\quad + 2\int_{-\infty}^{\infty} \frac{\cos(\lambda(r+s-2))}{1+\lambda^2}\, d\lambda + 2\int_{-\infty}^{\infty} \frac{\sin(\lambda(r+s-2))}{\lambda(1+\lambda^2)}\, d\lambda. \end{aligned}$$

We need to use the following basic integral identities for $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \frac{\cos(\lambda x) - 1}{\lambda^2}\, d\lambda = -\pi|x|, \quad \int_{-\infty}^{\infty} \frac{\cos(\lambda x)}{1+\lambda^2}\, d\lambda = \pi e^{-|x|},$$
$$\int_{-\infty}^{\infty} \frac{\sin(\lambda x)}{\lambda(1+\lambda^2)}\, d\lambda = \pi\int_0^x e^{-|y|}\, dy = \pi\big(1 - e^{-|x|}\big)\,\mathrm{sign}(x).$$

Here $\mathrm{sign}(x)$ is given by

$$\mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Noting that $r, s > 1$, we have

$$F_1(r,s) = \pi\big(-|r-s| + (r+s-2) + 2\big) = \pi\big(-|r-s| + r + s\big).$$

Plugging this into (2.3), we obtain

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi_\lambda(r)\phi_\lambda(s)\, d\lambda = \frac{1}{r^4}\delta(r-s).$$ ■

Next we prove the dispersive estimate. Recall the definition of Fourier transform:

$$\mathcal{F}_0 f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi_\lambda(r) f(r) r^4 \, dr.$$

Then $\mathcal{F}_0$ is an isometric map from $L^2([1,\infty); r^4 dr)$ to $L^2([0,\infty); d\lambda)$ with the inverse transform given by

$$\mathcal{F}_0^* g(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi_\lambda(r) g(\lambda) \, d\lambda.$$

The fundamental solution can be written as

$$(e^{it\Delta_D} f)(r,t) = \mathcal{F}_0^* \left(e^{-it\lambda^2} \mathcal{F}_0 f \right)(r) = \int_0^\infty K(r,s,t) f(s) s^4 \, ds,$$

where

$$K(r,s,t) = \frac{2}{\pi} \int_0^\infty \phi_\lambda(r) \phi_\lambda(s) e^{-it\lambda^2} \, d\lambda.$$

We will prove the following important decay estimate.

Lemma 2.2

$$\sup_{r,s>1} \big| K(r,s,t) \big| \lesssim |t|^{-5/2}, \quad t \neq 0.$$

Proof Proceeding similarly as in Lemma 2.1, we define

$$G(r,s,t) = \int_0^\infty \frac{\cos(\lambda(r-s)) - \cos(\lambda(r+s-2)+2a)}{\lambda^2} e^{-it\lambda^2} \, d\lambda,$$

and we can write

$$K(r,s,t) = \frac{1}{\pi r^2 s^2} \left(\frac{1}{rs} G - \frac{1}{s} \partial_r G - \frac{1}{r} \partial_s G + \partial_{rs} G \right).$$

To continue, we need to compute the explicit expression of G. Note that

$$\begin{aligned} G(r,s,t) = & \int_0^\infty \frac{\cos(\lambda(r-s)) - 1}{\lambda^2} e^{-it\lambda^2} \, d\lambda \\ & + \int_0^\infty \frac{1 - \cos(\lambda(r+s-2))}{\lambda^2} e^{-it\lambda^2} \, d\lambda \\ & + 2 \int_0^\infty \frac{\cos(\lambda(r+s-2))}{1+\lambda^2} e^{-it\lambda^2} \, d\lambda \\ & + 2 \int_0^\infty \frac{\sin(\lambda(r+s-2))}{\lambda(1+\lambda^2)} e^{-it\lambda^2} \, d\lambda. \end{aligned} \tag{2.4}$$

From the fundamental solution of 1D Schrödinger equation and the Fundamental Theorem of Calculus, we have

$$\int_0^\infty \cos(\lambda x) e^{-it\lambda^2} \, d\lambda = \sqrt{\frac{\pi}{4it}} e^{\frac{ix^2}{4t}},$$

$$\int_0^\infty \frac{\sin(\lambda x)}{\lambda} e^{-it\lambda^2} \, d\lambda = \sqrt{\frac{\pi}{4it}} \int_0^x e^{\frac{iy^2}{4t}} \, dy,$$

$$\int_0^\infty \frac{1 - \cos(\lambda x)}{\lambda^2} e^{-it\lambda^2} \, d\lambda = \sqrt{\frac{\pi}{4it}} \int_0^x \int_0^y e^{\frac{iz^2}{4t}} \, dz dy = \sqrt{\frac{\pi}{4it}} \int_0^x (x-y) e^{\frac{iy^2}{4t}} \, dy,$$

and (here $*$ denotes the usual spatial convolution on $\mathbb{R}$)

$$\begin{aligned} 2\int_0^\infty \frac{\cos(\lambda x)}{1+\lambda^2} e^{-it\lambda^2}\, d\lambda &= \Big(\int_{-\infty}^\infty e^{i\lambda x} e^{-it\lambda^2}\, d\lambda\Big) * \Big(\int_{-\infty}^\infty e^{i\lambda x}\frac{1}{1+\lambda^2}\, d\lambda\Big)(x) \\ &= \sqrt{\frac{\pi}{4it}} \int_{-\infty}^\infty e^{-|y|} e^{\frac{i(x-y)^2}{4t}}\, dy, \\ 2\int_0^\infty \frac{\sin(\lambda x)}{\lambda(1+\lambda^2)} e^{-it\lambda^2}\, d\lambda &= \sqrt{\frac{\pi}{4it}} \int_{-\infty}^\infty \int_0^x e^{-|y|} e^{\frac{i(z-y)^2}{4t}}\, dz dy. \end{aligned}$$

Inserting the above expressions into (2.4), we have

$$\begin{aligned} G(r,s,t) &= \sqrt{\frac{\pi}{4it}} \Big(-\int_0^{r-s} (r-s-y) e^{\frac{iy^2}{4t}}\, dy + \int_0^{r+s-2} (r+s-2-y) e^{\frac{iy^2}{4t}}\, dy \\ &\qquad + \int_{-\infty}^\infty e^{-|y|} e^{\frac{i(r+s-2-y)^2}{4t}}\, dy + \int_{-\infty}^\infty e^{-|y|} \int_0^{r+s-2} e^{\frac{i(z-y)^2}{4t}}\, dz dy \Big) \\ &=: \sqrt{\frac{\pi}{4it}} \sum_{k=1}^4 I_k, \end{aligned}$$

and therefore

$$K(r,s,t) = \frac{1}{\pi r^2 s^2} \sqrt{\frac{\pi}{4it}} \sum_{k=1}^4 \Big(\frac{1}{rs} - \frac{1}{s}\partial_r - \frac{1}{r}\partial_s + \partial_{rs}\Big) I_k. \tag{2.5}$$

We now compute each summand as follows:

$$\begin{aligned} &\Big(\frac{1}{rs} - \frac{1}{s}\partial_r - \frac{1}{r}\partial_s + \partial_{rs}\Big) I_1 \\ &\quad = -\Big(\frac{1}{rs} - \frac{1}{s}\partial_r - \frac{1}{r}\partial_s + \partial_{rs}\Big) \int_0^{r-s} (r-s-y) e^{\frac{iy^2}{4t}}\, dy \\ &\quad = -\frac{r-s}{rs} \int_0^{r-s} e^{\frac{iy^2}{4t}}\, dy + \frac{1}{rs}\int_0^{r-s} y e^{\frac{iy^2}{4t}}\, dy + \frac{1}{s}\int_0^{r-s} e^{\frac{iy^2}{4t}}\, dy \\ &\qquad - \frac{1}{r}\int_0^{r-s} e^{\frac{iy^2}{4t}}\, dy + e^{\frac{i(r-s)^2}{4t}} \\ &\quad = \frac{1}{rs}\frac{2t}{i} e^{\frac{i(r-s)^2}{4t}} + e^{\frac{i(r-s)^2}{4t}} = \Big(\frac{2t}{irs} + 1\Big) e^{\frac{i(r-s)^2}{4t}}. \end{aligned} \tag{2.6}$$

By the same token, we have

$$\begin{aligned} &\Big(\frac{1}{rs} - \frac{1}{s}\partial_r - \frac{1}{r}\partial_s + \partial_{rs}\Big) I_2 \\ &\quad = \Big(\frac{1}{rs} - \frac{1}{s}\partial_r - \frac{1}{r}\partial_s + \partial_{rs}\Big) \int_0^{r+s-2} (r+s-2-y) e^{\frac{iy^2}{4t}}\, dy \\ &\quad = \frac{r+s-2}{rs} \int_0^{r+s-2} e^{\frac{iy^2}{4t}}\, dy - \frac{1}{rs}\int_0^{r+s-2} y e^{\frac{iy^2}{4t}}\, dy - \frac{1}{s}\int_0^{r+s-2} e^{\frac{iy^2}{4t}}\, dy \\ &\qquad - \frac{1}{r}\int_0^{r+s-2} e^{\frac{iy^2}{4t}}\, dy + e^{\frac{i(r+s-2)^2}{4t}} \end{aligned} \tag{2.7}$$

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$$= -\frac{2}{rs}\int_0^{r+s-2} e^{\frac{iy^2}{4t}}\,dy - \frac{2t}{irs}e^{\frac{i(r+s-2)^2}{4t}} + e^{\frac{i(r+s-2)^2}{4t}}$$
$$= e^{\frac{i(r+s-2)^2}{4t}}\Big(1-\frac{2t}{irs}\Big) - \frac{2}{rs}\int_0^{r+s-2} e^{\frac{iy^2}{4t}}\,dy.$$

To compute the contribution from I_3, I_4 we need the following three estimates. First we have

$$(2.8)\quad \int_{-\infty}^{\infty} e^{-|y|}e^{\frac{i(x-y)^2}{4t}}\,dy$$
$$= e^{\frac{ix^2}{4t}}\int_{-\infty}^{\infty} e^{-|y|}e^{-\frac{ixy}{2t}+\frac{iy^2}{4t}}\,dy$$
$$= e^{\frac{ix^2}{4t}}\Big\{\int_{-\infty}^{\infty} e^{-|y|}\,dy + \int_{-\infty}^{\infty} e^{-|y|}\Big(\frac{-ixy}{2t}+\frac{iy^2}{4t}\Big)\,dy\Big\} + O\Big(\frac{x^2+1}{t^2}\Big)$$
$$= e^{\frac{ix^2}{4t}}\Big(2+\frac{i}{t}\Big) + O\Big(\frac{x^2+1}{t^2}\Big).$$

Secondly, we have

$$(2.9)\int_{-\infty}^{\infty} \operatorname{sign}(y)\, e^{-|y|}e^{\frac{i(x-y)^2}{4t}}\,dy$$
$$= e^{\frac{ix^2}{4t}}\int_{-\infty}^{\infty} \operatorname{sign}(y)\, e^{-|y|}e^{-\frac{ixy}{2t}+\frac{iy^2}{4t}}\,dy$$
$$= e^{\frac{ix^2}{4t}}\Big\{\int_{-\infty}^{\infty} \operatorname{sign}(y)\, e^{-|y|}\,dy + \int_{-\infty}^{\infty} \operatorname{sign}(y)\, e^{-|y|}\Big(\frac{-ixy}{2t}+\frac{iy^2}{4t}\Big)\,dy\Big\}$$
$$+ O\Big(\frac{x^2+1}{t^2}\Big)$$
$$= -\frac{i}{t}e^{\frac{ix^2}{4t}}x + O\Big(\frac{x^2+1}{t^2}\Big).$$

