Proceedings of the Edinburgh Mathematical Society (2013) **56**, 303–336
DOI:10.1017/S0013091512000223

THE LAITINEN CONJECTURE FOR FINITE NON-SOLVABLE GROUPS

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Dedicated to Bob Oliver and Ron Solomon on the occasion of their 60th birthdays

Abstract For any finite group G, we impose an algebraic condition, the G^{nil}-*coset condition*, and prove that any finite Oliver group G satisfying the G^{nil}-coset condition has a smooth action on some sphere with isolated fixed points at which the tangent G-modules are not isomorphic to each other. Moreover, we prove that, for any finite non-solvable group G not isomorphic to $\mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2,27)$, the G^{nil}-coset condition holds if and only if $r_G \geqslant 2$, where r_G is the number of real conjugacy classes of elements of G not of prime power order. As a conclusion, the Laitinen Conjecture holds for any finite non-solvable group not isomorphic to $\mathrm{Aut}(A_6)$.

Keywords: Smith equivalence; real G-module; Oliver group; Laitinen conjecture; non-solvable group

2010 *Mathematics subject classification:* Primary 57S17; 57S25
Secondary 55M35; 20C15

1. The Laitinen Conjecture

Let G be a finite group. A *real G-module* V is a finite-dimensional real vector space equipped with a linear action of G, i.e. the action is given by a representation $G \to \mathrm{GL}(V)$.

Let U and V be two real G-modules. Following [**48**], U and V are called *Smith equivalent* if there exists a smooth action of G on a homotopy sphere Σ with exactly two fixed points, say $\Sigma^G = \{x, y\}$, at which the tangent G-modules are isomorphic to U and V, respectively. The tangent G-modules are determined on the tangent spaces $T_x(\Sigma)$ and $T_y(\Sigma)$ at x and y by taking the derivatives at x and y of the diffeomorphisms $\Sigma \to \Sigma$, $z \mapsto gz$, for all $g \in G$.

Following [**40**], U and V are called *Laitinen–Smith equivalent* if U and V are Smith equivalent and the action of G on Σ is such that for any element $g \in G$ of order 2^a for $a \geqslant 3$, the set $\Sigma^g = \{z \in \Sigma \mid gz = z\}$ is connected. Here, G is not a cyclic group of order 2^a for $a \geqslant 3$.

In 1960, Paul A. Smith posed a question which can be restated as follows.*

* The question is posed in [**54**], in the footnote on p. 406.

Problem 1.1 (the Smith Equivalence Problem). *Let G be a finite group. Is it true that any two Smith equivalent (respectively, Laitinen–Smith equivalent) real G-modules are isomorphic?*

Following [**24**], for a finite group G, we denote by r_G the number of real conjugacy classes $(g)^{\pm} = (g) \cup (g^{-1})$ of elements $g \in G$ which are not of prime power order.

In August 1996, Erkki Laitinen posed the following conjecture (see [**24**, Appendix], in which the Laitinen–Smith equivalence is called 2-proper Smith equivalence).

Conjecture 1.2 (the Laitinen Conjecture). *A finite Oliver group G has two non-isomorphic Laitinen–Smith equivalent real G-modules if and only if* $r_G \geqslant 2$.

A finite group G is called a *Laitinen group* if G is not of prime power order and there exist two non-isomorphic Laitinen–Smith equivalent real G-modules. So, the Laitinen Conjecture predicts that a finite Oliver group G is a Laitinen group if and only if $r_G \geqslant 2$.

Let $\mathrm{Aut}(A_6)$ be the group of automorphism of the alternating group A_6 on six letters, and let $\mathrm{P\Sigma L}(2,27)$ be the splitting extension associated with the exact sequence

$$1 \to \mathrm{PSL}(2,27) \to \mathrm{P\Sigma L}(2,27) \to \mathrm{Aut}(\mathbb{F}_{27}) \to 1$$

for the projective special linear group $\mathrm{PSL}(2,27)$ and the group $\mathrm{Aut}(\mathbb{F}_{27})$ of automorphisms of the field $\mathbb{F}_{27}$ of 27 elements. The groups $\mathrm{Aut}(A_6)$ and $\mathrm{P\Sigma L}(2,27)$ are not solvable.

In Definition 5.1, for any finite group G, we impose the G^{nil}*-coset condition*, which implies that $r_G \geqslant 2$. But, it may be that $r_G \geqslant 2$ and G does not satisfy the G^{nil}-coset condition. In fact, for $G = \mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2,27)$, $r_G = 2$ (see [**40**, Proposition 3.1]) but neither $\mathrm{Aut}(A_6)$ nor $\mathrm{P\Sigma L}(2,27)$ satisfies the G^{nil}-coset condition (see Lemma 7.6).

Now, we are ready to state our main theorems (Theorems A, B and C).

Theorem A. *If a finite Oliver group G satisfies the* G^{nil}*-coset condition, then G is a Laitinen group.*

Theorem B. *Let G be a finite non-solvable group not isomorphic to* $\mathrm{Aut}(A_6)$ *or* $\mathrm{P\Sigma L}(2,27)$. *Then G satisfies the* G^{nil}*-coset condition if and only if* $r_G \geqslant 2$.

Theorem C. *Let G be a finite non-solvable group. Then G is a Laitinen group if and only if* $r_G \geqslant 2$ *and G is not isomorphic to* $\mathrm{Aut}(A_6)$.

According to [**29**], for $G = \mathrm{Aut}(A_6)$, any two Smith equivalent real G-modules are isomorphic. Therefore, the Laitinen Conjecture is not true for $G = \mathrm{Aut}(A_6)$.

By Theorem C, the Laitinen Conjecture holds for any finite non-solvable group that is not isomorphic to $\mathrm{Aut}(A_6)$. Theorem C was obtained earlier in the case where G is a finite perfect group [**24**, Theorem A] or, more generally, a finite non-solvable *gap group* [**40**, Theorem B3], except for $G = \mathrm{P\Sigma L}(2,27)$, the case covered by [**30**, Theorem 1.1].*

The results of [**24**, Theorem A], [**30**, Theorem 1.1] and [**40**, Theorems B1–B3, p. 851] can be restated in the following way (see Theorems 1.3, 1.4 and 1.5 herein, respectively).

* We refer the reader to [**35**, **57**, **58**, **62**, **63**] for information about gap groups. We recall that $\mathrm{Aut}(A_6)$ is not a gap group, while $\mathrm{P\Sigma L}(2,27)$ is a gap group.

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

Theorem 1.3 (Laitinen and Pawałowski [24]). *A finite perfect group G is a Laitinen group if and only if $r_G \geqslant 2$.*

Theorem 1.4 (Pawałowski and Solomon [40]). *If a finite Oliver group G is of odd order or has a cyclic quotient of order pq for two distinct odd primes p and q, then G is a Laitinen group. In particular, any finite abelian (more generally, nilpotent) Oliver group G is a Laitinen group.*

Theorem 1.5 (Pawałowski and Solomon [40]; Morimoto [30]). *A finite non-solvable gap group G, which is not isomorphic to $\mathrm{P\Sigma L}(2,27)$, is a Laitinen group if and only if $r_G \geqslant 2$. $\mathrm{P\Sigma L}(2,27)$ is a Laitinen group, and therefore a finite non-solvable gap group G is a Laitinen group if and only if $r_G \geqslant 2$.*

By [**43**, Propositions 5.3–5.6], the following proposition holds.

Proposition 1.6 (Pawałowski and Sumi [43]). *In every case below, G is a finite solvable Oliver group such that any two Smith equivalent real G-modules are isomorphic, and so G is not a Laitinen group.*

(i) $G = S_3 \times A_4$ *with* $r_G = 2$*, in the GAP libraries [**16**]:* $G = \mathrm{SG}(72, 44)$.

