The Absolute Summability (A) of Fourier Series

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Sufficient conditions for the absolute summability (A)[1] of a Fourier series have been given by J. M. Whittaker[2] and B. N. Prasad[3]. They obtained theorem 1 below in the cases $\alpha = 0$ and $\alpha = 1$ respectively. Theorem 1 is contained in theorem 2, which was given by Prasad[4] in the case $\alpha = 0$.

We suppose $f(t)$ to be defined and integrable L in $(-\pi, \pi)$ and periodic outside, and write $\phi(t) = \frac{1}{2}\{f(\theta + t) + f(\theta - t) - 2s\}$. We define $\Phi_\alpha(t)$ as the Riemann-Liouville integral of order α (not necessarily an integer) of $\phi(t)$, and we suppose throughout that $t > 0$. Then

$$\begin{cases} \Phi_\alpha(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-u)^{\alpha-1} \phi(u)\, du, & \alpha > 0, \\ \Phi_0(t) = \phi(t). \end{cases} \tag{1}$$

It is well known that, if $\beta > 0$,

$$\Phi_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} \Phi_\alpha(u)\, du. \tag{2}$$

Accordingly we define $\Phi_\alpha(t)$, for $-1 < \alpha < 0$, by

$$\Phi_\alpha(t) = \frac{d}{dt}\, \Phi_{\alpha+1}(t). \tag{3}$$

We define the mean value $\phi_\alpha(t)$ of $\phi(t)$ by

$$\phi_\alpha(t) = \Gamma(\alpha + 1)\, t^{-\alpha} \Phi_\alpha(t). \tag{4}$$

We also write

$$\begin{cases} \Psi_\alpha(t) = \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-u)^{\alpha-2} \{u\phi(u) - \Phi_1(u)\}\, du, & \alpha > 1, \\ \Psi_1(t) = t\phi(t) - \Phi_1(t), \end{cases} \tag{5}$$

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[1] Σa_n is said to be *absolutely summable* (A) if $\Sigma a_n x^n$ converges to $A(x)$ for $0 \leq x < 1$ and $A(x)$ is of *bounded variation* in $(0, 1)$. The sum is then $\lim_{x \to 1-0} A(x)$. See Whittaker, 4.

[2] Whittaker, 4.

[3] Prasad, 2, 3.

[4] Prasad, 2, 3.

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and extend this, for $0 < a < 1$, by

$$\Psi_a(t) = \frac{d}{dt}\Psi_{a+1}(t).[1] \tag{6}$$

Theorem 1. *If, for some* $a \geqq 0$ and $\eta > 0$,

$$\int_0^\eta \frac{|\phi_a(t)|}{t}\,dt < \infty, \tag{7}$$

then the Fourier series of $f(t)$ *is absolutely summable* (A) *for* $t = \theta$ *to the sum* s.

This is included in

Theorem 2. *If, for some* $a \geqq 0$, $\phi_a(t)$ *is of bounded variation in* $(0, \eta)$ *and* $\phi_a(t) \to 0$ *as* $t \to 0$, *then the Fourier series of* $f(t)$ *is absolutely summable* (A) *for* $t = \theta$ *to the sum* s.

We use the following lemmas.

Lemma 1. If $a > 0$, then

$$\Gamma(a+1)\,t^{-a}\,\Psi_a(t) = a\,\{\phi_{a-1}(t) - \phi_a(t)\} = t\phi'_a(t). \tag{8}$$

This follows easily from the definitions.[2]

Lemma 2. *If* $F_a(t)$ *is the* a*-th integral of* $f(t)$ *and* $a \geqq 0$, then

$$\Gamma(a+1)\int_0^\eta \frac{|F_a(t)|}{t^{1+a}}\,dt \tag{9}$$

is a non-increasing function of a.

We have, for $\beta > a$,

$$\begin{aligned}\Gamma(\beta+1)\int_0^\eta \frac{|F_\beta(t)|}{t^{1+\beta}}\,dt &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-a)}\int_0^\eta \frac{dt}{t^{1+\beta}}\left|\int_0^t (t-u)^{\beta-a-1}F_a(u)\,du\right| \\ &\leqq \frac{\Gamma(\beta+1)}{\Gamma(\beta-a)}\int_0^\eta \frac{dt}{t^{1+\beta}}\int_0^t (t-u)^{\beta-a-1}|F_a(u)|\,du \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-a)}\int_0^\eta |F_a(u)|\,du\int_u^\eta (t-u)^{\beta-a-1}t^{-1-\beta}\,dt \\ &\leqq \frac{\Gamma(\beta+1)}{\Gamma(\beta-a)}\int_0^\eta \frac{|F_a(u)|}{u^{1+a}}\,du\int_1^\infty (v-1)^{\beta-a-1}v^{-1-\beta}\,dv \\ &= \Gamma(a+1)\int_0^\eta \frac{|F_a(u)|}{u^{1+a}}\,du,\end{aligned}$$

which proves the lemma.