Finally, we have

$$(2.10)$$
$$\int_{-\infty}^{\infty} e^{-|y|}\int_0^{x} e^{\frac{i(z-y)^2}{4t}}\,dz dy$$
$$= \int_{-\infty}^{\infty} e^{-|y|}\int_0^{x} e^{\frac{iz^2}{4t}}e^{\frac{iy^2-2izy}{4t}}\,dz dy$$
$$= 2\int_0^{\infty} e^{-|y|}\int_0^{x} e^{\frac{iz^2}{4t}}\,dz dy + \int_{-\infty}^{\infty} e^{-|y|}\int_0^{x} e^{\frac{iz^2}{4t}}\frac{iy^2-2izy}{4t}\,dz dy + O\Big(\frac{x^3+1}{t^2}\Big)$$
$$= 2\int_0^{x} e^{\frac{iz^2}{4t}}\,dz + \frac{i}{t}\int_0^{x} e^{\frac{iz^2}{4t}}\,dz + O\Big(\frac{x^3+1}{t^2}\Big)$$
$$= 2\int_0^{x} e^{\frac{iz^2}{4t}}\,dz + \frac{i}{t}x + O\Big(\frac{x^3+1}{t^2}\Big).$$

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Now, plugging (2.8), (2.9) with $x = r+s-2$, we estimate

(2.11)
$$\begin{aligned}
&\Big(\frac{1}{rs}-\frac{1}{s}\partial_r-\frac{1}{r}\partial_s+\partial_{rs}\Big)I_3\\
&\quad=\Big(\frac{1}{rs}-\frac{1}{s}\partial_r-\frac{1}{r}\partial_s+\partial_{rs}\Big)\int_{-\infty}^{\infty}e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&\quad=\frac{1}{rs}\int_{-\infty}^{\infty}e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy-\Big(\frac{1}{s}+\frac{1}{r}\Big)\int_{-\infty}^{\infty}(-\operatorname{sign}(y))\,e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&\qquad+\int_{-\infty}^{\infty}\big(1-2\delta(y)\big)\,e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&\quad=\Big(1+\frac{1}{rs}\Big)\int_{-\infty}^{\infty}e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy+\Big(\frac{1}{s}+\frac{1}{r}\Big)\int_{-\infty}^{\infty}\operatorname{sign}(y)\,e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&\qquad-2e^{\frac{i(r+s-2)^2}{4t}}\\
&\quad=\Big(1+\frac{1}{rs}\Big)e^{\frac{i(r+s-2)^2}{4t}}\Big(2+\frac{i}{t}\Big)-\Big(\frac{1}{r}+\frac{1}{s}\Big)e^{\frac{i(r+s-2)^2}{4t}}\frac{i}{t}(r+s-2)\\
&\qquad-2e^{\frac{i(r+s-2)^2}{4t}}+O\Big(\frac{r^2+s^2}{t^2}\Big)\\
&\quad=e^{\frac{i(r+s-2)^2}{4t}}\Big\{\Big(1+\frac{1}{rs}\Big)\Big(2+\frac{i}{t}\Big)-\Big(\frac{1}{r}+\frac{1}{s}\Big)(r+s-2)\frac{i}{t}-2\Big\}+O\Big(\frac{r^2+s^2}{t^2}\Big).
\end{aligned}$$

For I_4 we need to use (2.8) through (2.10). This gives

$$\begin{aligned}
(2.12)\quad&\Big(\frac{1}{rs}-\frac{1}{s}\partial_r-\frac{1}{r}\partial_s+\partial_{rs}\Big)I_4\\
&=\frac{1}{rs}\int_{-\infty}^{\infty}e^{-|y|}\int_0^{r+s-2}e^{\frac{i(z-y)^2}{4t}}\,dzdy-\Big(\frac{1}{s}+\frac{1}{r}\Big)\int_{-\infty}^{\infty}e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&\quad-\int_{-\infty}^{\infty}\operatorname{sign}(y)\,e^{-|y|}e^{\frac{i(r+s-2-y)^2}{4t}}\,dy\\
&=\frac{1}{rs}\Big\{2\int_0^{r+s-2}e^{\frac{iz^2}{4t}}\,dz+\frac{i}{t}(r+s-2)\Big\}-\Big(\frac{1}{s}+\frac{1}{r}\Big)e^{\frac{i(r+s-2)^2}{4t}}\Big(2+\frac{i}{t}\Big)\\
&\quad+\frac{i}{t}e^{\frac{i(r+s-2)^2}{4t}}(r+s-2)+O\Big(\frac{r^2+s^2}{t^2}\Big).
\end{aligned}$$

Collecting the estimates (2.6)-(2.12), we get

$$\begin{aligned}
&\sum_{k=1}^{4}\Big(\frac{1}{rs}-\frac{1}{s}\partial_r-\frac{1}{r}\partial_s+\partial_{rs}\Big)I_k\\
&\quad=\Big(1+\frac{2t}{irs}\Big)e^{\frac{i(r-s)^2}{4t}}+e^{\frac{i(r+s-2)^2}{4t}}\Big(1-\frac{2t}{irs}\Big)-\frac{2}{rs}\int_0^{r+s-2}e^{\frac{iy^2}{4t}}\,dy\\
&\qquad+e^{\frac{i(r+s-2)^2}{4t}}\Big(\Big(1+\frac{1}{rs}\Big)\Big(2+\frac{i}{t}\Big)-\Big(\frac{1}{r}+\frac{1}{s}\Big)(r+s-2)\frac{i}{t}-2\Big)\\
&\qquad+\frac{2}{rs}\int_0^{r+s-2}e^{\frac{iz^2}{4t}}\,dz+\frac{i}{t}\frac{r+s-2}{rs}
\end{aligned}$$

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$$
\begin{aligned}
&\quad + e^{\frac{i(r+s-2)^2}{4t}}\Big(-\Big(\frac{1}{s}+\frac{1}{r}\Big)\Big(2+\frac{i}{t}\Big)+\frac{i}{t}\big(r+s-2\big)\Big)\\
&\quad + O\Big(\frac{r^2+s^2}{t^2}\Big)\\
&= \Big(1+\frac{2t}{irs}\Big)e^{\frac{i(r-s)^2}{4t}} + \frac{i}{t}\frac{r+s-2}{rs}\\
&\quad + e^{\frac{i(r+s-2)^2}{4t}}\Big\{1-\frac{2t}{irs}-2+(2+\frac{i}{t})(1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s})\\
&\qquad\qquad + \frac{i}{t}\Big[r+s-2-\Big(\frac{1}{r}+\frac{1}{s}\Big)(r+s-2)\Big]\Big\}\\
&\quad + O\Big(\frac{r^2+s^2}{t^2}\Big)\\
&= e^{\frac{i(r+s-2)^2}{4t}}\Big\{\Big(1+\frac{2t}{irs}\Big)e^{\frac{i(r-s)^2-i(r+s-2)^2}{4t}} + \frac{i}{t}\frac{r+s-2}{rs}e^{\frac{-i(r+s-2)^2}{4t}} - \frac{2t}{irs}\\
&\qquad\qquad -1+2\Big(1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s}\Big)\\
&\qquad\qquad + \frac{i}{t}\Big[1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s}+(r+s-2)\Big(1-\frac{1}{r}-\frac{1}{s}\Big)\Big]\Big\}\\
&\quad + O\Big(\frac{r^2+s^2}{t^2}\Big).
\end{aligned}
$$

Using the Taylor expansion, we compute

$$
\begin{aligned}
&\sum_{k=1}^{4}\Big(\frac{1}{rs}-\frac{1}{s}\partial_r-\frac{1}{r}\partial_s+\partial_{rs}\Big)I_k\\
&= e^{\frac{i(r+s-2)^2}{4t}}\Big\{t\Big(\frac{2}{irs}-\frac{2}{irs}\Big)+1+\frac{2t}{irs}\frac{i(r-s)^2-i(r+s-2)^2}{4t}-1\\
&\quad +2\Big(1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s}\Big)\\
&\qquad + \frac{i}{t}\Big[\frac{(r-s)^2-(r+s-2)^2}{4}+\frac{\big((r-s)^2-(r+s-2)^2\big)^2}{16rs}\\
&\qquad + \frac{r+s-2}{rs}+1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s}+(r+s-2)\Big(1-\frac{1}{r}-\frac{1}{s}\Big)\Big]\Big\}\\
&+O\Big(\frac{r^2+s^2}{t^2}\Big)\\
&= e^{\frac{i(r+s-2)^2}{4t}}\Big\{-\frac{2(r-1)(s-1)}{rs}+2\Big(1+\frac{1}{rs}-\frac{1}{r}-\frac{1}{s}\Big)\\
&\qquad + \frac{i}{t}\frac{1}{rs}\Big[-(r-1)(s-1)rs+(r-1)^2(s-1)^2\\
&\qquad + r+s-2+(r-1)(s-1)+(r+s-2)(rs-r-s)\Big]\Big\}\\
&\quad + O\Big(\frac{r^2+s^2}{t^2}\Big)\\
&= O\Big(\frac{r^2+s^2}{t^2}\Big).
\end{aligned}
$$

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Here, note that in obtaining the last equality, we have used the fact that the terms in the first block (enclosed in the curly bracket) all vanish due to massive cancellations. This together with (2.5) finally yields the desired estimate

$$\sup_{r,s>1} |K(r,s,t)| \lesssim |t|^{-5/2}. \qquad \blacksquare$$

Lemma 2.2 immediately implies the following corollary.

Corollary 2.3

$$\| e^{it\Delta_D} f \|_{L^\infty_x(\Omega)} \lesssim |t|^{-5/2} \| f \|_{L^1_x(\Omega)}.$$

3 Fundamental Solution and the Dispersive Estimate for $n = 7$

In dimension $n = 7$, we consider the radial eigenfunctions of Dirichlet Laplacian on the exterior domain $\overline{B(0,1)}^c$:

$$\Delta\phi_\lambda + \lambda^2\phi_\lambda = 0, \quad \lambda > 0.$$

Using the Sommerfeld radiation condition, $\phi_\lambda(r)$ is given by

$$\phi_\lambda(r) = \frac{1}{r^3}\left(H^{(2)}_{\frac{5}{2}}(\lambda) H^{(1)}_{\frac{5}{2}}(\lambda r) - H^{(1)}_{\frac{5}{2}}(\lambda) H^{(2)}_{\frac{5}{2}}(\lambda r) \right) (\lambda r)^{1/2} \left(\widetilde{a}(\lambda) \right)^{1/2},$$

where

$$H^{(1)}_{\frac{5}{2}}(z) = i\sqrt{\frac{2}{\pi}} \frac{e^{iz}(-3 + 3iz + z^2)}{z^{5/2}}, \quad H^{(2)}_{\frac{5}{2}}(z) = -i\sqrt{\frac{2}{\pi}} \frac{e^{-iz}(-3 - 3iz + z^2)}{z^{5/2}},$$

$$\widetilde{a}(\lambda) = \frac{\lambda}{(\frac{3}{\lambda^2} - 1)^2 + \left(\frac{3}{\lambda}\right)^2} = \frac{\lambda^5}{(\lambda^2 - 3)^2 + (3\lambda)^2}.$$

3.1 Dispersive Estimate

We first prove the dispersive estimate. Recall the definition of Fourier transform:

$$\mathcal{F}_0 f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi_\lambda(r) f(r) r^6 \, dr.$$

Then $\mathcal{F}_0$ is an isometric map $\mathcal{F}_0 \colon L^2\left([1,\infty); r^6 dr\right) \longrightarrow L^2([0,\infty); d\lambda)$ with the inverse transform given by

$$\mathcal{F}_0^* g(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi_\lambda(r) g(\lambda) \, d\lambda.$$

The fundamental solution can be written as

$$(e^{it\Delta_D} f)(r,t) = \mathcal{F}_0^*(e^{-it\lambda^2} \mathcal{F}_0 f)(r) = \int_0^\infty K(r,s,t) f(s) s^6 \, ds$$

where

$$K(r,s,t) = \frac{2}{\pi} \int_0^\infty \phi_\lambda(r)\phi_\lambda(s) e^{-it\lambda^2} \, d\lambda.$$