(ii) $G = (\mathbb{Z}_2^2 \times \mathbb{Z}_3)^2 \rtimes \mathbb{Z}_2$ *with* $r_G = 2$*, in the GAP libraries [**16**]:* $G = \mathrm{SG}(288, 1025)$.

(iii) $G = \mathrm{Aff}(2, 3)$ *with* $r_G = 2$*, in the GAP libraries [**16**]:* $G = \mathrm{SG}(432, 734)$.

(iv) $G = (A_4 \times A_4) \rtimes \mathbb{Z}_2^2$ *with* $r_G = 3$*, in the GAP libraries [**16**]:* $G = \mathrm{SG}(576, 8654)$.

We refer the reader to [**1, 2, 4, 7, 8, 10–14, 18, 24, 26–28, 32, 38–40, 44–51, 56–58**] for more results related to the Smith Equivalence Problem obtained before 2006. For the results of 2006–2010, see [**20, 22, 29, 30, 33, 34, 41–43, 59–63**].

In § 2, we recall the notions of Smith set $\mathrm{Sm}(G)$, primary Smith set $\mathrm{PSm}(G)$, and Laitinen–Smith set $\mathrm{LSm}(G)$ of G, and we describe classes of finite groups G where $\mathrm{Sm}(G) = 0$ for $r_G \leqslant 1$, and $\mathrm{LSm}(G) \neq 0$ for $r_G \geqslant 2$ (Theorems 2.6 and 2.10).

In § 3, for a finite group G and its normal subgroup H, we describe four subgroups $\mathrm{PO}(G)$, $\mathrm{PO}(G, H)$, $\mathrm{PLO}(G)$, and $\mathrm{PLO}(G, H)$ of the real representation ring $\mathrm{RO}(G)$, and we recall their basic properties (Lemmas 3.1, 3.4, 3.6, 3.8 and Corollary 3.9).

In § 4, we define a subgroup $\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0}$ of $\mathrm{RO}(G)$ and a subset $\mathrm{PSm}^{\mathrm{c}}(G)$ of $\mathrm{RO}(G)$ such that $\mathrm{PSm}^{\mathrm{c}}(G) \subseteq \mathrm{LSm}(G) \subseteq \mathrm{PSm}(G)$. Then, we prove the Smith Equivalence Theorem (Theorem 4.9) asserting that $\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} \subseteq \mathrm{PSm}^{\mathrm{c}}(G)$ for any finite Oliver group G.

In § 5, for $H \trianglelefteq G$, we introduce the *H-coset condition* (Definition 5.1) and state our first key algebraic result (Theorem 5.6), which we next use to construct smooth actions of G on spheres with isolated fixed points at which the tangent G-modules are not isomorphic to each other (Theorem 5.8). The result is a key ingredient in the proof of Theorem A.

In § 6, we prove Theorem 5.6. First, for $H \trianglelefteq G$, we define a subgroup $\mathrm{PO}(G, H)^{\mathrm{gap}}_{\geqslant 0}$ of the group $\mathrm{PO}(G, H)$ and we restate Theorem 5.6 by claiming that $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0$ for any finite Oliver group G satisfying the G^{nil}-coset condition (Theorem 6.1).

In § 7, we prove our second key algebraic result, asserting that, except for $G = \mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2, 27)$, any finite non-solvable group G with $r_G \geqslant 2$ satisfies the G^{sol}-coset condition (Theorem 7.1). The result is a key ingredient in the proof of Theorem B.

In § 8, by using the material of §§ 2–7, we prove Theorems A–C. Theorems A and B call for the new key algebraic results (Theorems 6.1 and 7.1, respectively), except for $G = \mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2, 27)$; Theorem C follows from Theorems A and B.

In Appendix A, we recall the notion of *Oliver group* and quote results from [**23**,**25**,**37**]. Then, we introduce the notion of the *Solomon group* and restate some results from [**40**,**43**]. At the end, we ask: *is* $\mathrm{Sm}(G) = 0$ *for any finite non-solvable Solomon group* G*?*

We refer the reader to the books of Bredon [**3**, Chapters III and VI], tom Dieck [**64**, Chapters I and III] and Kawakubo [**21**, Chapters 3–5] for the basic material on transformation groups that is needed in this paper.

2. The subsets $\mathrm{Sm}(G)$, $\mathrm{PSm}(G)$ and $\mathrm{LSm}(G)$ of $\mathrm{RO}(G)$

Let G be a finite group. Two real G-modules U and V are called 2*-matched* if the characters χ_U and χ_V of U and V, respectively, agree on any element of G of order 2^a for $a \geqslant 0$.

By character theory arguments, χ_U and χ_V agree on any element of G of order 1, 2 or 4 if U and V are Smith equivalent. Hence, if U and V are Laitinen–Smith equivalent, U and V are 2-matched. The results of Atiyah and Bott [**1**, (7.27)] or Milnor [**27**, (12.11)], as well as Sanchez [**51**, (1.11)] and the character theory arguments, yield the following corollary.

Corollary 2.1 (Atiyah and Bott [1]; Milnor [27]; Sanchez [51]). *Let G be a finite group. Then for any two Smith equivalent (respectively, 2-matched Smith equivalent) real G-modules U and V, $\chi_U(g) = \chi_V(g)$ for every element $g \in G$ of order 1, 2, 4 or p^a, where p is an odd prime (respectively, p is a prime) and $a \geqslant 1$.*

Let G be a finite group and let $\mathcal{P}(G)$ be the family of subgroups of G of prime power order. Two real G-modules U and V are called $\mathcal{P}(G)$*-matched* if for every $P \in \mathcal{P}(G)$, U and V are isomorphic as P-modules, i.e. $\chi_U(g) = \chi_V(g)$ for any $g \in G$ of prime power order.

Definition 2.2. For a finite group G, two real G-modules U and V are called *primary Smith equivalent* if U and V are Smith equivalent and $\chi_U(g) = \chi_V(g)$ whenever $\dim U^g = \dim V^g = 0$ for an element $g \in G$ of order 2^a for $a \geqslant 3$.

Note that two real G-modules U and V are primary Smith equivalent if and only if U and V are 2-matched and Smith equivalent, which is equivalent to saying that U and V are $\mathcal{P}(G)$-matched and Smith equivalent (cf. Corollary 2.1).

The *Smith set* $\mathrm{Sm}(G)$, the *primary Smith set* $\mathrm{PSm}(G)$ and the *Laitinen–Smith set* $\mathrm{LSm}(G)$ of G are the subsets of the real representation ring $\mathrm{RO}(G)$ consisting of the differences of two Smith equivalent, primary Smith equivalent and Laitinen–Smith equivalent real G-modules, respectively. The last equivalence is defined for $G \ncong \mathbb{Z}_{2^a}$, where $a \geqslant 3$.

Clearly, $\mathrm{Sm}(G) = 0$ (respectively, $\mathrm{PSm}(G) = 0$, $\mathrm{LSm}(G) = 0$) if and only if any two Smith equivalent (respectively, primary Smith equivalent, Laitinen–Smith equivalent) real G-modules are isomorphic. In the last case, we assume that $G \ncong \mathbb{Z}_{2^a}$, $a \geqslant 3$. In accordance with the fact that $\mathrm{LSm}(G) = 0$ for $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$, we set $\mathrm{LSm}(G) = 0$ for $G \cong \mathbb{Z}_{2^a}$, $a \geqslant 3$. Now, for any finite group G, the sets $\mathrm{Sm}(G)$, $\mathrm{PSm}(G)$ and $\mathrm{LSm}(G)$ all contain the zero of $\mathrm{RO}(G)$.

By the definition of Laitinen group given in § 1, a finite group G is a Laitinen group if and only if G is not of prime power order and $\mathrm{LSm}(G) \neq 0$.

By [**24**, Lemma 2.1] (cf. Lemma 3.1 of this paper), the following lemma holds and it shows that in the Laitinen Conjecture the condition that $r_G \geqslant 2$ is necessary.

