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M-COHYPONORMAL POWERS OF COMPOSITION OPERATORS

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Abstract

Let T_1, $i = 1, 2$ be measurable transformations which define bounded composition operators C_{T_i} on L^2 of a σ-finite measure space. Let us denote the Radon-Nikodym derivative of $m \circ T_i^{-1}$ with respect to m by h_i, $i = 1, 2$. The main result of this paper is that if $C_{T_1}^*$ and $C_{T_2}^*$ are both M-hyponormal with $h_1 \leq M^2(h_2 \circ T_2)$ a.e. and $h_2 \leq M^2(h_1 \circ T_1)$ a.e., then for all positive integers m, n and p, $[(C_{T_1}^m C_{T_2}^n)^p]^*$ is $M^{p^2(m+n)^2}$-hyponormal. As a consequence, we see that if C_T^* is an M-hyponormal composition operator, then $(C_T^*)^n$ is M^{n^2}-hyponormal for all positive integers n.

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1. Introduction

Let $(X, \sum, m)$ be a σ-finite measure space and let T be a measurable transformation from X into itself. Let $L^2 = L^2(X, \sum, m)$. Then the composition transformation C_T is defined by $C_T f = f \circ T$ for every $f \in L^2$. If C_T happens to be a bounded operator on L^2, then we call it the composition operator induced by T. C_T is a bounded linear operator on L^2 precisely when (i) $m \circ T^{-1}$ is absolutely continuous with respect to m and (ii) $h = dm \circ T^{-1}/dm$ is in $L^\infty(X, \sum, m)$. Let $R(C_T)$ denote the

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range of C_T and C_T^*, the adjoint of C_T. In what follows, $\mathbb{N}$ denotes the set of positive integers.

Let $B(H)$ denote the Banach algebra of all bounded linear operators on the Hilbert space H. An operator $T \in B(H)$ is called M-hyponormal if there exists some $M > 0$ such that $\|T^*x\| \leq M\|Tx\|$ for all $x \in H$.

Let T_1 and T_2 be measurable transformations of X into itself with Radon-Nikodym derivatives h_1 and h_2 respectively such that $C_{T_i} \in B(L^2)$ for $i = 1, 2$. It is shown in [2] that if $h_i \circ T_i \leq h_j$ for $i, j = 1, 2$, then for all positive integers m, n and p, the operator $(C_{T_1}^m C_{T_2}^n)^p$ is hyponormal. The aim of this paper is to obtain an analogous result when $C_{T_1}^*$ and $C_{T_2}^*$ are M-hyponormal operators.

2. Lemmas

LEMMA 2.1. *Let P be the projection of L^2 onto $\overline{R(C_T)}$. If C_T^* is M-hyponormal then*

$$((h \circ T)Pf, f) = ((h \circ T)f, f) \quad \textit{for all } f \in L^2.$$

PROOF. Since C_T^* is M-hyponormal, $\operatorname{Ker}(C_T^*) \subseteq \operatorname{Ker}(C_T)$. Also, for all $f \in L^2$, $Pf - f \in \operatorname{Ker}(P) = \operatorname{Ker}(C_T^*)$. Therefore,

$$\begin{aligned} ((h \circ T)Pf, f) &= ((h \circ T)(Pf - f), f) + ((h \circ T)f, f) \\ &= ((h \circ T)f, f) \quad \text{for all } f \in L^2, \end{aligned}$$

which proves the required result.

LEMMA 2.2. *If C_T^* is M-hyponormal, then*

$$h \leq M^2(h \circ T) \textit{ a.e.}$$

PROOF. Since C_T^* is M-hyponormal, we have

$$\|C_T f\|^2 \leq M^2 \|C_T^* f\|^2 \quad \text{for all } f \in L^2.$$

This implies that

$$(C_T f, C_T f) \leq M^2(C_T^* f, C_T^* f)$$

or

$$(C_T^* C_T f, f) \leq M^2(C_T C_T^* f, f)$$

or

$$(hf, f) \leq M^2((h \circ T)Pf, f) \quad \text{(by [2, Lemma 1.1(a)])}$$

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or

$$(hf, f) \leq M^2((h \circ T)f, f), \quad \text{(by Lemma 2.1)}$$

which yields

$$h \leq M^2(h \circ T) \text{ a.e.}$$

Lemma 2.1 can be extended to give the following result.