We write the kernel $K(r,s,t) = (e^{it\Delta_D})(r,s,t)$ as

$$K(r,s,t) = \frac{\text{Const}}{(rs)^3}\int_0^\infty \big(H^{(2)}_{\frac52}(\lambda)H^{(1)}_{\frac52}(\lambda r) - H^{(1)}_{\frac52}(\lambda)H^{(2)}_{\frac52}(\lambda r)\big) \times \big(H^{(2)}_{\frac52}(\lambda)H^{(1)}_{\frac52}(\lambda s) - H^{(1)}_{\frac52}(\lambda)H^{(2)}_{\frac52}(\lambda s)\big)(\lambda r)^{1/2}(\lambda s)^{1/2}\widetilde{a}(\lambda)e^{-i\lambda^2 t}\,d\lambda.$$

Using this expansion, we then obtain

$$\begin{aligned}
&\big(H^{(2)}_{\frac52}(\lambda)H^{(1)}_{\frac52}(\lambda r) - H^{(1)}_{\frac52}(\lambda)H^{(2)}_{\frac52}(\lambda r)\big)\\
&\quad\times \big(H^{(2)}_{\frac52}(\lambda)H^{(1)}_{\frac52}(\lambda s) - H^{(1)}_{\frac52}(\lambda)H^{(2)}_{\frac52}(\lambda s)\big)(\lambda r)^{1/2}(\lambda s)^{1/2}\\
&= \frac{4e^{-i(2+r+s)\lambda}\sqrt{rs}\sqrt{r\lambda}\sqrt{s\lambda}}{\pi^2 r^3 s^3\lambda^{10}}\\
&\quad\times\big(-e^{2i\lambda}(-3+3i\lambda+\lambda^2)(-3-3ir\lambda+r^2\lambda^2)\\
&\qquad\qquad + e^{2ir\lambda}(-3-3i\lambda+\lambda^2)(-3+3ir\lambda+r^2\lambda^2)\big)\\
&\quad\times\big(-e^{2i\lambda}(-3+3i\lambda+\lambda^2)(-3-3is\lambda+s^2\lambda^2)\\
&\qquad\qquad + e^{2is\lambda}(-3-3i\lambda+\lambda^2)(-3+3is\lambda+s^2\lambda^2)\big).
\end{aligned}$$

Denote $x_+ = r+s-2$, $x = r-s$; we then expand the above expression as

$$\frac{4}{\pi^2 r^2 s^2\lambda^9}\Big(e^{i\lambda x_+}\sum_{k=0}^{8}A_k(i\lambda)^k - e^{i\lambda x}\sum_{k=0}^{8}B_k(i\lambda)^k + \text{C. C.}\Big),$$

where C. C. denotes the complex conjugate terms (of the first two terms), and

$$\begin{aligned}
&A_0 = 81,\quad A_1 = -i(162i - 81ir - 81is),\\
&A_2 = 135 - 162r + 27r^2 - 162s + 81rs + 27s^2,\\
&A_3 = i(-54i + 135ir - 54ir^2 + 135is - 162irs + 27ir^2s - 54is^2 + 27irs^2),\\
&A_4 = 9 - 54r + 45r^2 - 54s + 135rs - 54r^2s + 45s^2 - 54rs^2 + 9r^2s^2,\\
&A_5 = -i(-9ir + 18ir^2 - 9is + 54irs - 45ir^2s + 18is^2 - 45irs^2 + 18ir^2s^2),\\
&A_6 = 3r^2 + 9rs - 18r^2s + 3s^2 - 18rs^2 + 15r^2s^2,\\
&A_7 = i(3ir^2s + 3irs^2 - 6ir^2s^2),\quad A_8 = r^2s^2,\\
&B_0 = 81,\quad B_1 = -i(-81ir + 81is),\\
&B_2 = -27 + 27r^2 - 81rs + 27s^2,\\
&B_3 = i(-27ir + 27is - 27ir^2s + 27irs^2),\\
&B_4 = 9 - 9r^2 + 27rs - 9s^2 + 9r^2s^2,\\
&B_5 = -i(-9ir + 9is - 9ir^2s + 9irs^2),\\
&B_6 = 3r^2 - 9rs + 3s^2 - 3r^2s^2,\\
&B_7 = i(-3ir^2s + 3irs^2),\quad B_8 = r^2s^2.
\end{aligned}$$

Denote

$$A(\lambda) = \frac{\widetilde{a}(\lambda)}{\lambda^5} = \frac{1}{(\lambda^2-3)^2 + (3\lambda)^2}.$$

We now simplify the expression for K and write

$$K(r,s,t) = \frac{\text{Const}}{(rs)^5}\int_0^\infty \frac{A(\lambda)e^{-i\lambda^2 t}}{\lambda^4}\Big(e^{i\lambda x_+}\sum_{k=0}^{8}A_k(i\lambda)^k - e^{i\lambda x}\sum_{k=0}^{8}B_k(i\lambda)^k + \text{C. C.}\Big)\,d\lambda. \tag{3.1}$$

Formally, the integrand in the above expression has a singularity near $\lambda = 0$ of order λ^{-4}. However, as we show below, the coefficients A_k, B_k for $0 \le k \le 2$ exhibit very nice cancellation properties, and terms of $O(\lambda^{-k})$ will never appear. Note also that, due to conjugation, for odd k only $\sin\lambda x_+$ and $\sin\lambda x$ appear; for even k, only $\cos\lambda x_+$ and $\cos\lambda x$ appear. Using this parity property, we can extend the integral of λ in (3.1) to the whole real axis. By a tedious cancellation, it is not difficult to check that the following lemma holds true for any $\lambda \in \mathbb{R}$.

Lemma 3.1 *The coefficients A_k, B_k, $0 \le k \le 2$ obey the relation*

$$A_0\Big(1-\frac{(\lambda x_+)^2}{2}\Big) + A_1(i\lambda)(i\lambda x_+) + A_2(i\lambda)^2 = B_0\Big(1-\frac{(\lambda x)^2}{2}\Big) + B_1(i\lambda)(i\lambda x) + B_2(i\lambda)^2.$$

Using Lemma 3.1, we can decompose the sum in (3.1) as

$$K(r,s,t) = \frac{\text{Const}}{(rs)^5}\Big(\sum_{k=0}^{8}S_k(x_+)A_k - \sum_{k=0}^{8}S_k(x)B_k + \text{C. C.}\Big), \tag{3.2}$$

where

$$\begin{aligned}
S_0(x_+) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda^2 t}A(\lambda)}{\lambda^4}\Big(\cos(\lambda x_+) - 1 + \frac{(\lambda x_+)^2}{2}\Big)\,d\lambda,\\
S_1(x_+) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda^2 t}A(\lambda)}{\lambda^3}\big(\sin(\lambda x_+) - \lambda x_+\big)\,d\lambda,\\
S_2(x_+) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda^2 t}A(\lambda)}{\lambda^2}\big(1-\cos(\lambda x_+)\big)\,d\lambda,\\
S_3(x_+) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\lambda^2 t}A(\lambda)}{\lambda}\sin(\lambda x_+)\,d\lambda,\\
S_k(x_+) &= \begin{cases}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}A(\lambda)\cos(\lambda x_+)i^k\lambda^{k-4}\,d\lambda, & k=4,6,8,\\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}A(\lambda)\sin(\lambda x_+)i^{k+1}\lambda^{k-4}\,d\lambda, & k=5,7.\end{cases}
\end{aligned} \tag{3.3}$$

A similar expression holds for $S_k(x)$ with variable x.

It is not difficult to check that as functions of x_+, we have

$$S_{k+1}(x_+) = \partial_{x_+}\big(S_k(x_+)\big), \quad \text{for } k = 0,1,2,\dots,7. \tag{3.4}$$

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Therefore we shall first derive the expression for $S_0(x_+)$. By (3.3), we have

$$S_0(x_+) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\lambda^2 t}}{\lambda^4}\Big(\cos(\lambda x_+) - 1 + \frac{1}{2}\lambda^2 x_+^2\Big)\big[A(\lambda) - A(0)\big]\ d\lambda \tag{3.5}$$

$$+ \frac{A(0)}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{\cos(\lambda x_+) - 1 + \frac{1}{2}\lambda^2 x_+^2}{\lambda^4}\ d\lambda. \tag{3.6}$$

We first consider (3.6). Recall that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\cos(\lambda y)\ d\lambda = \sqrt{\frac{1}{4\pi i t}}e^{\frac{iy^2}{4t}}.$$

Integrating repeatedly in y gives us the identities

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{\sin(\lambda y)}{\lambda}\ d\lambda = \sqrt{\frac{1}{4\pi i t}}\int_0^y e^{\frac{iz^2}{4t}}\ dz, \tag{3.7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{1-\cos(\lambda y)}{\lambda^2}\ d\lambda = \sqrt{\frac{1}{4\pi i t}}\int_0^y (y-z)e^{\frac{iz^2}{4t}}\ dz,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{\lambda y - \sin(\lambda y)}{\lambda^3}\ d\lambda = \sqrt{\frac{1}{4\pi i t}}\int_0^y \frac{(y-z)^2}{2}e^{\frac{iz^2}{4t}}\ dz,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{\cos(\lambda y) - 1 + \frac{1}{2}\lambda^2 y^2}{\lambda^4}\ d\lambda = \sqrt{\frac{1}{4\pi i t}}\int_0^y \frac{(y-z)^3}{6}e^{\frac{iz^2}{4t}}\ dz,$$

Hence by (3.7), we have

$$(3.6) = \sqrt{\frac{1}{4\pi i t}}A(0)\int_0^{x_+} \frac{(x_+ - z)^3}{6}e^{\frac{iz^2}{4t}}\ dz.$$

For (3.5), we write

$$\begin{aligned} A(\lambda) - A(0) &= \frac{1}{9}\Big(\frac{9}{\lambda^4 + 3\lambda^2 + 9} - 1\Big) \\ &= -\frac{1}{9}\Big(\frac{\lambda^4}{\lambda^4 + 3\lambda^2 + 9} + \frac{3\lambda^2}{\lambda^4 + 3\lambda^2 + 9}\Big) \\ &= -\frac{1}{9}\lambda^4 A(\lambda) - \frac{1}{3}\lambda^2 A(\lambda). \end{aligned}$$

Therefore,

$$(3.5) = -\frac{1}{9}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\Big(\cos(\lambda x_+) - 1 + \frac{1}{2}\lambda^2 x_+^2\Big)A(\lambda)\ d\lambda \tag{3.8}$$

$$-\frac{1}{3}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{\cos(\lambda x_+) - 1}{\lambda^2}A(\lambda)\ d\lambda \tag{3.9}$$

$$-\frac{1}{3}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda^2 t}\frac{1}{2}x_+^2 A(\lambda)\ d\lambda. \tag{3.10}$$

Now denote

$$F(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda y}A(\lambda)\ d\lambda.$$

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By the convolution property of the Fourier transform, we have

$$(3.8) = -\frac{1}{9}\sqrt{\frac{1}{4\pi it}}\int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy + \frac{1}{9}\sqrt{\frac{1}{4\pi it}}\int_{-\infty}^{\infty} F(y)e^{\frac{iy^2}{4t}}\,dy + \frac{1}{9}\sqrt{\frac{1}{4\pi it}}\frac{1}{2}x_+^2\int_{-\infty}^{\infty} F''(y)e^{\frac{iy^2}{4t}}\,dy.$$

By using a derivation similar to that in (3.7), we also have

$$(3.9) = \frac{1}{3}\sqrt{\frac{1}{4\pi it}}\int_{-\infty}^{\infty} F(y)\Big(\int_0^{x_+}(x_+-z)e^{\frac{i(z-y)^2}{4t}}\,dz\Big)\,dy.$$

Note that

$$(3.10) = -\frac{1}{3}\frac{1}{2}x_+^2\sqrt{\frac{1}{4\pi it}}\int_{-\infty}^{\infty} F(y)e^{\frac{iy^2}{4t}}\,dy.$$