Lemma 2.3 (Laitinen and Pawałowski [24]). *Let G be a finite group with $r_G \leqslant 1$. Then* $\mathrm{LSm}(G) = \mathrm{PSm}(G) = 0$.

Following [**40**, p. 853], we say that a finite group G satisfies the 8-*condition* if G does not contain an element of order 8, or for any element $g \in G$ of order 2^a with $a \geqslant 3$, $\dim V^g > 0$ for any irreducible real G-module V (see [**24**, Example 2.5] and [**40**, Examples E1–E3]).*

In general, $\mathrm{LSm}(G) \subseteq \mathrm{PSm}(G) \subseteq \mathrm{Sm}(G)$, but if G satisfies the 8-condition, the converse inclusions also hold by [**24**, Lemma 2.6] or [**40**, the 8-condition lemma, p. 854].

Lemma 2.4 (Laitinen and Pawałowski [24]; Pawałowski and Solomon [40]). *If a finite group G satisfies the* 8*-condition, any two Smith equivalent real G-modules are Laitinen–Smith equivalent, and so* $\mathrm{LSm}(G) = \mathrm{PSm}(G) = \mathrm{Sm}(G)$.

Lemmas 2.3 and 2.4 yield immediately the following corollary.

Corollary 2.5. *If a finite group G satisfies the* 8*-condition and $r_G \leqslant 1$,* $\mathrm{Sm}(G) = 0$.

We wish to find classes of finite groups G such that $\mathrm{Sm}(G) = 0$ for $r_G \leqslant 1$, and, for $r_G \geqslant 2$, $\mathrm{LSm}(G) \neq 0$, and therefore $\mathrm{PSm}(G) \neq 0$ and $\mathrm{Sm}(G) \neq 0$.

Theorem 2.6 (Atiyah and Bott [1]; Laitinen and Pawałowski [24]; Pawałowski and Solomon [40]). *Let G be a finite simple group. Then the Smith set* $\mathrm{Sm}(G) = 0$ *for $r_G \leqslant 1$, and the Laitinen–Smith set* $\mathrm{LSm}(G) \neq 0$ *for $r_G \geqslant 2$.*

Proof. According to [**1**, (7.27)] or [**27**, (12.11)], $\mathrm{Sm}(G) = 0$ and $r_G = 0$ for $G = \mathbb{Z}_p$, where p is a prime. Now, assume that G is a finite non-abelian simple group.

If $r_G \leqslant 1$, G satisfies the 8-condition by [**40**, Theorem C1, p. 851, and Example E1, p. 854], and therefore $\mathrm{Sm}(G) = 0$ by Corollary 2.5.

If $r_G \geqslant 2$, then $\mathrm{LSm}(G) \neq 0$ by [**24**, Theorem A] (cf. Theorem 1.3 herein). □

Now, we shall describe other classes of finite groups G such that $\mathrm{Sm}(G) = 0$ for $r_G \leqslant 1$ and $\mathrm{LSm}(G) \neq 0$ for $r_G \geqslant 2$. First, we focus on two finite groups G with $r_G = 1$, which do not satisfy the 8-condition, and hence we cannot apply Corollary 2.5 to prove that $\mathrm{Sm}(G) = 0$. The groups of interest are the general linear group $\mathrm{GL}(2,3)$ and the projective general linear group $\mathrm{PGL}(2,7)$ of two-by-two matrixes with coefficients in the fields $\mathbb{F}_3$ and $\mathbb{F}_7$, which consist of three and seven elements, respectively.

* In [**24**], a finite group G satisfying the 8-condition is called 2-proper.

Proposition 2.7. *Let $G = \mathrm{GL}(2,3)$. Then $r_G = 1$, G does not satisfy the 8-condition, and any two Smith equivalent real G-modules are isomorphic, i.e.* $\mathrm{Sm}(G) = 0$.

Proof. According to [**15**, § 5.2] or [**19**, Chapter 28], the group $G = \mathrm{GL}(2,3)$ of order 48 has eight conjugacy classes of elements of orders 1, 2, 2, 3, 4, 6, 8 and 8, respectively, where the last two classes can be represented by the following two elements of order 8:

$$h = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad h^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}.$$

It follows that there exists exactly one real conjugacy class $(h)^{\pm}$ in G of elements of order 8, and $r_G = 1$ due to the unique real conjugacy class of elements of order 6.

By looking at the character table of G (see, for example, [**15**, p. 70] or [**19**, p. 327]) and computing the dimensions $\dim \chi^h = \frac{1}{8}\sum_{n=1}^{8} \chi(h^n)$ for the irreducible characters χ of G, we see that there exist characters χ with $\dim \chi^h = 0$. Therefore, G does not satisfy the 8-condition. However, G satisfies the 2-condition of [**43**, Definition 2.4], and so $\mathrm{Sm}(G) = 0$ by [**43**, Theorem 2.5]. □

Proposition 2.8. *Let $G = \mathrm{PGL}(2,7)$. Then $r_G = 1$, G does not satisfy the 8-condition, and any two Smith equivalent real G-modules are isomorphic, i.e.* $\mathrm{Sm}(G) = 0$.

Proof. According to [**55**], the group $G = \mathrm{PGL}(2,7)$ of order 336 has nine conjugacy classes of elements of orders 1, 2, 2, 3, 4, 6, 7, 8 and 8, respectively, where the last two classes can be represented by the elements $hZ(\mathrm{GL}(2,7))$ and $h^3Z(\mathrm{GL}(2,7))$ of order 8, where

$$h = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

It follows that there are exactly two real conjugacy classes in G of elements of order 8, and $r_G = 1$ due to the unique real conjugacy class of elements of order 6.

Let N be the subgroup of the real representation ring of G consisting of the differences $U - V$ of real G-modules U and V with characters $\chi = \chi_U - \chi_V$ such that $\chi(g) = 0$ for any element $g \in G$ of order $|g| \notin \{6, 8\}$. Then N is spanned by the following three elements:

$$(2V_{1a} \oplus V_{8a}) - (2V_{1b} \oplus V_{8b}), \quad V_{6a} - V_{6b} \quad \text{and} \quad (2V_{7a} \oplus V_{8a}) - (2V_{7b} \oplus V_{8b}),$$

where V_{na} and V_{nb} denote two distinct n-dimensional irreducible real G-modules and V_{1a} is the trivial G-module $\mathbb{R}$. Just two of the G-modules V_{1a}, V_{1b}, V_{6a}, V_{6b}, V_{7a}, V_{7b}, V_{8a}, V_{8b}, namely V_{1b} and V_{7a}, have zero-dimensional h-fixed point sets. So, G does not satisfy the 8-condition.

If two real G-modules U and V are Smith equivalent, it follows from the description of N and Corollary 2.1 that $U - V \in N$ and thus, $\dim U^h = \dim V^h > 0$ and $\dim U^{h^3} = \dim V^{h^3} > 0$. Hence, U and V are Laitinen–Smith equivalent, proving that $\mathrm{LSm}(G) = \mathrm{PSm}(G) = \mathrm{Sm}(G)$.* Consequently, as $r_G = 1$, it follows from Lemma 2.3 that $\mathrm{Sm}(G) = 0$. □

* By [**61**, Theorems 4.3 and 5.3], $\mathrm{LSm}(G) = \mathrm{PSm}(G) = \mathrm{Sm}(G)$ for $G = \mathrm{PGL}(2,q)$, q prime power.

Consider the projective general linear groups $\mathrm{PGL}(n,q)$, the general linear groups $\mathrm{GL}(n,q)$, and the affine groups $\mathrm{Aff}(n,q)$. By using the two canonical epimorphisms $\mathrm{Aff}(n,q) \to \mathrm{GL}(n,q)$ and $\mathrm{GL}(n,q) \to \mathrm{PGL}(n,q)$, we see that $r_{\mathrm{Aff}(n,q)} \geqslant r_{\mathrm{GL}(n,q)} \geqslant r_{\mathrm{PGL}(n,q)}$.