LEMMA 2.3. *Let P denote the projection of L^2 onto $\overline{R(C_T)}$. If C_T^* is M-hyponormal, then*

$$((h^n \circ T)Pf, f) = ((h^n \circ T)f, f) \quad \text{for all } f \in L^2 \text{ and } n \in \mathbb{N}.$$

LEMMA 2.4. *If $h \leq M^2(h \circ T)$ a.e., for all $r, m \in \mathbb{N}$ and $f \in L^2$, then*

$$\text{(2.4.1)} \qquad ((h \circ T)^r C_T^m f, C_T^m f) \leq M^{(m-1)(2r+m)}(h^{r+m} f, f).$$

PROOF. We shall prove the result by induction on m and fixed r. For $m = 1$ and $f \in L^2$,

$$\begin{aligned} ((h \circ T)^r C_T f, C_T f) &= \int (h \circ T)^r (|f|^2 \circ T)\, dm \\ &= \int h^r |f|^2\, dm \circ T^{-1} \\ &= \int h^r |f|^2 h\, dm \\ &= (h^{r+1} f, f), \end{aligned}$$

which shows that (2.4.1) holds for $m = 1$. Now assuming that (2.4.1) holds for $m = 1, 2, \ldots, k$ and $f \in L^2$, we have

$$\begin{aligned} ((h \circ T)^r C_T^{k+1} f, C_T^{k+1} f) &= ((h \circ T)^r C_T^k C_T f, C_T^k C_T f) \\ &\leq M^{(k-1)(2r+k)}(h^{r+k} C_T f, C_T f) \\ &\qquad \text{(by the induction hypothesis)} \\ &= M^{(k-1)(2r+k)} \int h^{r+k} (|f|^2 \circ T)\, dm \\ &\leq M^{(k-1)(2r+k)+2(r+k)} \int (h^{r+k} \circ T)(|f|^2 \circ T)\, dm \\ &\qquad \text{(since } h \leq M^2(h \circ T) \text{ a.e.)} \\ &= M^{k(2r+k+1)} \int h^{r+k} |f|^2 h\, dm \\ &= M^{k(2r+k+1)}(h^{r+k+1} f, f), \end{aligned}$$

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which completes the induction step and (2.4.1) holds for all $r, m \in \mathbb{N}$ and $f \in L^2$.

LEMMA 2.5. *If C_T^* is M-hyponormal, then for all $r, m \in$ and $f \in L^2$,*

$$M^{(m-1)(2r+m)}(h^r (C_T^m)^* f, (C_T^m)^* f) \geq ((h \circ T)^{r+m} f, f). \tag{2.5.1}$$

PROOF. We fix r and induct on m. For $m = 1$ and $f \in L^2$,

$$\begin{aligned}(h^r C_T^* f, C_T^* f) &= (C_T h^r C_T^* f, f)\\ &= ((h^r \circ T) C_T C_T^* f, f)\\ &= ((h^r \circ T)(h \circ T) P f, f) \quad \text{(by [2, Lemma 1.1(a)])}\\ &= ((h^{r+1} \circ T) P f, f)\\ &= ((h^{r+1} \circ T) f, f), \quad \text{(by Lemma 2.3)}\end{aligned}$$

which shows that the result holds for $m = 1$. Let us suppose that the result holds for $m = 1, 2, \ldots, k$ and $f \in L^2$. Then

$$\begin{aligned}(h^r (C_T^{k+1})^* f, (C_T^{k+1})^* f) &= (h^r (C_T^k)^* C_T^* f, (C_T^k)^* C_T^* f)\\ &\geq \frac{1}{M^{(k-1)(2r+k)}} ((h \circ T)^{r+k} C_T^* f, C_T^* f)\end{aligned} \tag{2.5.2}$$

(by induction hypothesis).

But $M^2(h \circ T) \geq h$ a.e., so that $M^{2(r+k)}(h \circ T)^{r+k} \geq h^{r+k}$ a.e. Thus

(2.5.3)

$$\begin{aligned}((h \circ T)^{r+k} C_T^* f, C_T^* f) &\geq \frac{1}{M^{2(r+k)}} (h^{r+k} C_T^* f, C_T^* f)\\ &= \frac{1}{M^{2(r+k)}} ((h^{r+k} \circ T) C_T C_T^* f, f)\\ &= \frac{1}{M^{2(r+k)}} ((h^{r+k} \circ T)(h \circ T) P f, f)\\ &= \frac{1}{M^{2(r+k)}} ((h^{r+k+1} \circ T) P f, f)\\ &= \frac{1}{M^{2(r+k)}} ((h^{r+k+1} \circ T) f, f) \quad \text{(by Lemma 2.3)}\\ &= \frac{1}{M^{2(r+k)}} ((h \circ T)^{r+k+1} f, f).\end{aligned}$$