Collecting all the expressions, we obtain

$$\begin{aligned}
\sqrt{4\pi it}S_0(x_+) = {} & A(0)\int_0^{x_+}\frac{(x_+-z)^3}{6}e^{\frac{iz^2}{4t}}\,dy \\
& + \frac{1}{3}\int_{-\infty}^{\infty} F(y)\Big(\int_0^{x_+}(x_+-z)e^{\frac{i(z-y)^2}{4t}}\,dz\Big)\,dy && (3.11)\\
& - \frac{1}{9}\int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy && (3.12)\\
& + \Big(\frac{1}{9}-\frac{1}{6}x_+^2\Big)\int_{-\infty}^{\infty} F(y)e^{\frac{iy^2}{4t}}\,dy \\
& + \frac{1}{18}x_+^2\int_{-\infty}^{\infty} F''(y)e^{\frac{iy^2}{4t}}\,dy.
\end{aligned}$$

By (3.4), we then have

$$\begin{aligned}
\sqrt{4\pi it}S_1(x_+) = {} & A(0)\int_0^{x_+}\frac{(x_+-z)^2}{2}e^{\frac{iz^2}{4t}}\,dz && (3.13)\\
& + \frac{1}{3}\int_{-\infty}^{\infty} F(y)\Big(\int_0^{x_+}e^{\frac{i(z-y)^2}{4t}}\,dz\Big)\,dy \\
& - \frac{1}{9}\int_{-\infty}^{\infty} F'(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy \\
& - \frac{1}{3}x_+\int_{-\infty}^{\infty} F(y)e^{\frac{iy^2}{4t}}\,dy \\
& + \frac{1}{9}x_+\int_{-\infty}^{\infty} F''(y)e^{\frac{iy^2}{4t}}\,dy,
\end{aligned}$$

$$\begin{aligned}\sqrt{4\pi i t}S_2(x_+) = A(0)\int_0^{x_+}(x_+ - z)e^{\frac{iz^2}{4t}}\,dz\\ &+ \frac{1}{3}\int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy\\ &- \frac{1}{9}\int_{-\infty}^{\infty} F''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy\\ &- \frac{1}{3}\int_{-\infty}^{\infty} F(y)e^{\frac{iy^2}{4t}}\,dy\\ &+ \frac{1}{9}\int_{-\infty}^{\infty} F''(y)e^{\frac{iy^2}{4t}}\,dy.\end{aligned}$$

$$\begin{aligned}\sqrt{4\pi i t}S_3(x_+) = A(0)\int_0^{x_+} e^{\frac{iz^2}{4t}}\,dz + \frac{1}{3}\int_{-\infty}^{\infty} F'(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy\\ &- \frac{1}{9}\int_{-\infty}^{\infty} F'''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy,\end{aligned}$$

$$\begin{aligned}\sqrt{4\pi i t}S_4(x_+) = A(0)e^{\frac{ix_+^2}{4t}} + \frac{1}{3}\int_{-\infty}^{\infty} F''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy\\ &- \frac{1}{9}\int_{-\infty}^{\infty} F^{(4)}(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy.\end{aligned} \tag{3.14}$$

By the definition of $A(\lambda)$, we have

$$A(0) - \frac{1}{3}\lambda^2 A(\lambda) - \frac{1}{9}\lambda^4 A(\lambda) = A(\lambda). \tag{3.15}$$

Therefore, (3.14) is simply

$$\sqrt{4\pi i t}S_4(x_+) = \int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy. \tag{3.16}$$

Continuing the differentiation, we have

$$\begin{aligned}\sqrt{4\pi i t}S_5(x_+) &= \int_{-\infty}^{\infty} F'(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy,\\ \sqrt{4\pi i t}S_6(x_+) &= \int_{-\infty}^{\infty} F''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy,\\ \sqrt{4\pi i t}S_7(x_+) &= \int_{-\infty}^{\infty} F'''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy,\\ \sqrt{4\pi i t}S_8(x_+) &= \int_{-\infty}^{\infty} F^{(4)}(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy\\ &= 9A(0)e^{\frac{ix_+^2}{4t}} + 3\int_{-\infty}^{\infty} F''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy - 9\int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy,\end{aligned}$$

where in the last identity we have used the equivalence of the expressions (3.14) and (3.16).

We now estimate the terms $S_0(x_+), \dots, S_8(x_+)$. Note the important factor $(rs)^{-5}$ in front of the expression for the kernel $K(r, s, t)$. In view of this, to derive the dispersive

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estimate, we can safely discard terms of order $O(\frac{r}{t^3})$, $O(\frac{s}{t^3})$, $O(\frac{(rs)^5}{t^3})$ in the expressions of $\sqrt{4\pi it}S_j(x_+)$ for $0 \le j \le 8$.

We first consider $\sqrt{4\pi it}S_0(x_+)$. Denote

$$H_+ = \int_0^{x_+} e^{\frac{iz^2}{4t}}\, dz. \tag{3.17}$$

By using integration by parts, we have

$$\begin{aligned}
\int_0^{x_+} (x_+ - z)\, e^{\frac{iz^2}{4t}}\, dz &= 2i\Big(-1 + e^{\frac{ix_+^2}{4t}}\Big)t + H_+x_+,\\
\int_0^{x_+} \frac{(x_+ - z)^2}{2} e^{\frac{iz^2}{4t}}\, dz &= \frac{1}{2}H_+x_+^2 + 2it\Big(\frac{H_+}{2} - x_+ + \frac{1}{2}e^{\frac{ix_+^2}{4t}}x_+\Big),\\
\int_0^{x_+} \frac{(x_+ - z)^3}{6} e^{\frac{iz^2}{4t}}\, dz &= i\big(-1 + e^{\frac{i(r-s)_+^2}{4t}}\big)tx_+^2 + \frac{1}{6}H_+x_+^3\\
&\quad + x_+\big(itH_+ - ie^{\frac{ix_+^2}{4t}}tx_+\big)\\
&\quad - \frac{4}{3}\big(-1 + e^{\frac{ix_+^2}{4t}}\big)t^2 + \frac{1}{3}ie^{\frac{ix_+^2}{4t}}tx_+^2.
\end{aligned}$$

For (3.11), we write

$$\begin{aligned}
&\int_0^{x_+} (x_+ - z)e^{\frac{i(z-y)^2}{4t}}\, dz\\
&\quad = \int_0^{x_+} (x_+ - z)e^{\frac{iz^2}{4t}} e^{\frac{i(-2zy+y^2)}{4t}}\, dz\\
&\quad = \int_0^{x_+} (x_+ - z)e^{\frac{iz^2}{4t}}\Big\{1 + \frac{i(-2zy + y^2)}{4t} + \frac{-(-2zy + y^2)^2}{32t^2}\\
&\qquad\qquad + O\Big(\frac{(|z||y| + y^2)^3}{|t|^3}\Big)\Big\}\, dz.
\end{aligned}$$

The last error term is acceptable for us, since

$$\int_{-\infty}^{\infty} |F(y)|\Big|\int_0^{x_+} |x_+ - z|\frac{(|z||y| + y^2)^3}{|t|^3}\, dz\Big|\, dy \lesssim \frac{x_+^5}{|t|^3} \lesssim \frac{r^5 + s^5}{|t|^3}.$$

Therefore we can safely discard the error term and write

$$\begin{aligned}
(3.11) &= \frac{1}{3}\int_{-\infty}^{\infty} F(y)\Big\{\int_0^{x_+} (x_+ - z)e^{\frac{iz^2}{4t}}\Big(1 + \frac{i(-2zy + y^2)}{4t}\\
&\qquad - \frac{(-2zy + y^2)^2}{32t^2}\Big)\, dz\Big\}\, dy\\
&= \frac{1}{3}\int_0^{x_+} (x_+ - z)e^{\frac{iz^2}{4t}}\Big\{m_0 + \Big(\frac{i}{4t} - \frac{z^2}{8t^2}\Big)m_2 - \frac{1}{32t^2}m_4\Big\}\, dz, \qquad (3.18)
\end{aligned}$$

where

$$m_0 = \int_{-\infty}^{\infty} F(y)\, dy = \frac{1}{9}, \quad m_2 = \int_{-\infty}^{\infty} F(y)y^2\, dy = \frac{2}{27},$$
$$m_4 = \int_{-\infty}^{\infty} F(y)y^4\, dy = 0,$$

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and we have used the fact that F is an even function. We can use the notation (3.17) to simplify (3.18) further as

$$(3.11) = \frac{1}{3}\Big(\frac{H_+x_+}{9} + \frac{ie^{\frac{ix_+^2}{4t}}x_+^2}{54t} - \frac{2(1+6it)t - ie^{\frac{ix_+^2}{4t}}\big(2t(-i+6t) - x_+^2\big)}{54t}\Big).$$

In a similar way, we also have

$$(3.13) = \frac{1}{3}\Big(\frac{H_+}{9} + \frac{ie^{\frac{ix_+^2}{4t}}x_+}{54t}\Big).$$

Now for (3.12), we can expand the function $e^{\frac{i(x_+-y)^2}{4t}}$ as

$$\begin{aligned} e^{\frac{i(x_+-y)^2}{4t}} &= e^{\frac{ix_+^2}{4t}}e^{\frac{i(-2x_+y+y^2)}{4t}} \\ &= e^{\frac{ix_+^2}{4t}}\Big(1 + \frac{i(-2x_+y+y^2)}{4t} - \frac{(-2x_+y+y^2)^2}{32t^2} + O\Big(\frac{x_+^3y^3+y^6}{|t|^3}\Big)\Big). \end{aligned}$$

The error term is again acceptable for us, since

$$\int_{-\infty}^{\infty} |F(y)|\cdot O\Big(\frac{x_+^3y^3+y^6}{|t|^3}\Big)\,dy \lesssim \frac{x_+^3}{|t|^3} \lesssim \frac{r^3+s^3}{|t|^3}.$$

Therefore, neglecting the error term, we have

$$\begin{aligned} \int_{-\infty}^{\infty} F(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy &= e^{\frac{ix_+^2}{4t}}\int_{-\infty}^{\infty} F(y)\Big(1 + \frac{i(-2x_+y+y^2)}{4t} - \frac{(-2x_+y+y^2)^2}{32t^2}\Big)\,dy \\ &= -e^{\frac{ix_+^2}{4t}}\frac{-2t(i+6t)+x_+^2}{108t^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{-\infty}^{\infty} F'(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy &= e^{\frac{ix_+^2}{4t}}\frac{(-1+2it)x_+}{36t^2}, \\ \int_{-\infty}^{\infty} F''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy &= -e^{\frac{ix_+^2}{4t}}\frac{1-2it+x_+^2}{36t^2}, \\ \int_{-\infty}^{\infty} F'''(y)e^{\frac{i(x_+-y)^2}{4t}}\,dy &= -e^{\frac{ix_+^2}{4t}}\frac{x_+}{12t^2}. \end{aligned}$$

Collecting all the estimates, we have

$$\begin{aligned} \sqrt{4\pi it}S_0(x_+) = &-\frac{(1-2it)x_+^2}{648t^2} + \frac{(i+6t)\Big(\frac{1}{9}-\frac{x_+^2}{6}\Big)}{54t} + \frac{e^{\frac{ix_+^2}{4t}}\big(-2t(i+6t)+x_+^2\big)}{972t^2} \\ &+ \frac{1}{9}\Big(i\big(-1+e^{\frac{ix_+^2}{4t}}\big)tx_+^2 + \frac{1}{6}H_+x_+^3 + \frac{1}{2}x_+\big(2itH_+ - 2ie^{\frac{ix_+^2}{4t}}tx_+\big) \\ &\qquad + \frac{1}{6}\big(-8(-1+e^{\frac{ix_+^2}{4t}})t^2 + 2ie^{\frac{ix_+^2}{4t}}tx_+^2\big)\Big) \\ &+ \frac{1}{3}\Big(\frac{x_+^2}{9} + \frac{ie^{\frac{ix_+^2}{4t}}x_+^2}{54t} - \frac{2(1+6it)t - ie^{\frac{ix_+^2}{4t}}(2t(-i+6t)-x_+^2)}{54t}\Big), \end{aligned}$$