Proposition 2.9. *Let $G = \mathrm{PGL}(n,q)$, $\mathrm{GL}(n,q)$ or $\mathrm{Aff}(n,q)$ for any integer $n \geqslant 2$ and any prime power $q \geqslant 2$. Assume that $r_G = 0$ or 1. Then $\mathrm{Sm}(G) = 0$ and, except for the case where $G = \mathrm{GL}(2,3)$ or $\mathrm{PGL}(2,7)$, G satisfies the 8-condition. Moreover, the following hold:*

$$\begin{aligned} r_G = 0: \ & G = \mathrm{PGL}(2,2), \mathrm{PGL}(2,3), \mathrm{PGL}(2,4), \mathrm{PGL}(2,8), \mathrm{PGL}(3,2), \\ & \qquad \mathrm{GL}(2,2), \mathrm{GL}(3,2) \text{ or } \mathrm{Aff}(2,2), \\ r_G = 1: \ & G = \mathrm{PGL}(2,5), \mathrm{PGL}(2,7), \mathrm{PGL}(3,3), \mathrm{GL}(2,3) \text{ or } \mathrm{Aff}(3,2). \end{aligned}$$

Proof. By straightforward computation, or using [**9**], we obtain the complete list of groups G with $r_G = 0$ or 1 as in the conclusion, and we see that, except for the two cases where $G = \mathrm{GL}(2,3)$ or $\mathrm{PGL}(2,7)$, every G listed above satisfies the 8-condition, and therefore $\mathrm{Sm}(G) = 0$ by Corollary 2.5. In the two exceptional cases, G does not satisfy the 8-condition, and $\mathrm{Sm}(G) = 0$ by Propositions 2.7 and 2.8, respectively. □

Theorem 2.10. *In each of the cases (i)–(v), the Smith set $\mathrm{Sm}(G) = 0$ for $r_G \leqslant 1$, and the Laitinen–Smith set $\mathrm{LSm}(G) \neq 0$ for $r_G \geqslant 2$.*

(i) *$G = \mathrm{PSL}(n,q)$ or $\mathrm{SL}(n,q)$ for any $n \geqslant 2$ and any prime power q.*

(ii) *$G = \mathrm{PSp}(n,q)$ or $\mathrm{Sp}(n,q)$ for any even $n \geqslant 2$ and any prime power q.*

(iii) *$G = A_n$ or S_n for any $n \geqslant 2$.*

(iv) *$G = \mathrm{PGL}(n,q)$ or $\mathrm{GL}(n,q)$ for any $n \geqslant 2$ and any prime power q.*

(v) *$G = \mathrm{Aff}(n,q)$ for any $n \geqslant 2$ and any prime power q, except for $(n,q) = (2,3)$.*

Proof. Cases (i)–(iii) are covered by [**40**]. For $r_G \leqslant 1$, every group G in (i)–(iii) satisfies the 8-condition by [**40**, Theorems C1–C3, pp. 851–852, and Examples E1–E3, pp. 854–855], and thus $\mathrm{Sm}(G) = 0$ by Corollary 2.5. For $r_G \geqslant 2$, every group G in (i)–(iii) is a non-solvable gap group, and thus $\mathrm{LSm}(G) \neq 0$ by [**40**, Theorem B3, p. 851] (cf. Theorem 1.5 herein).

Now, we deal with cases (iv) and (v). According to Proposition 2.9, $\mathrm{Sm}(G) = 0$ when $r_G = 0$ or 1. In [**42**, Proposition 5.5], it has been checked that the affine group $G = \mathrm{Aff}(2,3)$ is a finite solvable Oliver group such that $r_G = 2$ and $\mathrm{Sm}(G) = 0$.

As $\mathrm{PGL}(n,q)$ is solvable only for $(n,q) \in \{(2,2),(2,3)\}$, the groups $\mathrm{PGL}(n,q)$, $\mathrm{GL}(n,q)$ and $\mathrm{Aff}(n,q)$ are non-solvable when $(n,q) \notin \{(2,2),(2,3)\}$. So, if $r_G \geqslant 2$ for G in (iv) or (v), then G is non-solvable, and thus $\mathrm{LSm}(G) \neq 0$ by Theorem C, completing the proof. □

Theorems 2.6 and 2.10 immediately yield the following corollary.

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

Corollary 2.11. *For G as in Theorems 2.6 or 2.10, the following two claims hold:*

(i) *if $r_G \leqslant 1$, then* $\mathrm{LSm}(G) = 0$, $\mathrm{PSm}(G) = 0$ *and* $\mathrm{Sm}(G) = 0$*;*

(ii) *if $r_G \geqslant 2$, then* $\mathrm{LSm}(G) \neq 0$, $\mathrm{PSm}(G) \neq 0$ *and* $\mathrm{Sm}(G) \neq 0$.

3. The subgroups $\mathrm{PO}(G, H)$ and $\mathrm{PLO}(G, H)$ of $\mathrm{RO}(G)$

Let G be a finite group and let $\mathrm{RO}(G)$ be the Grothendieck ring of the differences $U - V$ of real G-modules U and V. Recall that, as a group, the real representation ring $\mathrm{RO}(G)$ is a finitely generated free abelian group whose rank, $\mathrm{rk}\,\mathrm{RO}(G)$, is equal to the number of real conjugacy classes $(g)^{\pm} = (g) \cup (g^{-1})$ of elements $g \in G$.

Let $\mathrm{PO}(G)$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences $U - V$ of real G-modules U and V which are $\mathcal{P}(G)$-matched, i.e. U and V are isomorphic as real P-modules for any $P \in \mathcal{P}(G)$. By [**24**, Lemma 2.1], $\mathrm{PO}(G) = 0$ for $r_G = 0$ and $\mathrm{rk}\,\mathrm{PO}(G) = r_G$ for $r_G \geqslant 1$.

Let $\mathrm{RO}(G, G)$ be the kernel of the homomorphism $\mathrm{RO}(G) \to \mathbb{Z}$ that maps the difference $U - V$ into the difference $\dim U^G - \dim V^G$. Set $\mathrm{PO}(G, G) = \mathrm{PO}(G) \cap \mathrm{RO}(G, G)$.

Lemma 3.1 (Laitinen and Pawałowski [24, Lemma 2.1]). *For a finite group G, the following two conclusions hold:*

(i) $\mathrm{PO}(G, G) = 0$ *for $r_G = 0$ or 1;*

(ii) $\mathrm{rk}\,\mathrm{PO}(G, G) = r_G - 1$ *for $r_G \geqslant 2$.*

Lemma 3.2. *Let G be a finite group acting smoothly on a disc (respectively, sphere) M with two (respectively, three) or more isolated fixed points. If $r_G \leqslant 1$, then at any two points x and y fixed by the action of G on M the tangent G-modules $T_x(M)$ and $T_y(M)$ are isomorphic.*

Proof. Set $U = T_x(M)$ and $V = T_y(M)$. For any prime p dividing $|G|$, consider an element $g \in G$ of order p^a for $a \geqslant 1$. Then, by the Smith Theory, the set $M^g = \{z \in M \mid gz = z\}$ is connected, and thus U and V are isomorphic as $\langle g \rangle$-modules, where $\langle g \rangle$ is the cyclic subgroup of G generated by g. Therefore, U and V are $\mathcal{P}(G)$-matched, i.e. $U - V \in \mathrm{PO}(G)$.

As $\dim U^G = \dim V^G = 0$ by the Slice Theorem, $U - V \in \mathrm{PO}(G, G)$, and hence if $r_G \leqslant 1$, U and V are isomorphic by Lemma 3.1 (i). □

For $H \trianglelefteq G$, let $\mathrm{PO}(G, H)$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences $U - V$ of real G-modules U and V which are $\mathcal{P}(G)$-matched, i.e. $U - V \in \mathrm{PO}(G)$, and which are also G/H*-matched*, i.e. the fixed point sets U^H and V^H are isomorphic as real G/H-modules, where G/H acts on U^H and V^H in the standard way.*

* In [**40**], $\mathrm{PO}(G)$ and $\mathrm{PO}(G, H)$ are denoted by $IO(G)$ and $IO(G, H)$, respectively.