Hence, by the use of (2.5.2) and (2.5.3), we have

$$M^{k(2r+k+1)}(h^r (C_T^{k+1})^* f, (C_T^{k+1})^* f) \geq ((h \circ T)^{r+k+1} f, f),$$

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which shows that the result holds for $m = k+1$. Thus the result holds for all $r, m \in \mathbb{N}$ and $f \in L^2$.

LEMMA 2.6. *If $h \le M^2(h \circ T)$ a.e., then for all $n \in \mathbb{N}$ and $f \in L^2$,*

$$((C_T^n)^* C_T^n f, f) \le M^{n(n-1)}(h^n f, f).$$

PROOF. For $n = 1$, the result is true since $C_T^* C_T f = hf$. Let us suppose that the result is true for $n = r$ and $f \in L^2$. Then

$$\begin{aligned}((C_T^{r+1})^* C_T^{r+1} f, f) &= ((C_T^r C_T)^* C_T^r C_T f, f) = ((C_T^r)^* C_T^r C_T f, C_T f) \\ &\le M^{r(r-1)}(h^r C_T f, C_T f) \quad \text{(by the induction hypothesis)}.\end{aligned}$$

Now, since $h \le M^2(h \circ T)$ a.e., $h^r \le M^{2r}(h^r \circ T)$ a.e. and so

$$\begin{aligned}(h^r C_T f, C_T f) &= \int h^r(|f|^2 \circ T)\, dm \\ &\le M^{2r} \int (h^r \circ T)(|f|^2 \circ T)\, dm \\ &= M^{2r} \int h^r |f|^2 h\, dm = M^{2r}(h^{r+1} f, f).\end{aligned}$$

Hence

$$((C_T^{r+1})^* C_T^{r+1} f, f) \le M^{r(r-1)} M^{2r}(h^{r+1} f, f) = M^{r(r+1)}(h^{r+1} f, f),$$

which completes the induction step and the result follows.

LEMMA 2.7. *If C_T^* is M-hyponormal, then*

$$M^{n(n-1)}(C_T^n (C_T^n)^* f, f) \ge ((h \circ T)^n f, f)$$

for all $n \in \mathbb{N}$ and $f \in L^2$.

PROOF. The result can be proved using induction on n by applying similar techniques as in Lemma 2.6.

3. Main results

In this section we shall prove our main results.

THEOREM 3.1. *If C_T^* is M-hyponormal, then $(C_T^*)^n$ is M^{n^2}-hyponormal for all $n \in \mathbb{N}$.*

PROOF. Since C_T^* is M-hyponormal, for all $n \in \mathbb{N}$ and $f \in L^2$,

$$M^{n(n+1)}(C_T^n (C_T^n)^* f, f) \ge (h^n f, f) \quad \text{(by Lemmas 2.7 and 2.2)}.$$

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Also, by the use of Lemma 2.6, for all $n \in \mathbb{N}$ and $f \in L^2$,

$$((C_T^n)^* C_T^n f, f) \leq M^{n(n-1)}(h^n f, f).$$

Hence, for all $n \in \mathbb{N}$ and $f \in L^2$, we have either

$$((C_T^n)^* C_T^n f, f) \leq M^{2n^2}(C_T^n (C_T^n)^* f, f)$$

or

$$\|C_T^n f\|^2 \leq M^{2n^2} \|(C_T^n)^* f\|^2$$

or

$$\|C_T^n f\| \leq M^{n^2} \|(C_T^n)^* f\|,$$

which proves the required result.

THEOREM 3.2. *With T_1, T_2, h_1 and h_2 as above, let $A = C_{T_1}$ and $B = C_{T_2}$. If A^* and B^* are M-hyponormal such that*

$$h_1 \leq M^2(h_2 \circ T_2) \quad a.e.,$$

and

$$h_2 \leq M^2(h_1 \circ T_1) \quad a.e.,$$

then $(A^m B^n)^*$ is $M^{(m+n)^2}$-hyponormal for all $m, n \in \mathbb{N}$.