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$$\sqrt{4\pi it}S_1(x_+) = -\frac{(1-2it)x_+}{324t^2} - \frac{e^{\frac{ix_+^2}{4t}}(-1+2it)x_+}{324t^2} - \frac{(i+6t)x_+}{162t}$$
$$+\frac{1}{3}\Big(\frac{x_+}{9} + \frac{ie^{\frac{ix_+^2}{4t}}x_+}{54t}\Big) + \frac{1}{9}\Big(\frac{1}{2}H_+x_+^2 + 2it\Big(\frac{H_+}{2} - x_+ + \frac{1}{2}e^{\frac{ix_+^2}{4t}}x_+\Big)\Big),$$
$$\sqrt{4\pi it}S_2(x_+) = -\frac{1-2it}{324t^2} - \frac{i+6t}{162t} + \frac{1}{9}\big(2i(-1+e^{\frac{ix_+^2}{4t}})t + H_+x_+\big)$$
$$+\frac{e^{\frac{ix_+^2}{4t}}(1-2it+x_+^2)}{324t^2} - \frac{e^{\frac{ix_+^2}{4t}}\big(-2t(i+6t)+x_+^2\big)}{324t^2},$$
$$\sqrt{4\pi it}S_3(x_+) = \frac{H_+}{9} + \frac{e^{\frac{ix_+^2}{4t}}x_+}{108t^2} + \frac{e^{\frac{ix_+^2}{4t}}(-1+2it)x_+}{108t^2},$$
$$\sqrt{4\pi it}S_4(x_+) = -\frac{e^{\frac{ix_+^2}{4t}}(-2t(i+6t)+x_+^2)}{108t^2},$$
$$\sqrt{4\pi it}S_5(x_+) = \frac{e^{\frac{ix_+^2}{4t}}(-1+2it)x_+}{36t^2},$$
$$\sqrt{4\pi it}S_6(x_+) = -\frac{e^{\frac{ix_+^2}{4t}}(1-2it+x_+^2)}{36t^2},$$
$$\sqrt{4\pi it}S_7(x_+) = -\frac{e^{\frac{ix_+^2}{4t}}x_+}{12t^2},$$
$$\sqrt{4\pi it}S_8(x_+) = e^{\frac{ix_+^2}{4t}} - \frac{e^{\frac{ix_+^2}{4t}}(1-2it+x_+^2)}{12t^2} + \frac{e^{\frac{ix_+^2}{4t}}(-2t(i+6t)+x_+^2)}{12t^2}.$$

In the same way, we have the expression for $S_k(x)$ for $0 \le k \le 8$.

By (3.2) and neglecting the acceptable error terms, we then have

$$\begin{aligned}
&\frac{\sqrt{4\pi it}(rs)^5}{\text{Const}}K(r,s,t)\\
&= \sqrt{4\pi it}\sum_{k=0}^{8} A_kS_k(x_+) - \sqrt{4\pi it}\sum_{k=0}^{8} B_kS_k(x)\\
&= \frac{1}{12t^2}\Big\{e^{\frac{i(r-s)^2}{4t}}(1-12r^2s^2t^2+72irst^3+144t^4)\\
&\quad + e^{\frac{i(-2+r+s)^2}{4t}}[-7-6r^4(-1+s)^2-6s^4-12r^3(2-3s+s^2)(-1+s-it)\\
&\quad + 24s^3(1+it)-24it+72t^2+144it^3-144t^4\\
&\quad - 6r^2(6-6s+s^2)(1+s^2+s(-2-2it)+2it-2t^2)\\
&\quad + 36s^2(-1-2it+2t^2)+24s(1+3it-6t^2-6it^3)\\
&\quad + 12r(-2+s)(-1+s^3+s^2(-3-3it)-3it+6t^2+6it^3+s(3+6it-6t^2))]\Big\}\\
&=: I_1.
\end{aligned}$$

We multiply I_1 by the factor $e^{-\frac{i(r+s-2)^2}{4t}}$, and this gives us

$$
\begin{aligned}
e^{-\frac{i(r+s-2)^2}{4t}} I_1 = {} & 6 - 12r + 6r^2 - 12s + 18rs - 6r^2 s + 6s^2 - 6rs^2 + r^2 s^2 \\
& - e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} r^2 s^2 - \frac{7}{12t^2} + \frac{e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}}}{12t^2} \qquad (3.19)\\
& + \frac{2r}{t^2} - \frac{3r^2}{t^2} + \frac{2r^3}{t^2} - \frac{r^4}{2t^2} \\
& + \frac{2s}{t^2} - \frac{7rs}{t^2} + \frac{9r^2 s}{t^2} - \frac{5r^3 s}{t^2} + \frac{r^4 s}{t^2} \\
& - \frac{3s^2}{t^2} + \frac{9rs^2}{t^2} - \frac{19r^2 s^2}{2t^2} + \frac{4r^3 s^2}{t^2} - \frac{r^4 s^2}{2t^2} \\
& + \frac{2s^3}{t^2} - \frac{5rs^3}{t^2} + \frac{4r^2 s^3}{t^2} - \frac{r^3 s^3}{t^2} \\
& - \frac{s^4}{2t^2} + \frac{rs^4}{t^2} - \frac{r^2 s^4}{2t^2} \\
& - \frac{2i}{t} + \frac{6ir}{t} - \frac{6ir^2}{t} + \frac{2ir^3}{t} \\
& + \frac{6is}{t} - \frac{15irs}{t} + \frac{12ir^2 s}{t} - \frac{3ir^3 s}{t} \\
& - \frac{6is^2}{t} + \frac{12irs^2}{t} - \frac{7ir^2 s^2}{t} + \frac{ir^3 s^2}{t} \\
& + \frac{2is^3}{t} - \frac{3irs^3}{t} + \frac{ir^2 s^3}{t} \\
& + 12it - 12irt - 12ist + 6irst \\
& + 6ie^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} rst - 12t^2 + 12e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} t^2. \qquad (3.20)
\end{aligned}
$$

Now note that

$$e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} = e^{-\frac{i(r-1)(s-1)}{t}}.$$

For the term $e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} r^2 s^2$, we expand it as

$$
\begin{aligned}
e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} r^2 s^2 &= e^{-\frac{i(r-1)(s-1)}{t}} r^2 s^2 \\
&= \sum_{k=0}^{2} \frac{1}{k!} \Big(-\frac{i(r-1)(s-1)}{t} \Big)^k r^2 s^2 + O\Big(\frac{r^5 s^5}{t^3} \Big),
\end{aligned}
$$

which is acceptable for us. Similarly,

$$
\begin{aligned}
\frac{e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}}}{12t^2} &= \frac{1}{12t^2} + O\Big(\frac{rs}{t^3} \Big), \\
12e^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} t^2 &= 12t^2 \sum_{k=0}^{4} \frac{1}{k!} \Big(-\frac{i(r-1)(s-1)}{t} \Big)^k + O\Big(\frac{r^5 s^5}{t^3} \Big), \\
6ie^{\frac{i(r-s)^2}{4t} - \frac{i(-2+r+s)^2}{4t}} rst &= 6irst \sum_{k=0}^{3} \frac{1}{k!} \Big(-\frac{i(r-1)(s-1)}{t} \Big)^k + O\Big(\frac{r^5 s^5}{t^3} \Big).
\end{aligned}
$$

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Plugging the above expressions into (3.19) and (3.20), we obtain (after a long and tedious computation) that

$$e^{-\frac{i(r+s-2)^2}{4t}} I_1 = O\Big(\frac{r^5 s^5}{t^3}\Big).$$

This concludes the proof of the dispersive estimate. Therefore we have proved the following lemma.

Lemma 3.2

$$\sup_{r,s>1}\big|K(r,s,t)\big| \lesssim |t|^{-7/2}, \quad t \neq 0.$$

Consequently, we have the following corollary.

Corollary 3.3

$$\big\|e^{it\Delta_D} f\big\|_{L_x^\infty(\Omega)} \lesssim |t|^{-7/2}\big\|f\big\|_{L_x^1(\Omega)}, \quad t \neq 0.$$

As a sanity check, we now verify that the eigenfunctions $\phi_\lambda(r)$ form a complete basis.

3.2 Resolution of Identity

Lemma 3.4

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi_\lambda(r)\phi_\lambda(s)\, d\lambda = \frac{1}{r^6}\delta(r-s), \quad r,s>1.$$

Adopting the same notations as in the proof of the dispersive estimate (see (3.2)), we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi_\lambda(r)\phi_\lambda(s)\, d\lambda = \frac{\text{Const}}{(rs)^5}\Big(\sum_{k=0}^{8} T_k(x_+)A_k - \sum_{k=0}^{8} T_k(x)B_k + \text{C. C.}\Big),$$

where

$$T_0(x_+) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda^4}\Big(\cos(\lambda x_+) - 1 + \frac{(\lambda x_+)^2}{2}\Big)\, d\lambda,$$

$$T_1(x_+) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda^3}\big(\sin(\lambda x_+) - \lambda x_+\big)\, d\lambda,$$

$$T_2(x_+) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda^2}\big(1 - \cos(\lambda x_+)\big)\, d\lambda,$$

$$T_3(x_+) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A(\lambda)}{\lambda}\sin(\lambda x_+)\, d\lambda,$$

$$T_k(x_+) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\lambda)\cos(\lambda x_+) i^k \lambda^{k-4}\, d\lambda, & k = 4,6,8, \\ \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\lambda)\sin(\lambda x_+) i^{k+1}\lambda^{k-4}\, d\lambda, & k = 5,7. \end{cases}$$

Note that

$$T_k(x_+) = \partial_{x_+}^k\big(T_0(x_+)\big), \quad \text{for } k = 0,1,2,\dots,7. \tag{3.21}$$

We first derive the expression for $T_0(x_+)$. By a derivation similar to (3.5), (3.6), we have

$$T_0(x_+) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda^4}\Big(\cos(\lambda x_+) - 1 + \frac{1}{2}\lambda^2 x_+^2\Big)\big(A(\lambda) - A(0)\big)\ dx \tag{3.22}$$

$$+\frac{A(0)}{2\pi}\int_{-\infty}^{\infty}\frac{\cos(\lambda x_+) - 1 + \frac{1}{2}\lambda^2 x_+^2}{\lambda^4}\ d\lambda. \tag{3.23}$$

Now recall

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(\lambda y)\ d\lambda = \delta(y), \tag{3.24}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin(\lambda y)}{\lambda}\ d\lambda = \frac{1}{2}\operatorname{sign}(y), \tag{3.25}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\cos(\lambda y)}{\lambda^2}\ d\lambda = \frac{1}{2}|y|,$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda y - \sin(\lambda y)}{\lambda^3}\ d\lambda = \frac{1}{4}y^2\operatorname{sign}(y),$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos(\lambda y) - 1 + \frac{1}{2}(\lambda y)^2}{\lambda^4}\ d\lambda = \frac{1}{12}|y|^3.$$

Therefore

$$(3.23) = \frac{|x_+|^3}{12}A(0).$$

From (3.25) and (3.24) , we also obtain

$$\int_0^x \delta(y)\ dy = \frac{1}{2}\operatorname{sign}(x). \tag{3.26}$$

This useful identity will be used below.