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Definition 3.3. Let G be a finite group. For a subgroup $H \trianglelefteq G$, let $r_{(G,H)}$ be the number of real conjugacy classes $(gH)^{\pm}$ in G/H such that the coset gH contains an element of G that is not of prime power order. By the definition, $r_{(G,H)} \geqslant r_{G/H}$.*

Lemma 3.4 (Pawałowski and Solomon [40, Second Rank Lemma, p. 856]). *For a finite group G and any $H \trianglelefteq G$, the inequality $r_G \geqslant r_{(G,H)}$ is true and the following two conclusions hold:*

(i) $\mathrm{PO}(G,H) = 0$ *for* $r_G = r_{(G,H)}$;

(ii) $\mathrm{rk}\,\mathrm{PO}(G,H) = r_G - r_{(G,H)}$ *for* $r_G > r_{(G,H)}$.

Note that $r_{(G,H)} = 0$ if and only if $r_G = 0$, and $r_G = 0$ if and only if each element of G has prime power order. Moreover, for $H = G$, one of the following three cases occurs:

$$r_G = r_{(G,G)} = 0, \quad r_G = r_{(G,G)} = 1 \quad \text{or} \quad r_G > r_{(G,G)} = 1.$$

Therefore, $r_G = r_{(G,G)}$ if and only if $r_G = 0$ or 1. So, Lemma 3.4 generalizes Lemma 3.1.

For two subgroups $H \trianglelefteq G$ and $K \leqslant G$, consider the homomorphism

$$\mathrm{Fix}_K^{H\cap K} \colon \mathrm{RO}(K) \to \mathrm{RO}(K/(H \cap K))$$

given by $\mathrm{Fix}_K^{H\cap K}(U - V) = U^{H\cap K} - V^{H\cap K}$ for two real K-modules U and V.

Recall that $\mathrm{PO}(G,H)$ consists of the differences $U - V$ of $\mathcal{P}(G)$-matched and G/H-matched real G-modules U and V. So,

$$\mathrm{PO}(G,H) = \mathrm{PO}(G) \cap \mathrm{Ker}(\mathrm{Fix}_G^H \colon \mathrm{RO}(G) \to \mathrm{RO}(G/H)).$$

Now, consider the induction homomorphism

$$\mathrm{Ind}_K^G \colon \mathrm{RO}(K, H\cap K) \to \mathrm{RO}(G,H).$$

We wish to compute the rank of the image of $\mathrm{PO}(K, H \cap K)$ under the map Ind_K^G.

Definition 3.5. Let G be a finite group. For two subgroups $H \trianglelefteq G$ and $K \leqslant G$, define two numbers r_G^K and $r_{(G,H)}^K$ as follows.

(i) r_G^K is the number of real conjugacy classes in G represented by elements of K not of prime power order. In particular, $r_G^G = r_G$.

(ii) $r_{(G,H)}^K$ is the number of real conjugacy classes $(gH)^{\pm}$ in G/H such that the coset gH has an element of K not of prime power order. In particular, $r_{(G,H)}^G = r_{(G,H)}$.

Lemma 3.6. *For a finite group G, $H \trianglelefteq G$ and $K \leqslant G$, the inequality $r_G^K \geqslant r_{(G,H)}^K$ is true and the following two conclusions hold:*

(i) $\mathrm{Ind}_K^G(\mathrm{PO}(K, H\cap K)) = 0$ *for* $r_G^K = r_{(G,H)}^K$;

(ii) $\mathrm{rk}\,\mathrm{Ind}_K^G(\mathrm{PO}(K, H\cap K)) = r_G^K - r_{(G,H)}^K$ *for* $r_G^K > r_{(G,H)}^K$.

* In [**40**], r_G and $r_{(G,H)}$ are denoted by a_G and $b_{G/H}$, respectively.

Proof. For a real K-module W, the Frobenius reciprocity law yields the equality

$$\dim(\mathrm{Ind}_K^G(W))^H = |G/HK| \dim(W^{H\cap K}).$$

Therefore, the homomorphism $\mathrm{Ind}_K^G\colon \mathrm{RO}(K) \to \mathrm{RO}(G)$ maps $\mathrm{RO}(K, H\cap K)$ to $\mathrm{RO}(G, H)$. As $\mathrm{Ind}_K^G(\mathrm{PO}(K)) \leqslant \mathrm{PO}(G)$, it follows that $\mathrm{Ind}_K^G(\mathrm{PO}(K, H\cap K)) \leqslant \mathrm{PO}(G, H)$.

By comparing the character values, we obtain that $(\mathrm{Ind}_K^G(W))^H \cong \mathrm{Ind}_{HK}^G(U)$, where U is regarded as the (HK)-module $W^{H\cap K}$ with $U^H = U$. The following diagram commutes:

$$\begin{array}{ccccccc}
\mathrm{PO}(K, H\cap K) & \longrightarrow & \mathrm{PO}(K) & \xrightarrow{\mathrm{Fix}_K^{H\cap K}} & \mathrm{RO}(K/(H\cap K)) & \xrightarrow{\cong} & \mathrm{RO}(HK/H) \\
\downarrow{\scriptstyle \mathrm{Ind}_K^G} & & & & \downarrow{\scriptstyle \mathrm{Ind}_K^G} & & \downarrow{\scriptstyle \mathrm{Ind}_{HK}^G} \\
\mathrm{PO}(G, H) & \longrightarrow & \mathrm{PO}(G) & \xrightarrow{\mathrm{Fix}_G^H} & \mathrm{RO}(G/H) & \xrightarrow{=} & \mathrm{RO}(G/H)
\end{array}$$

As the left-hand diagram above commutes, the following diagram also commutes:

$$\begin{array}{ccccc}
\mathrm{PO}(K, H\cap K) & \longrightarrow & \mathrm{PO}(K) & \xrightarrow{\mathrm{Fix}_K^{H\cap K}} & \mathrm{RO}(HK/H) \\
\downarrow{\scriptstyle \mathrm{Ind}_K^G} & & \downarrow{\scriptstyle \mathrm{Ind}_K^G} & & \downarrow{\scriptstyle \mathrm{Ind}_{HK}^G} \\
\mathrm{Ind}_K^G(\mathrm{PO}(K, H\cap K)) & \xrightarrow{\text{inj}} & \mathrm{Ind}_K^G(\mathrm{PO}(K)) & \xrightarrow{\text{surj}} & (\mathrm{Ind}_{HK}^G \circ \mathrm{Fix}_K^{H\cap K})(\mathrm{PO}(K))
\end{array}$$

By [**24**, Lemma 2.1], $\mathrm{PO}(G) = 0$ for $r_G = 0$ and $\mathrm{rk}\,\mathrm{PO}(G) = r_G$ for $r_G \geqslant 1$. Moreover, by the arguments at the end of the proof on [**40**, p. 857], the homomorphism

$$\mathrm{Fix}_G^H\colon \mathrm{PO}(G) \to \mathrm{RO}(G/H)$$

has image of rank $r_{(G,H)}$, $\mathrm{rk}\,\mathrm{Fix}_G^H(\mathrm{PO}(G)) = r_{(G,H)}$. More generally, for $K \leqslant G$,

$$\mathrm{rk}\,\mathrm{Fix}_K^{H\cap K}(\mathrm{PO}(K)) = r_{(HK,H)}$$

and $\mathrm{rk}\,\mathrm{Ind}_K^G(\mathrm{PO}(K)) = r_G^K$. From the commutative diagram above, it follows that

$$\mathrm{rk}(\mathrm{Fix}_G^H \circ \mathrm{Ind}_K^G)(\mathrm{PO}(K)) = \mathrm{rk}(\mathrm{Ind}_{HK}^G \circ \mathrm{Fix}_K^{H\cap K})(\mathrm{PO}(K)) = r_{(G,H)}^K.$$

Therefore, $\mathrm{rk}\,\mathrm{Ind}_K^G(\mathrm{PO}(K, K\cap K)) = r_G^K - r_{(G,H)}^K$, completing the proof. □

For a finite group G, let G^{nil} (respectively, G^{sol}) be the smallest normal subgroup of G such that G/G^{nil} is nilpotent (respectively, G/G^{sol} is solvable). Clearly, $G^{\mathrm{sol}} \leqslant G^{\mathrm{nil}}$. Recall that

$$G^{\mathrm{nil}} = \bigcap_{p\in\pi(G)} O^p(G),$$

where $\pi(G)$ is the set of prime divisors p of $|G|$, and $O^p(G)$ is the smallest normal subgroup of G such that $|G/O^p(G)| = p^a$ for an integer $a \geqslant 0$.