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[1] If $\phi(t)$ is the integral of its derivative, $\Psi_a(t)$ is the a-th integral of $t\phi'(t)$. The relation of the function $t\phi'(t)$ to $\phi(t)$ is analogous to that of the sequence na_n to $a_0 + \ldots + a_n$.

[2] *Cf.* Bosanquet, 1, lemma 2.

Lemma 3. *If $\phi_\alpha(t)$ is of bounded variation in* $(0, \eta)$ *for a given* $\alpha \geqq 0$, *then this is true if α is replaced by $\beta > \alpha$.*

If $\alpha > 0$, we have, by lemma 1,

$$\int_0^\eta |\phi'_\alpha(t)|\, dt = \Gamma(\alpha+1) \int_0^\eta \frac{|\Psi_\alpha(t)|}{t^{1+\alpha}}\, dt,$$

and the result follows from lemma 2. The case $\alpha = 0$ may be left to the reader.

Let

$$K(x, t) = \frac{1-x^2}{1-2x\cos t + x^2}. \tag{10}$$

Then we have the following two lemmas.

Lemma 4. *If* $0 < t < \frac{1}{2}\pi$, $0 < x < 1$, *and r is a non-negative integer, then*

$$\left| \frac{\partial^{r+1}}{\partial x\, \partial t^r} K(x, t) \right| \begin{cases} \leqq A t^{-2-r} \\ \leqq A(1-x)^{-2} t^{-r}, \end{cases} \tag{11}$$

when A is independent of x and t.

It is easily verified by induction that

$$\frac{\partial^r}{\partial t^r} K(x, t) = (1-x^2) \sum_{\lambda=0}^{r} P_\lambda(t) \frac{x^{r-\lambda} \sin^{r-2\lambda} t}{(1-2x\cos t + x^2)^{1+r-\lambda}},$$

where $P_\lambda(t)$ is a trigonometrical polynomial. Hence it is easy to see[1] that

$$\frac{\partial^{r+1}}{\partial x\, \partial t^r} K(x, t) = \sum_{\lambda=0}^{r} O\left\{\frac{\sin^{r-2\lambda} t}{(1-2x\cos t + x^2)^{1+r-\lambda}}\right\}$$

uniformly in x and t.

Now

$$1 - 2x\cos t + x^2 = (x - \cos t)^2 + \sin^2 t \geqq \sin^2 t. \tag{12}$$

Therefore

$$\frac{\partial^{r+1}}{\partial x\, \partial t^r} K(x, t) = \sum_{\lambda=0}^{r} O\{\sin^{r-2\lambda-2(1+r-\lambda)} t\} = O(t^{-2-r}).$$

On the other hand

$$1 - 2x\cos t + x^2 = (1-x)^2 + 4x\sin^2 \tfrac{1}{2}t \geqq (1-x)^2. \tag{13}$$

Therefore by (12) and (13),

$$\begin{aligned} \frac{\partial^{r+1}}{\partial x\, \partial t^r} K(x, t) &= \sum_{\lambda=0}^{r} O\{(1-x)^{-2} \sin^{r-2\lambda-2(r-\lambda)} t\} \\ &= O\{(1-x)^{-2} t^{-r}\}, \end{aligned}$$

which completes the proof.

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[1] Since $P_\lambda(t)$ $(\lambda = 0, 1, \ldots r)$ and $\dfrac{(1-x)(x-\cos t)}{1-2x\cos t + x^2}$ are bounded. The last fact follows from (12) and (13).

Lemma 5. *If* $0 < t < \frac{1}{2}\pi$, $0 < x < 1$, *and* r *is a non-negative integer, then*

$$\left| \int_0^t u^r \frac{\partial^{r+1}}{\partial x \, \partial u^r} K(x, u)\, du \right| \begin{cases} \leqq A t^{-1} \\ \leqq A(1-x)^{-2} t, \end{cases} \tag{14}$$

where A *is independent of* x *and* t.

If $r = 0$ we have

$$\left| \int_0^t \frac{\partial}{\partial x} K(x, u)\, du \right| = \left| \frac{2 \sin t}{1 - 2x \cos t + x^2} \right| \begin{cases} \leqq A (\sin t)^{-1} \\ \leqq A (1-x)^{-2} \sin t, \end{cases}$$

from which the result follows. On the other hand, if $r > 0$,

$$\int_0^t u^r \frac{\partial^{r+1}}{\partial x \, \partial u^r} K(x, u)\, du = t^r \frac{\partial^r}{\partial x \, \partial t^{r-1}} K(x, t) - r \int_0^t u^{r-1} \frac{\partial^r}{\partial x \, \partial u^{r-1}} K(x, u)\, du,$$

and the result follows from lemma 4 by induction.