Now, using a similar analysis as in (3.8)–(3.10), we have

$$(3.22) = -\frac{1}{9}\Big(F(x_+) - F(0) - \frac{1}{2}F''(0)x_+^2\Big) + \frac{1}{3}\int_0^{x_+}(x_+ - y)F(y)\ dy - \frac{1}{6}x_+^2 F(0).$$

Hence

$$\begin{aligned} T_0(x_+) = {}& -\frac{1}{9}F(x_+) + \Big(\frac{1}{9} - \frac{1}{6}x_+^2\Big)F(0) + \frac{1}{18}F''(0)x_+^2 \\ & + \frac{1}{3}\int_0^{x_+}(x_+ - y)F(y)\ dy + \frac{A(0)}{12}|x_+|^3. \end{aligned}$$

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By (3.21), we then have

$$T_1(x_+) = -\frac{1}{9}F'(x_+) - \frac{1}{3}x_+F(0) + \frac{1}{9}F''(0)x_+ + \frac{1}{3}\int_0^{x_+} F(y)\,dy + \frac{A(0)}{4}x_+^2 \operatorname{sign}(x_+), \tag{3.27}$$

$$T_2(x_+) = -\frac{1}{9}F''(x_+) - \frac{1}{3}F(0) + \frac{1}{3}F(x_+) + \frac{A(0)}{2}|x_+| + \frac{1}{9}F''(0),$$

$$T_3(x_+) = -\frac{1}{9}F'''(x_+) + \frac{1}{3}F'(x_+) + \frac{A(0)}{2}\operatorname{sign}(x_+),$$

$$T_4(x_+) = -\frac{1}{9}F^{(4)}(x_+) + \frac{1}{3}F''(x_+) + A(0)\delta(x_+) = F(x_+),$$

where we have used the identity (3.15). Continuing the differentiation, we have

$$T_5(x_+) = F'(x_+), \quad T_6(x_+) = F''(x_+), \quad T_7(x_+) = F'''(x_+),$$
$$T_8(x_+) = F^{(4)}(x_+) = 9A(0)\delta(x_+) + 3F''(x_+) - 9F(x_+).$$

We now compute the explicit expression for T_k, $0 \le k \le 8$. By using the definition of $A(\lambda)$ and a contour integration, we have

$$F(y) = \frac{1}{18}e^{-\frac{3}{2}|y|}\Big(\cos\Big(\frac{\sqrt{3}}{2}|y|\Big) + \sqrt{3}\sin\Big(\frac{\sqrt{3}}{2}|y|\Big)\Big). \tag{3.28}$$

By using the identities (3.27), (3.26), we have

$$\int_0^{x_+} F(y)\,dy = -\frac{1}{9}\int_0^{x_+} F^{(4)}(y)\,dy + \frac{1}{3}\int_0^{x_+} F^{(2)}(y)\,dy + A(0)\int_0^{x_+}\delta(y)dy = -\frac{1}{9}F^{(3)}(x_+) + \frac{1}{3}F'(x_+) + \frac{A(0)}{2}\operatorname{sign}(x_+). \tag{3.29}$$

Similarly,

$$\int_0^{x_+}(x_+ - y)F(y)\,dy = -\frac{1}{9}\int_0^{x_+}(x_+ - y)F^{(4)}(y)\,dy + \frac{1}{3}\int_0^{x_+}(x_+ - y)F''(y)\,dy + A(0)\int_0^{x_+}(x_+ - y)\delta(y)dy$$
$$= -\frac{1}{9}\big(F''(x_+) - F''(0)\big) + \frac{1}{3}\big(F(x_+) - F(0)\big) + \frac{A(0)}{2}|x_+|. \tag{3.30}$$

Substituting (3.28) – (3.30) into the expressions for T_k, $0 \le k \le 8$, we then obtain

$$\sum_{k=0}^{8} A_k T_k(x_+)$$
$$= r^2s^2\delta(x_+) - r^2s^2\cos\big(\tfrac{\sqrt{3}}{2}x_+\big)\big(e^{\frac{3}{2}x_+}\mathcal{H}[-x_+] + e^{-\frac{3}{2}x_+}\mathcal{H}[x_+]\big)$$
$$+ \frac{1}{4}\Big\{6\big(6 + r^2 + 3r(-2 + s) - 6s + s^2\big)|x_+| + 3|x_+|^3$$

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$$
\begin{aligned}
&- 3\big[-32 + 3r^3 + 11r^2(-2+s) + 48s - 22s^2 + 3s^3 \\
&\qquad + r(48 - 48s + 11s^2)\big] \operatorname{sign}(x_+) \\
&+ 4\Bigg[1 + r^2 + s^2 - e^{\frac{3}{2}x_+}\Big(\\
&\qquad [-18r(-1+s) - 6(2-3s+s^2) + r^2(-6+5s^2)] \cos\big(\tfrac{\sqrt{3}}{2}x_+\big) \\
&\qquad + 6\sqrt{3}(-1+r)(r(-1+s)-s)(-1+s)\sin\big(\tfrac{\sqrt{3}}{2}x_+\big)\Big)\mathcal{H}[-x_+] \\
&\qquad + e^{-\frac{3}{2}x_+} r^2 s^2 \cos\big(\tfrac{\sqrt{3}}{2}x_+\big)\mathcal{H}[x_+]\Bigg]\Bigg\} \\
&= r^2s^2\delta(x_+) - e^{-\frac{3}{2}x_+} r^2 s^2 \cos\big(\tfrac{\sqrt{3}}{2}x_+\big) \\
&\quad + \frac{3}{2}\big(6 + r^2 + 3r(-2+s) - 6s + s^2\big)x_+ + \frac{3}{4}x_+^3 \\
&\quad - \frac{3}{4}\big(-32 + 3r^3 + 11r^2(-2+s) + 48s - 22s^2 + 3s^3 + r(48 - 48s + 11s^2)\big) \\
&\quad + 1 + r^2 + s^2 + e^{-\frac{3}{2}x_+} r^2 s^2 \cos\big(\tfrac{\sqrt{3}}{2}x_+\big)
\end{aligned}
$$

where $\mathcal{H}$ is the Heaviside step function defined as

$$
\mathcal{H}[y] = \begin{cases} 1, & y > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

Similarly,

$$
\begin{aligned}
&\sum_{k=0}^{8} B_k T_k(x) \\
&\quad = r^2s^2\delta(x) + r^2s^2\cos\Big(\frac{\sqrt{3}}{2}x\Big)\big(e^{-\frac{3}{2}x}\mathcal{H}[x] + e^{\frac{3}{2}x}\mathcal{H}[-x]\big) \\
&\qquad + \frac{1}{4}\big(6(r^2 - 3rs + s^2)|x| + 3|x|^3 + (-9r^3 + 33r^2s - 33rs^2 + 9s^3)\operatorname{sign}(x)\big) \\
&\qquad + 1 + r^2 + s^2 + e^{-\frac{3}{2}x} r^2s^2\cos\Big(\frac{\sqrt{3}}{2}x\Big)\mathcal{H}[x] + e^{\frac{3}{2}x} r^2s^2\cos\Big(\frac{\sqrt{3}}{2}x\Big)\mathcal{H}[-x].
\end{aligned}
$$

After a long and tedious computation, we arrive at

$$
\sum_{k=0}^{8} A_k T_k(x_+) - \sum_{k=0}^{8} B_k T_k(x) = r^2s^2\delta(x_+) - r^2s^2\delta(x) = -r^2s^2\delta(r-s), \quad r, s > 1.
$$

Here in the last equality we have used again the fact that $\delta(x_+) = \delta(r+s-2) = 0$ for $r > 1$, $s > 1$. We have proved the resolution of identity.

4 Littlewood–Paley Operators and L^p Based Sobolev Spaces

In this section, we introduce the Littlewood–Paley operators and L^p based Sobolev spaces on exterior domains. These will be needed later for the nonlinear estimates.

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First, given a bounded function $m(\ell)$, which for convenience we assume to be defined on all of $\mathbb{R}$ and even in ℓ, we define

$$m\big(\sqrt{-\Delta_D}\,\big) f = \mathcal{F}_0^*\big(m(\,\cdot\,)\,\mathcal{F}_0 f\big).$$

This defines a functional calculus on $L^2 + \dot H_0^1$. In particular, for $N > 0$, we can define Littlewood–Paley projectors P_N by taking $m = \psi(N^{-1}\lambda)$, for suitable ψ compactly supported away from 0. We similarly define $P_{\le N}$ using $m = \phi(N^{-1}\lambda)$, where $\phi \in C_c^\infty(\mathbb{R})$ equals 1 on a neighborhood of 0. We also set $P_{\ge N} = 1 - P_{\le N}$.

As in the whole space case, we have the following proposition.

Proposition 4.1 (Bernstein inequality [14, 16]) *Let $1 \le p \le q \le \infty$, and suppose $\sigma \in \mathbb{R}$. Then for any $N > 0$*

$$\|P_{\le N} f\|_{L^q(\Omega)} \lesssim N^{n(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)},$$
$$\|(-\Delta_D)^{\frac{\sigma}{2}} P_N f\|_{L^p(\Omega)} \approx N^\sigma \|P_N f\|_{L^p(\Omega)}.$$

Now we introduce the L^p based Sobolev spaces on exterior domains. In [16], the authors first noticed that in the obstacle case, Sobolev spaces defined via the usual Laplacian Δ and the Dirichlet Laplacian Δ_D are not always equivalent unless one works with $L^p(\mathbb{R}^n)$, $1 < p < n$. For $p \ge n$, counterexamples can be constructed by modifying the eigenfunctions of the Dirichlet Laplacian. This is the first evidence of the subtle difference between the obstacle case and the whole space case. In a subsequent paper [14] the authors proved the equivalence between two Sobolev spaces in the general nonradial setting on the exterior domain of a strictly convex obstacle. Here we will just record a typical version of these results that is good enough to use in this paper.

Lemma 4.2 ([14]) *Let $0 < s \le 1$, $1 < p < \frac{n}{s}$ Then for any $C_c^\infty(\Omega)$ function f, we have*

$$\big\|\,|\nabla|^s f\big\|_p \sim \big\|(-\Delta_D)^{s/2} f\big\|_p.$$

Here $|\nabla|^s$ is defined for functions $f\colon \mathbb{R}^n \to \mathbb{C}$ by using the Fourier transform

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat f(\xi), \quad \xi \in \mathbb{R}^n.$$

Let I be a time interval. We now define several function spaces $S^0(I)$, $Z(I)$, $W(I)$ and $N(I)$ with the norms given by

$$\|u\|_{S^0(I)} := \|u\|_{L_t^\infty L_x^2 \cap L_t^2 L_x^{\frac{2n}{n-2}}(I\times\Omega)},$$
$$\|u\|_{Z(I)} := \|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I\times\Omega)},$$
$$\|u\|_{W(I)} := \big\|(-\Delta_D)^{\frac12-\frac1n} u\big\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n^2(n+2)}{n^3-8}}(I\times\Omega)},$$
$$\|u\|_{N(I)} := \big\|(-\Delta_D)^{\frac12-\frac1n} u\big\|_{L_t^2 L_x^{\frac{2n^2}{n^2+2n-4}}(I\times\Omega)}.$$

Keeping in mind that the exponents fall into restrictions defined in Lemma 4.2, we estimate

$$\begin{aligned}\|F(u)\|_{N(I)} &\lesssim \big\|(-\Delta_D)^{\frac12-\frac1n}F(u)\big\|_{L_t^2L_x^{\frac{2n^2}{n^2+2n-4}}(I\times\Omega)}\\ &\lesssim \big\||\nabla|^{1-\frac2n}F(u)\big\|_{L_t^2L_x^{\frac{2n^2}{n^2+2n-4}}(I\times\Omega)}\\ &\lesssim \|u\|_{Z(I)}^{\frac{4}{n-2}}\big\||\nabla|^{1-\frac2n}u\big\|_{L_t^{\frac{2(n+2)}{n-2}}L_x^{\frac{2n^2(n+2)}{n^3-8}}(I\times\Omega)} \lesssim \|u\|_{W(I)}^{\frac{n+2}{n-2}}.\end{aligned}$$

This important estimate will be used several times without explicit mention in Section 5 (*cf.* (5.8)).

5 Proof of Theorem 1.2

A direct consequence of our dispersive estimates is the following standard Strichartz estimates. Although we state it only for dimensions $n = 5, 7$, the lemma actually holds for all dimensions $n \geq 3$ provided the corresponding dispersive estimate holds.