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A subgroup H of a finite group G is called a *large subgroup* of G if $O^p(G) \leqslant H$ for some prime p. Let $\mathcal{L}(G)$ denote the family of large subgroups of G. A real G-module V is called $\mathcal{L}(G)$-*free* if $\dim V^H = 0$ for all $H \in \mathcal{L}(G)$.

For a finite group G, let $\mathrm{PLO}(G)$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences $U - V$ of two $\mathcal{P}(G)$-matched and $\mathcal{L}(G)$-free real G-modules U and V.*

More generally, for a normal subgroup H of G, let $\mathrm{PLO}(G,H)$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences $U - V$ of two $\mathcal{P}(G)$-matched, G/H-matched and $\mathcal{L}(G)$-free real G-modules U and V. Clearly, $\mathrm{PLO}(G,G) = \mathrm{PLO}(G)$.

The following two lemmas essentially go back to [**40**, Subgroup Lemma, p. 858].

Lemma 3.7 (Pawałowski and Solomon [40, p. 858]). *For a finite group G and two subgroups $H, K \trianglelefteq G$ with $H \leqslant K$,*

(i) $\mathrm{PO}(G,H) \leqslant \mathrm{PO}(G,K) \leqslant \mathrm{PO}(G,G)$ *and*

(ii) $\mathrm{PLO}(G,H) \leqslant \mathrm{PLO}(G,K) \leqslant \mathrm{PLO}(G)$.

Lemma 3.8 (Pawałowski and Solomon [40, p. 858]). *For a finite group G and a subgroup $H \trianglelefteq G$ with $H \leqslant G^{\mathrm{nil}}$,*

$$\mathrm{PLO}(G,H) = \mathrm{PO}(G,H) \leqslant \mathrm{PO}(G,G^{\mathrm{nil}}) \leqslant \mathrm{PLO}(G) \leqslant \bigcap_{p\in\pi(G)} \mathrm{PO}(G,O^p(G)).$$

Lemmas 3.4 and 3.8 yield the following corollary.

Corollary 3.9 (Pawałowski and Solomon [40, p. 859]). *For a finite group G, the following two inequalities hold:*

$$r_G - r_{(G,G^{\mathrm{nil}})} \leqslant \mathrm{rk}\,\mathrm{PLO}(G) \leqslant \min\{r_G - r_{(G,O^p(G))} \mid p \in \pi(G)\}.$$

4. The Smith Equivalence Theorem

Let G be a finite group. Let $\mathcal{PH}(G)$ be the set of pairs (P,H) of subgroups $P < H \leqslant G$ with $P \in \mathcal{P}(G)$. For a real G-module V, consider the *gap function* $d_V \colon \mathcal{PH}(G) \to \mathbb{Z}$ given by

$$d_V(P,H) = \dim V^P - 2\dim V^H \quad \text{for any } (P,H) \in \mathcal{PH}(G).$$

Definition 4.1. A real G-module V is called *gap-positive* (respectively, *gap-non-negative*) if the gap function $d_V \colon \mathcal{PH}(G) \to \mathbb{Z}$ is positive (respectively, non-negative) on $\mathcal{PH}(G)$.

Definition 4.2. Let W be a real G-module. We say that W satisfies the *weak gap condition* if W is gap-non-negative and in the case $d_W(P,H) = 0$ for some $(P,H) \in \mathcal{PH}(G)$, $[H : P] = 2$ and the following three additional conditions hold:

* In [**40**], the subgroup $\mathrm{PLO}(G)$ of $\mathrm{RO}(G)$ is denoted by $\mathrm{LO}(G)$.

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(i) $\dim W^H > \dim W^K + 1$ for any group K with $H < K \leqslant G$;

(ii) W^H is oriented so that the map $W^H \to W^H$, $x \mapsto gx$ is orientation preserving for any element $g \in N_G(H)$, the normalizer of H in G;

(iii) if $d_W(P, H') = 0$ for some $H' \leqslant G$, then $\langle H, H' \rangle \notin \mathcal{L}(G)$, where $\langle H, H' \rangle$ is the smallest subgroup of G containing the subgroups H and H' of G.

For a real G-module V and a G-submodule W of V, denote by $V - W$ the G-orthogonal complement of W in V. Clearly, $V \cong W \oplus (V - W)$ as real G-modules. Set

$$V_{\mathcal{L}(G)} = (V - V^G) - \bigoplus_{p \in \pi(G)} (V - V^G)^{O^p(G)}.$$

The real G-module $V_{\mathcal{L}(G)}$ is the maximal $\mathcal{L}(G)$-free G-submodule of V.

For a finite Oliver group G, set $V(G) = \mathbb{R}[G]_{\mathcal{L}(G)}$, where $\mathbb{R}[G]$ is the regular real G-module. Let $\mathcal{PH}_1(G) = \mathcal{PH}(G) \setminus \mathcal{PH}_2(G)$, where $\mathcal{PH}_2(G)$ consists of $(P, H) \in \mathcal{PH}(G)$ such that

$$[H : P] = [\mathrm{HO}^2(G) : \mathrm{PO}^2(G)] = 2$$

and $\mathrm{PO}^p(G) = G$ for all odd primes p dividing the order of G.

Following [**28**], we say that V is G-*oriented* if for any $H \leqslant G$, subgroup V^H is oriented and the map $g \colon V^H \to V^H$, $x \mapsto gx$ is orientation preserving for any $g \in N_G(H)$.

Theorem 4.3 (Laitinen and Morimoto [23, Theorem 2.3]). *Let G be a finite Oliver group. Then the gap function*

$$d_{V(G)} \colon \mathcal{PH}(G) \to \mathbb{Z}$$

is positive on $\mathcal{PH}_1(G)$ and vanishes on $\mathcal{PH}_2(G)$. Moreover, the real G-module

$$W = \ell V(G) = V(G) \oplus \cdots \oplus V(G),\ \ell \text{ times},$$

satisfies the weak gap condition for any even integer $\ell \geqslant 2$. Also, $\dim W^H = 0$ if and only if $H \in \mathcal{L}(G)$. In particular, the real G-module W is G-oriented and $\mathcal{L}(G)$-free.

As in [**32**] and [**40**], we say that a real G-module V is $\mathcal{P}(G)$-*oriented* if for any $P \in \mathcal{P}(G)$, V^P is oriented and the map $g \colon V^P \to V^P$ is orientation preserving for any $g \in N_G(P)$.

Theorem 4.4. *Let G be a finite Oliver group. Let V be a $\mathcal{P}(G)$-oriented and $\mathcal{L}(G)$-free real G-module satisfying the weak gap condition. Then there exists a smooth action of G on some sphere with exactly one fixed point at which the tangent G-module is isomorphic to $V \oplus \ell V(G)$ for any sufficiently large even integer ℓ.*

If V is G-oriented, Theorem 4.4 follows from [**23**, Theorem 4.1]. If V is only $\mathcal{P}(G)$-oriented, Theorem 4.4 follows from [**32**, Theorem 36], which generalizes [**28**, Theorem 0.1].