Proof of theorem 2. If

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{15}$$

is the Fourier series of $f(t)$, and

$$P(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) x^n, \tag{16}$$

then, for $0 < x < 1$,

$$\begin{aligned} Q(x) &= P(x) - s \\ &= \frac{1}{\pi} \int_0^\pi \phi(t) K(x, t)\, dt. \end{aligned} \tag{17}$$

To show that (15) is absolutely summable (A) for $t = \theta$ we must show that

$$\int_0^1 | Q'(x) |\, dx < \infty. \tag{18}$$

Let

$$Q(x) = \frac{1}{\pi} \left\{ \int_0^\eta + \int_\eta^\pi \right\} \phi(t) K(x, t)\, dt = Q_1(x) + Q_2(x).$$

Then Whittaker and Prasad have remarked that we need only prove that

$$\int_0^1 | Q'_1(x) |\, dx < \infty, \tag{19}$$

and we may suppose that $0 < \eta \leqq \frac{1}{2}\pi$.

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By lemma 3 we may suppose that $\alpha = \kappa$ is a positive integer. Now integration by parts gives

$$Q'_1(x) = \int_0^\eta \phi(t) \frac{\partial K}{\partial x} \partial t$$

$$= \left[\sum_{\rho=0}^{\kappa} (-1)^{\rho-1} \Phi_\rho(t) \frac{\partial^\rho}{\partial x\, \partial t^{\rho-1}} K(x, t) \right]_{t=0}^{\eta}$$

$$+ (-1)^\kappa \int_0^\eta \Phi_\kappa(t) \frac{\partial^{\kappa+1}}{\partial x\, \partial t^\kappa} K(x, t)\, dt,$$

and

$$\int_0^\eta \phi_\kappa(t)\, t^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial t^\kappa} K(x, t)\, dt = \left[\phi_\kappa(t) \int_0^t u^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du \right]_{t=0}^{\eta}$$

$$- \int_0^\eta \phi'_\kappa(t)\, dt \int_0^t u^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du.$$

Hence

$$\int_0^1 | Q'_1(x) |\, dx \leqq \int_0^1 dx \sum_{\rho=0}^{\kappa} \left| \Phi_\rho(\eta) \frac{\partial^\rho}{\partial x\, \partial \eta^{\rho-1}} K(x, \eta) \right|$$

$$+ \frac{1}{\Gamma(\kappa+1)} \int_0^1 dx \left| \phi_\kappa(\eta) \int_0^\eta u^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du \right|$$

$$+ \frac{1}{\Gamma(\kappa+1)} \int_0^1 dx \int_0^\eta | \phi'_\kappa(t) |\, dt \left| \int_0^t u \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du \right|.$$

The first two terms are finite by the first inequalities in (11) and (14) respectively, with $t = \eta$. Therefore

$$\int_0^1 | Q'_1(x) |\, dx \leqq A + \frac{1}{\Gamma(\kappa+1)} \int_0^\eta | \phi'_\kappa(t) |\, dt \int_0^1 dx \left| \int_0^t u^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du \right|,$$

where A denotes some constant.

Now write

$$\int_0^1 dx \left| \int_0^t u^\kappa \frac{\partial^{\kappa+1}}{\partial x\, \partial u^\kappa} K(x, u)\, du \right| = \int_0^{1-t} dx + \int_{1-t}^1 dx = I_1 + I_2.$$

Then, by lemma 5, we have, uniformly for $0 < t < \eta$,

$$I_1 = \int_0^{-t} t\ O\{(1-x)^{-2}\}\, dx = O(1)$$

and

$$I_2 = \int_{1-t}^1 t^{-1} O(1)\, dx = O(1).$$

Since

$$\int_0^\eta | \phi'_\kappa(t) |\, dt < \infty$$

(19) now follows.

To complete the proof we must show that

$$\lim_{x \to 1-0} Q(x) = 0. \tag{20}$$

This may be deduced from the hypothesis $\phi_a(t) \to s$ as $t \to 0$ by analysis of a similar nature to that already used. On the other hand it is well known that this hypothesis implies the summability (C) of the Fourier series to the sum s, and thus its summability (A) to the same sum.

Finally, to deduce theorem 1 from theorem 2, it is only necessary to observe that, by lemma 1,

$$\begin{aligned} \int_0^\eta |\phi'_{a+1}(t)|\,dt &= (a+1)\int_0^\eta \frac{|\phi_a(t) - \phi_{a+1}(t)|}{t}\,dt \\ &\leqq (a+1)\left\{\int_0^\eta \frac{|\phi_a(t)|}{t}\,dt + \int_0^\eta \frac{|\phi_{a+1}(t)|}{t}\,dt\right\} \\ &\leqq 2(a+1)\int_0^\eta \frac{|\phi_a(t)|}{t}\,dt, \end{aligned}$$

by lemma 2. Thus $\phi_{a+1}(t)$ is of bounded variation in $(0, \eta)$, and so tends to a limit as $t \to 0$. Since $\phi_{a+1}(t)/t$ is integrable this limit must be zero.

Added 13*th November* 1933. More precise results may be obtained by employing absolute summability (C), which itself implies absolute summability (A). It is true, for instance, that *a necessary and sufficient condition that the Fourier series should be absolutely summable* (C) *for* $t = \theta$ *is that* $\phi_a(t)$ *should be of bounded variation in* $(0, \eta)$ *for some* a.[1]

[1] A preliminary account of these results will be published in the *Mathematical Gazette*.

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