Lemma 5.1 *Let I be a time interval containing* 0*. Let the dimension $n = 5, 7$. Let $\Omega = \mathbb{R}^n \setminus \overline{B(0,1)}$. Let $u = u(t,x) : I \times \Omega \to \mathbb{C}$ be radial and satisfy*

$$u(t,\,\cdot\,) = e^{it\Delta_D}u_0 - i\int_0^t e^{i(t-s)\Delta_D}f(s,\,\cdot\,)ds, \quad t \in I.$$

Here $u_0 = u(0,\,\cdot\,) \in L^2 + \dot H_0^1$ and $f \in L_t^1(L^2 + \dot H_0^1)$ are both radial.

Let (q_i, r_i), $i = 1, 2$ be admissible pairs such that $2 \leq q_i \leq +\infty$, $\frac{2}{q_i} + \frac{n}{r_i} = \frac{n}{2}$. Then

$$\|u\|_{L_t^{q_1}L_x^{r_1}(I\times\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} + \|f\|_{L_t^{q_2'}L_x^{r_2'}(\Omega)}.$$

Here (q_2', r_2') are the Hölder conjugate exponents of (q_2, r_2).

We will use Lemma 5.1 many times below without explicit mention. For convenience we shall refer to it simply as "the Strichartz estimate" or "Strichartz estimates".

For the nonlinear problem, we need to make the definition of the solution more precise. Let I be a finite time interval containing 0. As is well known (*cf.* [16, (2.3), p. 5]), the radial Sobolev inequality

$$\||x|^{\frac{n-2}{2}}f\|_{L^\infty(\Omega)} \lesssim \|f\|_{\dot H_0^1(\Omega)}$$

implies that radial $\dot H_0^1(\Omega)$ forms an algebra. So by $\dot H_0^1 \to \dot H_0^1$ boundedness of $\exp(it\Delta_D)$, we have

$$\Big\|\int_0^t e^{i(t-s)\Delta_D}\big(|u|^{\frac{4}{n-2}}u\big)(s)\,ds\Big\|_{L_t^\infty \dot H_0^1(I\times\Omega)} \lesssim |I|\cdot\|u\|_{L_t^\infty\dot H_0^1(I\times\Omega)}^{\frac{n+2}{n-2}}. \tag{5.1}$$

Therefore, if $u \in C(I; \dot H_0^1(\Omega))$, then the inhomogeneous term will also be in $\dot H_0^1(\Omega)$. This naturally motivates the following definition.

Definition 5.2 (Solution) Denote $F(u) = |u|^{\frac{4}{n-2}}u$. A radial function $u\colon I\times\Omega \to \mathbb{C}$ on a non-empty time interval $I \subset \mathbb{R}$ (possibly infinite or semi-infinite) is a *strong*

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$\dot{H}^1_0(\Omega)$ *solution* (or *solution* for short) to (1.1) if it lies in the class $C^0_t\dot{H}^1_0(I\times\Omega)$, and we have the Duhamel formula

$$u(t_1)=e^{i(t_1-t_0)\Delta_D}u(t_0)-i\int_{t_0}^{t_1}e^{i(t_1-t)\Delta_D}F(u(t))\,dt$$

for all $t_0,t_1\in I$. We refer to the interval I as the *lifespan* of u. We say that u is a *maximal-lifespan solution* if the solution cannot be extended to any strictly larger interval. We say that u is a *global solution* if $I=\mathbb{R}$.

Using (5.1) we can easily construct the local solution of (1.1) using a fixed point argument in $C^0_t\dot{H}^1_0(\Omega)$. Moreover, (5.1) shows that the lifespan of the local solution depends on the $\dot{H}^1_0(\Omega)$ norm of the initial data. Hence, the existence of the global solution then follows quickly from the energy conservation of the defocusing equation (1.1). More precisely, we have the following theorem.

Theorem 5.3 (Global well-posedness) *Let $u_0\in\dot{H}^1_0(\Omega)$ be spherically symmetric. Then there exists a unique global solution*

$$u\in C(\mathbb{R};\dot{H}^1_0(\Omega))\cap L_{t,loc}^{\frac{2(n+2)}{n}}\dot{H}_0^{1,\frac{2(n+2)}{n}}(\mathbb{R}\times\Omega).$$

Moreover, $\nabla u\in L^q_{t,loc}L^r_x(\mathbb{R}\times\Omega)$ for any admissible pair (q,r). For any $t\in\mathbb{R}$, we have

$$E(u(t))=\int_\Omega\Big(\frac{1}{2}|\nabla u(t,x)|^2\,dx+\frac{n-2}{2n}|u(t,x)|^{\frac{2n}{n-2}}\Big)\,dx=E(u_0).$$

For this global solution, scattering holds provided the global space-time $L_{t,x}^{\frac{2(n+2)}{n-2}}$ norm is bounded. Precisely, suppose that u satisfies

$$\|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([0,\infty)\times\Omega)}<\infty.$$

Then u scatters forward in time, i.e., *there exists a unique radial function $v_+\in\dot{H}^1_0(\Omega)$ such that*

$$\lim_{t\to\infty}\|e^{it\Delta_D}v_+-u(t)\|_{\dot{H}^1_0(\Omega)}=0.$$

The same statement holds backward in time.

The fact that global control of $L_{t,x}^{2(n+2)/(n-2)}$ norm implies finiteness of Strichartz norms and scattering is established by similar steps to those leading from (5.3) to (5.4). Furthermore, a standard continuity argument shows that if $\|u_0\|_{\dot{H}^1_0}<\epsilon$ for small ϵ, then the corresponding solution scatters in both time directions. (see [7] for instance).

Due to Theorem 5.3, the proof of Theorem 1.2 is reduced to showing that the $L_{t,x}^{2(n+2)/(n-2)}$ norm of the solution over any compact time interval is bounded by a constant which should only depend on the energy. Theorem 1.2 is thus a consequence of the following theorem.

Theorem 5.4 *Assume that $u \in \dot{H}^1_0(\Omega)$ is a spherically symmetric solution of (1.1) on a compact interval $[t_-, t_+]$. Suppose $E(u_0) \le E$. Then*

$$\|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([t_-,t_+]\times\Omega)} < C(E).$$

The rest of this section will be devoted to the proof of Theorem 5.4. We begin with some useful conventions.

Conventions Let $0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll \eta_0 \ll 1$ be small constants to be determined. We use $c(\eta_i)$ to denote a small constant depending on η_i such that $\eta_{i+1} \ll c(\eta_i) \ll \eta_i$. We use $C(\eta_i)$ to denote a large constant such that $\frac{1}{\eta_i} \ll C(\eta_i) \ll \frac{1}{\eta_{i+1}}$. The constants $c(\eta_i)$ and $C(\eta_i)$ will sometimes vary from line to line, but the dependence is clear from the context. The notation $a \lesssim b$ will be used to mean that $a \le C(E)\, b$, where $C(E)$ may depend on the energy upper-bound E.

We will use $\phi(x)$ to denote a radial smooth cutoff function such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| > 2. \end{cases} \tag{5.2}$$

We also denote $\phi_{<C}(x) = \phi(\frac{x}{C})$, $\phi_{>C} = 1 - \phi_{<C}$ for any $C > 0$.

We decompose

$$[t_-, t_+] = \bigcup_{j=1}^{J} I_j$$

such that $\eta_0 < \|u\|_{W(I_j)} \le 2\eta_0$. By Strichartz estimates [13], we have

$$\begin{aligned} \|\nabla u\|_{S^0(I_j\times\Omega)} &\lesssim \|u(t_i)\|_{\dot{H}^1_0(\Omega)} + \|\nabla(|u|^{\frac{4}{n-2}}u)\|_{L_{t,x}^{\frac{2(n+2)}{n+4}}(I_j\times\Omega)} \\ &\lesssim 1 + \|u\|^{\frac{4}{n-2}}_{W(I_j\times\Omega)} \|\nabla u\|_{L_{t,x}^{\frac{2(n+2)}{n}}(I_j\times\Omega)} \lesssim 1 + \eta_0^{\frac{4}{n-2}} \|\nabla u\|_{S^0(I_j\times\Omega)}. \end{aligned} \tag{5.3}$$

By taking η_0 small we have

$$\|\nabla u\|_{S^0(I_j\times\Omega)} \lesssim 1. \tag{5.4}$$

From interpolation we have

$$\|u\|_{W(I_j)} \lesssim \|u\|^{\frac{2}{n}}_{Z(I_j)} \|\nabla u\|^{1-\frac{2}{n}}_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(I_j\times\Omega)} \lesssim \|u\|^{\frac{2}{n}}_{Z(I_j)}.$$

Thus

$$\|u\|_{Z(I_j)} \gtrsim \eta_0^{\frac{n}{2}}. \tag{5.5}$$

We now use (5.5) and

$$\|f\|_{L_x^{\frac{2(n+2)}{n-2}}(\Omega)} \lesssim \|\nabla f\|_{L^2_x(\Omega)}$$

for $f \in \dot{H}^1_{0,rad}(\Omega)$ to show the uniform lower bound for the interval I_j:

$$\eta_0^{\frac{n}{2}} \lesssim \|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}(I_j\times\Omega)} \lesssim |I_j|^{\frac{n-2}{2(n+2)}} \|u\|_{L_t^\infty L_x^{\frac{2(n+2)}{n-2}}(I_j\times\Omega)} \lesssim |I_j|^{\frac{n-2}{2(n+2)}}.$$

We have

$$|I_j| \ge \eta_1, \quad j = 1, \dots, J.$$

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Now let

$$u_+(t) = e^{i(t-t_+)\Delta_D}u(t_+),\ u_-(t) = e^{i(t-t_-)\Delta_D}u(t_-).$$

We consider two cases for each I_j:

- I_j is called exceptional if either

$$\|u_+\|_{W(I_j\times\Omega)} > \eta_0^{10} \quad \text{or} \quad \|u_-\|_{W(I_j\times\Omega)} > \eta_0^{10};$$

- I_j is called unexceptional if

$$\|u_\pm\|_{W(I_j\times\Omega)} \le \eta_0^{10}.$$

Since

$$\|u_\pm\|_{W([t_-,t_+])\times\Omega)} \lesssim \|\nabla u_\pm\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}([t_-,t_+])\times\Omega)} \lesssim \|u(t_\pm)\|_{\dot H_0^1},$$

the number of exceptional intervals is bounded by $C(\eta_0, E)$. We thus need to control only the number of unexceptional intervals.

We now focus on the mass concentration property of the unexceptional intervals. We begin with local mass concentration.

Let ϕ be the smooth cutoff function defined in (5.2), and $\phi_R(x) = \phi(x/R)$. Define the local mass of u to be

$$M_R(t) = \int_{|x|\ge 1} |u(t,x)|^2 \phi_R^2(x)\,dx.$$

We have the following lemma.

Lemma 5.5 (Local mass concentration)

$$\Big|\frac{d}{dt}M_R^{\frac12}(t)\Big| \lesssim \frac{1}{R}. \tag{5.6}$$

Proof Using equation in (1.1), we compute

$$\begin{aligned}\frac{d}{dt}M_R(t) &= 2\Re\int_{|x|\ge1} u_t\,\bar u\,\phi_R^2(x)\,dx = -2\,\Im\int_{|x|\ge 1}\Delta u\cdot\bar u\,\phi_R^2(x)\,dx\\ &= \frac{2}{R}\,\Im\int_{|x|\ge1}\nabla u\,\bar u\,\phi_R(x)(\nabla\phi)(\frac{x}{R})\,dx\,.\end{aligned}$$

Therefore

$$\Big|\frac{d}{dt}M_R(t)\Big| \lesssim \frac{2}{R}\,\|\nabla u\|_{L^2(|x|\ge1)}\,\|u\,\phi_R\|_{L^2(|x|\ge1)} \lesssim \frac{1}{R}M_R^{\frac12}(t).$$

Equation (5.6) follows directly. ∎

We next establish the following important lemma.