Lemma 4.5. *Let G be a finite Oliver group. Let U and V be two $\mathcal{P}(G)$-matched and $\mathcal{L}(G)$-free, gap-non-negative, real G-modules. Then, for any even integer $\ell \geqslant 2$, the G-modules*

$$X = U \oplus V \oplus \ell V(G) \quad \text{and} \quad Y = V \oplus V \oplus \ell V(G)$$

are both $\mathcal{P}(G)$-oriented and $\mathcal{L}(G)$-free, and both satisfy the weak gap condition.

Proof. As U, V and $V(G)$ are all $\mathcal{L}(G)$-free, X and Y are $\mathcal{L}(G)$-free. As U and V are $\mathcal{P}(G)$-matched, $U \oplus V$ is $\mathcal{P}(G)$-oriented by [**40**, Key Lemma, p. 887]. Clearly, $V \oplus V$ is G-oriented and so is $\ell V(G)$, if ℓ is even. Therefore, the G-modules X and Y are both $\mathcal{P}(G)$-oriented.

At $(P, H) \in \mathcal{PH}(G)$, the gap functions $d_X, d_Y : \mathcal{PH}(G) \to \mathbb{Z}$ take the values

$$d_X(P, H) = d_U(P, H) + d_V(P, H) + \ell d_{V(G)}(P, H),$$
$$d_Y(P, H) = d_V(P, H) + d_V(P, H) + \ell d_{V(G)}(P, H).$$

As the gap functions d_U, d_V, and $d_{V(G)}$ are non-negative on $\mathcal{PH}(G)$, so are d_X and d_Y. Clearly, by Theorem 4.3, d_X and d_Y are positive on $\mathcal{PH}_1(G)$, and in the case where $d_X(P, H) = 0$ or $d_Y(P, H) = 0$ for some $(P, H) \in \mathcal{PH}_2(G)$, $d_{V(G)}(P, H) = 0$ and thus $[H : P] = 2$.

We claim that conditions (i)–(iii) in Definition 4.2 all hold for $W = X$ or Y. In fact, if $W = X$ or Y, then for any subgroups $H < K \leqslant G$ either $\dim W^H = 0$, and thus $\dim W^K = 0$ for $H \in \mathcal{L}(G)$, or otherwise $\dim W^H \geqslant \dim W^K + \ell$, proving that (i) holds, if ℓ is even.

Since U and V are $\mathcal{P}(G)$-matched and $U \oplus V$ is $\mathcal{P}(G)$-oriented, (ii) holds for $W = U \oplus V$. Moreover, if ℓ is even, $\ell V(G)$ and Y are G-oriented and so, (ii) holds for $W = X$ or Y.

Finally, (iii) holds for $W = X$ or Y, because (iii) holds for $W = V(G)$, proving the claim and completing the proof of the lemma. □

The result of [**36**, Theorem 0.4] also holds when the corresponding G-modules are $\mathcal{L}(G)$-*free*. This statement appears in [**40**, Theorem 4.1] but now, we wish to show explicitly how the version with $\mathcal{L}(G)$-free G-modules follows from [**31**, Theorem 0.3].

Theorem 4.6. *Let G be a finite Oliver group. For an integer $k \geqslant 1$, let $V_1, \dots, V_k$ be a list of $\mathcal{L}(G)$-free real G-modules such that V_i and V_j are $\mathcal{P}(G)$-matched for all $1 \leqslant i, j \leqslant k$. Then there exists a smooth action of G on a disc D^n with exactly k fixed points at which the tangent G-modules are isomorphic to $V_1 \oplus \ell V(G), \dots, V_k \oplus \ell V(G)$ for sufficiently large ℓ.*

Proof. Set $M = \{x_1, \dots, x_k\}$, a manifold consisting of exactly k points $x_1, \dots, x_k$. Let τ_M be the tangent bundle to M, and let ν be the G-vector bundle over M with fibres $V_1, \dots, V_k$ over $x_1, \dots, x_k$, respectively, for the given $\mathcal{L}(G)$-free real G-modules $V_1, \dots, V_k$.

We claim that the conditions (B1) and (B2) posed in [**31**, p. 280] both hold. In fact, (B1) holds because, as a non-equivariant bundle, the Whitney sum $\tau_M \oplus \nu$ is a product bundle, and thus $\tau_M \oplus \nu = 0$ in the reduced KO-theory $\widetilde{\mathrm{KO}}(M)$ of M.

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Also, (B2) holds because the assumption that V_i and V_j are $\mathcal{P}(G)$-matched implies that for any $P \in \mathcal{P}(G)$ the Whitney sum $\tau_M \oplus \nu$ is a product P-vector bundle, and thus $\tau_M \oplus \nu = 0$ in the reduced P-equivariant KO-theory $\widetilde{\mathrm{KO}}_P(M)$ of M.

As (B1) and (B2) both hold, the conclusion follows by [**31**, Theorem 0.3]. □

As claimed in [**61**, Theorem 3.3], the result of [**40**, Theorem 4.4] obtained for finite Oliver *gap* groups, holds in fact for *all* finite Oliver groups. Now, we wish to prove the claim.

Theorem 4.7. *Let G be a finite Oliver group. For an integer $k \geqslant 1$, let $V_1, \dots, V_k$ be a list of gap-non-negative $\mathcal{L}(G)$-free real G-modules such that V_i and V_j are $\mathcal{P}(G)$-matched for all $1 \leqslant i, j \leqslant k$. Then there exists a smooth action of G on a sphere S^n with exactly k fixed points $x_1, \dots, x_k$ such that for any $1 \leqslant i \leqslant k$, the tangent G-module $T_{x_i}(S^n)$ is isomorphic to*

$$V_i \oplus V_0 \oplus \ell V(G)$$

for any G-module V_0 chosen from the list $V_1, \dots, V_k$ and any sufficiently large even integer ℓ. In particular, the P-fixed point set $(S^n)^P$ is connected for any $P \in \mathcal{P}(G)$.

Proof. The proof goes *mutatis mutandis* as in the paper [**40**, Theorems 4.3 and 4.4], where the acting Oliver group G is a gap group.

For $i = 1, \dots, k$, by Lemma 4.5, the G-module $V_i \oplus V_0$ is $\mathcal{P}(G)$-oriented and $\mathcal{L}(G)$-free, and satisfies the weak gap condition. So, by Theorem 4.4, there exists a smooth action of G on a copy S_i^n of the sphere S^n, where $n = 2 \dim V_0 + \ell \dim V(G)$ for any sufficiently large even integer ℓ, such that $(S_i^n)^G = \{z_i\}$ and such that, as real G-modules,

$$T_{z_i}(S_i^n) \cong V_i \oplus V_0 \oplus \ell V(G).$$

By Theorem 4.6, if ℓ is sufficiently large, there exists a smooth action of G on the n-disc D^n such that $(D^n)^G = \{x_1, \dots, x_k\}$ and such that, as real G-modules,

$$T_{x_i}(D^n) \cong V_i \oplus V_0 \oplus \ell V(G).$$

By taking the double $\partial(D^n \times D^1)$ of D^n, we obtain a smooth action of G on the n-sphere S^n such that $(S^n)^G = \{x_1, \dots, x_k, y_1, \dots, y_k\}$ and such that, as real G-modules,

$$T_{x_i}(S^n) \cong T_{y_i}(S^n) \cong V_i \oplus V_0 \oplus \ell V(G).$$

By forming the equivariant connected sum of S^n and the spheres $S_1^n, \dots, S_k^n$ around $y_i \in S^n$ and $z_i \in S_i^n$ for $i = 1, \dots, k$, we obtain a new smooth action of G on S^n such that

$$(S^n)^G = \{x_1, \dots, x_n\} \quad \text{and} \quad T_{x_i}(S^n) \cong V_i \oplus V_0 \oplus \ell V(G)$$

for $i = 1, \dots, k$. As $\dim(\ell V(G))^P > 0$ for any $P \in \mathcal{P}(G)$, the set $(S^n)^P$ is connected. □

For a finite group G and $H \trianglelefteq G$, let $\mathrm{PLO}(G, H)^{\mathrm{gap}}_{\geqslant 0}$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences $U - V$ of two gap-non-negative $\mathcal{P}(G)$-matched, G/H-matched and $\mathcal{L}(G)$-free real G-modules U and V. Set $\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PLO}(G, G)^{\mathrm{gap}}_{\geqslant 0}$.