PROOF. Since A^* and B^* are M-hyponormal, by Lemma 2.2,

$$h_i \leq M^2(h_i \circ T_i) \quad \text{for } i = 1, 2.$$

Now for $f \in L^2$,

$$\begin{aligned}((A^m B^n)^*(A^m B^n)f, f) &= ((A^m)^* A^m B^n f, B^n f) \\ &\leq M^{m(m-1)}(h_1^m B^n f, B^n f) \quad \text{(by Lemma 2.6)} \\ &\leq M^{m(m+1)}((h_2 \circ T_2)^m B^n f, B^n f) \\ &\qquad \text{(since } h_1 \leq M^2(h_2 \circ T_2)\text{a.e.)} \\ &\leq M^{m(m+1)+(n-1)(2m+n)}(h_2^{m+n} f, f) \quad \text{(by Lemma 2.4)} \\ &= M^{(m+n)^2-(m+n)}(h_2^{m+n} f, f).\end{aligned}$$

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On the other hand,

$$\begin{aligned}((A^mB^n)(A^mB^n)^*f, f) &= (A^mB^n(B^n)^*(A^m)^*f, f) \\ &= (B^n(B^n)^*(A^m)^*f, (A^m)^*f) \\ &\geq \frac{1}{M^{n(n-1)}}((h_2\circ T_2)^n(A^m)^*f, (A^m)^*f) \\ &\qquad \text{(by Lemma 2.7)} \\ &\geq \frac{1}{M^{n(n+1)}}(h_1^n(A^m)^*f, (A^m)^*f) \quad \text{(by hypothesis)} \\ &\geq \frac{1}{M^{n(n+1)+(m-1)(2n+m)}}((h_1\circ T_1)^{n+m}f, f) \\ &\qquad \text{(by Lemma 2.5)} \\ &= \frac{1}{M^{(m+n)^2-(m+n)}}((h_1\circ T_1)^{n+m}f, f) \\ &\geq \frac{1}{M^{(m+n)^2+(m+n)}}(h_2^{m+n}f, f) \quad \text{(by hypothesis)}.\end{aligned}$$

Thus, for all $f\in L^2$, we have either

$$((A^mB^n)^*(A^mB^n)f, f) \leq M^{2(m+n)^2}((A^mB^n)(A^mB^n)^*f, f)$$

or

$$\|(A^mB^n)f\|^2 \leq M^{2(m+n)^2}\|(A^mB^n)^*f\|^2$$

or

$$\|(A^mB^n)f\| \leq M^{(m+n)^2}\|(A^mB^n)^*f\|,$$

so that $(A^mB^n)^*$ is $M^{(m+n)^2}$-hyponormal.

Following the same lines as in the proof of Theorem 3.2 and induction on p, we can prove the following theorem.

THEOREM 3.3. *Under the hypothesis of Theorem* 3.2, *we have*

(3.3.1) $$([(A^mB^n)^p]^*(A^mB^n)^pf, f) \leq M^{p^2(m+n)^2-p(m+n)}(h_2^{p(m+n)}f, f)$$

and

(3.3.2) $$M^{p^2(m+n)^2+p(m+n)}((A^mB^n)^p[(A^mB^n)^p]^*f, f) \geq (h_2^{p(m+n)}f, f),$$

for all m, n *and* $p\in\mathbb{N}$ *and* $f\in L^2$.

With the help of Theorem 3.3, *we can generalize Theorem* 3.2 *in the following form.*

THEOREM 3.4. *Under the hypothesis of Theorem* 3.2, $[(A^mB^n)^p]^*$ *is* $M^{p^2(m+n)^2}$*-hyponormal.*

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PROOF. Using (3.3.1) and (3.3.2), for all m, n and $p \in \mathbb{N}$ and $f \in L^2$ we have that

$$([(A^m B^n)^p]^*(A^m B^n)^p f, f) \leq M^{2p^2(m+n)^2}((A^m B^n)^p[(A^m B^n)^p]^* f, f)$$

or

$$\|(A^m B^n)^p f\|^2 \leq M^{2p^2(m+n)^2}\|[(A^m B^n)^p]^* f\|^2$$

or

$$\|(A^m B^n)^p f\| \leq M^{p^2(m+n)^2}\|[(A^m B^n)^p]^* f\|,$$

which completes the proof of the theorem.

COROLLARY 3.5. *Under the hypothesis of Theorem* 3.2, $[(AB)^p]^*$ *is* M^{4p}-*hyponormal. In particular,* $(AB)^*$ *is* M^4-*hyponormal.*

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