Lemma 5.6 (Mass concentration on unexceptional intervals) *Let I be an unexceptional interval. Then for any $t\in I$,*

$$\int_{|x|<\frac{1}{\eta_3}|I|^{\frac12}} |u(t,x)|^2 dx \ge c(\eta_2)|I|.$$

Proof Denote $I = [a, b]$. Without loss of generality, we assume[1]

$$\|u\|_{W([\frac{a+b}{2},b]\times\Omega)} \geq \frac{\eta_0}{2}.$$

By the Duhamel formula

$$u(t) = u_-(t) - i\int_{t_-}^{a} e^{i(t-s)\Delta_D}F(u)(s)ds - i\int_a^t e^{i(t-s)\Delta_D}F(u)(s)ds.$$

We define

$$w(t) := i\int_{t_-}^{a} e^{i(t-s)\Delta_D}F(u)(s)ds = -u(t) + u_-(t) - i\int_a^t e^{i(t-s)\Delta_D}F(u)(s)ds.$$

We next observe that w has certain bounds. By Strichartz estimates and the steps leading from (5.3) to (5.5),

$$\sup_t \|w(t,\cdot)\|_{\dot H_0^1(\Omega)} \lesssim 1, \quad t \in I.$$

Moreover, we have

$$\|u_-\|_{W([\frac{a+b}{2},b]\times\Omega)} \leq \eta_0^{10}, \tag{5.7}$$

$$\Big\|\int_a^t e^{i(t-s)\Delta_D}F(u)(s)ds\Big\|_{W([\frac{a+b}{2},b])} \lesssim \|u\|_{W([\frac{a+b}{2},b])}^{\frac{n+2}{n-2}} \leq \eta_0^{\frac{n+2}{n-2}}. \tag{5.8}$$

By the triangle inequality we then have

$$\|w\|_{W([\frac{a+b}{2},b])} \geq \frac{\eta_0}{4},$$

and so

$$\|w\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)} \geq \Big(\frac{\eta_0}{4}\Big)^{\frac{n}{2}} \geqslant \eta_0^n.$$

We now show that the $L_{t,x}^{\frac{2(n+2)}{n-2}}$ norm of high frequency of w is negligible. To this end, we use the dispersive estimate to obtain

$$\begin{aligned}\tag{5.9} &\big\|P_{>C(\eta_2)|I|^{-\frac12}}w\big\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)}\\ &\quad= \Big\|\int_{t_-}^{a} e^{i(t-s)\Delta_D}P_{>C(\eta_2)|I|^{-\frac12}}F(u(s,\cdot))ds\Big\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)}\\ &\quad\lesssim |I|^{\frac{n-2}{2(n+2)}}\int_{t_-}^{a}|t-s|^{-\frac{2n}{n+2}}\big\|P_{>C(\eta_2)|I|^{-\frac12}}F(u(s,\cdot))\big\|_{L_x^{\frac{2(n+2)}{n+6}}(\Omega)}ds\\ &\quad\lesssim |I|^{-\frac{n-2}{2(n+2)}}\|P_{>C(\eta_2)|I|^{-\frac12}}F(u)\|_{L_t^\infty L_x^{\frac{2(n+2)}{n+6}}([t_-,a]\times\Omega)}.\end{aligned}$$

First consider the case that $\eta_2|I|^{\frac12} \geq 2$. We estimate the norm of $F(u)$ in different spatial regimes. First,

$$\begin{aligned}&\big\|P_{>C(\eta_2)|I|^{-\frac12}}\phi_{<\eta_2|I|^{\frac12}}F(u)\big\|_{L_x^{\frac{2(n+2)}{n+6}}(\Omega)}\\ &\quad\lesssim \|\phi_{<\eta_2|I|^{\frac12}}F(u)\|_{L_x^{\frac{2(n+2)}{n+6}}(\Omega)}\end{aligned}$$

[1]Otherwise this holds for $[a, \frac{a+b}{2}]$, and we apply a similar argument by just reversing the time direction.

$$\lesssim \|F(u)\|_{L_x^{\frac{2n}{n+2}}(\Omega)} \|\phi_{<\eta_2|I|^{\frac12}}\|_{L_x^{\frac{n(n+2)}{n-2}}(\Omega)}$$
$$\lesssim \eta_2^{\frac{n-2}{n+2}} |I|^{\frac{n-2}{2(n+2)}} \|u\|^{\frac{n+2}{n-2}}_{L_x^{\frac{2n}{n-2}}(\Omega)} \lesssim \eta_2^{\frac{n-2}{n+2}} |I|^{\frac{n-2}{2(n+2)}}.$$

Next, using the radial Sobolev embedding $\big\| |x|^{\frac{n-2}{n+2}} f\big\|_{L_x^{\frac{2(n+2)}{n-2}}} \lesssim \big\|\nabla f\big\|_{L_x^2}$, we estimate

$$\begin{aligned}
&\Big\| P_{>C(\eta_2)|I|^{-\frac12}}\big(\phi_{>\eta_2|I|^{\frac12}} F(u)\big)\Big\|_{L_x^{\frac{2(n+2)}{n+6}}(\Omega)} \\
&\quad\lesssim c(\eta_2)|I|^{\frac12}\Big\|\nabla\big(\phi_{>\eta_2|I|^{\frac12}}F(u)\big)\Big\|_{L_x^{\frac{2(n+2)}{n+6}}(\Omega)}\\
&\quad\lesssim c(\eta_2)|I|^{\frac12}\Big(\|\nabla\phi_{>\eta_2|I|^{\frac12}}\|_{L_x^{\frac{n(n+2)}{n-2}}(\Omega)}\|u\|^{\frac{n+2}{n-2}}_{L_x^{\frac{2n}{n-2}}(\Omega)} + \|\phi_{>\eta_2|I|^{\frac12}}u\|^{\frac{4}{n-2}}_{L_x^{\frac{2(n+2)}{n-2}}(\Omega)}\|\nabla u\|_{L_x^2(\Omega)}\Big)\\
&\quad\lesssim c(\eta_2)|I|^{\frac12}\big(\eta_2|I|^{\frac12}\big)^{-\frac{4}{n+2}}\big(\|u\|_{L_x^{\frac{2n}{n-2}}(\Omega)}+\|\nabla u\|_{L_x^2(\Omega)}\big)^{\frac{n+2}{n-2}}\\
&\quad\lesssim c(\eta_2)\,\eta_2^{-\frac{4}{n+2}}|I|^{\frac{n-2}{2(n+2)}} \le \eta_2^2\,|I|^{\frac{n-2}{2(n+2)}}.
\end{aligned}$$

Putting these two pieces back into (5.9), we have

$$\|P_{>C(\eta_2)|I|^{-\frac12}}w\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)} \le \eta_0^{2n}. \tag{5.10}$$

In the case where $\eta_2|I|^{\frac12} \le 2$, applying the same argument without spatial cutoff yields the better bound $c(\eta_2)|I|^{\frac12}$; we again get (5.10).

From (5.10) and interpolation, we see that

$$\|P_{>C(\eta_2)|I|^{-\frac12}}w\|_{W([\frac{a+b}{2},b])} \lesssim \eta_0^{2n\cdot\frac{2}{n}} \lesssim \eta_0^4.$$

This together with (5.7), (5.8), and the definition of unexceptional interval gives

$$\|P_{\ge C(\eta_2)|I|^{-\frac12}}u\|_{W([\frac{a+b}{2},b])} \lesssim \eta_0^{\frac{n+2}{n-2}}.$$

Thus,

$$\|P_{<C(\eta_2)|I|^{-\frac12}}u\|_{W([\frac{a+b}{2},b])} \ge \frac{\eta_0}{2}.$$

By interpolation, we have

$$\|P_{<C(\eta_2)|I|^{-\frac12}}u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)} \ge \Big(\frac{\eta_0}{2}\Big)^{\frac{n}{2}} \ge \eta_0^n. \tag{5.11}$$

On the other hand, using interpolation, radial Sobolev embedding and the lower bound for $|I|$ yield

$$\begin{aligned}
&\|\phi_{>\frac{1}{\eta_3}|I|^{\frac12}}u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)}\\
&\quad\lesssim |I|^{\frac{n-2}{2(n+2)}}\|\phi_{>\frac{1}{\eta_3}|I|^{\frac12}}u\|_{L_t^\infty L_x^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)}\\
&\quad\lesssim |I|^{\frac{n-2}{2(n+2)}}\Big(\frac{1}{\eta_3}|I|^{\frac12}\Big)^{-\frac{n-2}{n+2}}\big\|\nabla u\big\|_{L_x^2(\Omega)}\\
&\quad\le \frac{1}{100}\eta_0^n.
\end{aligned}$$

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Thus, (5.11) can be improved to

$$\big\| P_{<C(\eta_2)|I|^{-\frac{1}{2}}} \phi_{<\frac{1}{\eta_3}|I|^{\frac{1}{2}}} u \big\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)} \geq \frac{1}{2}\eta_0^n.$$

From this, the mass concentration follows quickly. Indeed, using the Bernstein and Hölder inequalities we have

$$\begin{aligned}
\frac{1}{2}\eta_0^n &\leq \big\| P_{<C(\eta_2)|I|^{-\frac{1}{2}}} \phi_{<\frac{1}{\eta_3}|I|^{\frac{1}{2}}} u \big\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}([\frac{a+b}{2},b]\times\Omega)} \\
&\lesssim |I|^{\frac{n-2}{2(n+2)}} \big(C(\eta_2)|I|^{-\frac{1}{2}} \big)^{n\left(\frac{1}{2}-\frac{n-2}{2(n+2)}\right)} \|\phi_{<\frac{1}{\eta_3}|I|^{\frac{1}{2}}} u\|_{L_t^\infty L_x^2([\frac{a+b}{2},b]\times\Omega)} \\
&\lesssim C(\eta_2)|I|^{-\frac{1}{2}} \|\phi_{<\frac{1}{\eta_3}|I|^{\frac{1}{2}}} u\|_{L_t^\infty L_x^2([\frac{a+b}{2},b]\times\Omega)}.
\end{aligned}$$

Thus, there exists $t_0 \in I$ such that

$$\|u(t_0)\|_{L^2(1\leq|x|<\frac{1}{\eta_3}|I|^{\frac{1}{2}})} > c(\eta_2)|I|^{\frac{1}{2}}.$$

Using (5.6) we get

$$\|u(t)\|_{L^2(1\leq|x|<\frac{1}{\eta_3}|I|^{\frac{1}{2}})} > c(\eta_2)|I|^{\frac{1}{2}}, \quad t \in I. \qquad \blacksquare$$

Next by applying an appropriate spatial cutoff, we can obtain the same space-localized Morawetz estimate

$$\int_I \int_{|x|\leq A|I|^{1/2}} \frac{|u(t,x)|^{2n/(n-2)}}{|x|}\, dt dx \lesssim A|I|^{1/2}$$

for all $A \geq 1$ as that in [6, 10, 16]. Combining the mass concentration on unexceptional intervals with the localized Morawetz estimate, we have the following lemma.

Lemma 5.7 ([6, 19]) *Let J be an interval that contains a contiguous collection $\bigcup_j I_j$ of unexceptional intervals. Then we have*

$$\sum_j |I_j|^{1/2} \leq C(\eta_2,\eta_3)|J|^{1/2}.$$

From this result, we can repeat an ingenious argument of Bourgain [6] to get the upper bound of the number of unexceptional intervals.

Proposition 5.8 ([6, 19]) *There exists $C(E,\eta_0,\eta_1,\eta_2,\eta_3)$ such that*

$$\#\{I_j, I_j \textit{ is unexceptional}\} \leq C(E,\eta_0,\eta_1,\eta_2,\eta_3).$$

This, together with the fact that the number of the exceptional intervals is finite, completes the proof of Theorem 1.2. ■

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