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Lemma 4.8. *Let G be a finite gap group and let H be a normal subgroup of G. Then*

$$\mathrm{PLO}(G,H)^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PLO}(G,H) \quad \textit{and} \quad \mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PLO}(G).$$

Proof. An element of $\mathrm{PLO}(G,H)$ is the difference $U-V$ of two $\mathcal{P}(G)$-matched, G/H-matched and $\mathcal{L}(G)$-free real G-modules U and V. By the notion of gap group, there exists a gap-positive $\mathcal{L}(G)$-free real G-module W. Therefore, $U \oplus \ell W$ and $V \oplus \ell W$ are both gap-positive for any sufficiently large integer ℓ. As $U - V = (U \oplus \ell W) - (V \oplus \ell W) \in \mathrm{PLO}(G,H)^{\mathrm{gap}}_{\geqslant 0}$, it follows that $\mathrm{PLO}(G,H)^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PLO}(G,H)$, completing the proof. □

Let G be a finite group not of prime power order. Two real G-modules U and V are called *c-primary Smith equivalent* if there exists a smooth action of G on a homotopy sphere Σ with $\Sigma^G = \{x,y\}$, such that $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ (i.e. U and V are Smith equivalent) and for any $P \in \mathcal{P}(G)$, Σ^P is connected which (by the Slice Theorem and Smith Theory) amounts to saying that $\Sigma^P \supset \Sigma^G$ as a proper subset or, equivalently, $\dim U^P = \dim V^P > 0$.

The *c-primary Smith set* $\mathrm{PSm}^{\mathrm{c}}(G)$ of G is the subset of $\mathrm{RO}(G)$ consisting of the differences $U-V$ of two c-primary Smith equivalent real G-modules U and V. In general,

$$\mathrm{PSm}^{\mathrm{c}}(G) \subseteq \mathrm{LSm}(G) \subseteq \mathrm{PSm}(G) = \mathrm{PO}(G,G) \cap \mathrm{Sm}(G).$$

Theorem 4.9 (the Smith Equivalence Theorem). *Let G be a finite Oliver group. Then*

$$\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} \subseteq \mathrm{PSm}^{\mathrm{c}}(G),$$

i.e. the difference of any two gap-non-negative $\mathcal{P}(G)$-matched and $\mathcal{L}(G)$-free real G-modules is also the difference of two c-primary Smith equivalent real G-modules.

Theorem 4.9 follows from Theorem 4.7 in the special case where $k = 2$. The conclusion of Theorem 4.9 generalizes that one of [**40**, Realization Theorem, p. 850] asserting that if a finite Oliver group G is a gap group, then $\mathrm{PLO}(G) \subseteq \mathrm{LSm}(G)$ and, in fact, $\mathrm{PLO}(G) \subseteq \mathrm{PSm}^{\mathrm{c}}(G)$. Recall that by Lemma 4.8, $\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PLO}(G)$ for a finite gap group G.

Theorem 4.9 was claimed earlier in [**41**, Theorem 2.1], [**42**, Theorem 2.1], [**59**, Lemma 3.1], [**60**, Theorem 2.2], [**61**, Theorem 3.2], and also in [**33**, Theorem 11], wherein Morimoto and Qi describe the corresponding result without the $\mathcal{P}(G)$-orientation condition.

In Lemma 4.5, we have shown how to ensure for real G-modules that the $\mathcal{P}(G)$-orientation condition and the weak gap condition both hold. Then, in the proof of Theorem 4.7, for a list of gap-non-negative $\mathcal{L}(G)$-free real G-modules $V_1, \dots, V_k$, where V_i and V_j are $\mathcal{P}(G)$-matched, we were able to conclude that for some real G-module W, the G-modules $V_1 \oplus W, \dots, V_k \oplus W$ satisfy the $\mathcal{P}(G)$-orientation condition and the weak gap condition. Here, the bottom line is that we did not change the difference, $V_i - V_j = (V_i \oplus W) - (V_j \oplus W)$, in $\mathrm{RO}(G)$.

5. The G^{nil}-Coset Theorem and its applications

For a finite group G, we impose a condition (stronger than $r_G \geqslant 2$) which, in the case where G is an Oliver group, is sufficient for the existence of a smooth action of G on a sphere with isolated fixed point at which the tangent G-modules are not isomorphic to each other.

An element $g \in G$ is called an *NPP element* if the order of g is not a prime power.

Definition 5.1. Let G be a finite group and let H be a normal subgroup of G. We say that G *satisfies the H-coset condition* if there exists an H-coset of G containing two NPP elements x and y that are not real conjugate in G (and thus $r_G \geqslant 2$) and, in addition,

(i) the elements x and y are both in a gap subgroup of G, or

(ii) the orders of x and y are even and the involutions of $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G, where $\langle x \rangle$ and $\langle y \rangle$ are the cyclic groups generated by x and y, respectively.

We recall that $\mathrm{PO}(G, H) \neq 0$ if and only if $r_G > r_{(G,H)}$. Moreover, if $r_G > r_{(G/H)}$, then $r_G \geqslant 2$, and if $r_G \leqslant 1$, then $r_G = r_{(G/H)}$ (see Definition 3.3 and Lemma 3.4).

Lemma 5.2. *If G is a finite group and $H \trianglelefteq G$, then* $\mathrm{PO}(G, H) \neq 0$ *if and only if there exists an H-coset of G containing two* NPP *elements which are not real conjugate in G.*

Proof. By Definition 3.3, $r_G > r_{(G,H)}$ if and only if there exists an H-coset of G containing two NPP elements $x, z \in G$ such that $(x)^{\pm} \neq (z)^{\pm}$ and $(xH)^{\pm} = (zH)^{\pm}$. Now, the equality $(xH)^{\pm} = (zH)^{\pm}$ holds if and only if there exists an NPP element $y \in G$ such that $xH = yH$ and $(y)^{\pm} = (z)^{\pm}$. So, for the two NPP elements x and y, $xH = yH$ and $(x)^{\pm} \neq (y)^{\pm}$. □

Definition 5.1 and Lemma 5.2 immediately yield the following two lemmas.

Lemma 5.3. *Let G be a finite group and let $H \trianglelefteq G$. If G satisfies the H-coset condition, then* $\mathrm{PO}(G, H) \neq 0$.

Lemma 5.4. *Let G be a finite gap group and let $H \trianglelefteq G$. If* $\mathrm{PO}(G, H) \neq 0$, *then G satisfies the H-coset condition.*

Corollary 5.5. *Let G be a finite gap group and let $H \trianglelefteq G$. Then* $\mathrm{PO}(G, H) \neq 0$ *if and only if G satisfies the H-coset condition.*

Now, we state our first key algebraic result, which we shall prove in the next section.

Theorem 5.6 (the G^{nil}-Coset Theorem). *If G is a finite Oliver group satisfying the G^{nil}-coset condition, then there exist two gap-non-negative $\mathcal{P}(G)$-matched G/G^{nil}-matched real G-modules that are not isomorphic to each other.*

Theorem 5.6 allows us to obtain mutually non-isomorphic real G-modules $V_1, \dots, V_k$ that are gap-non-negative, $\mathcal{L}(G)$-free and such that V_i and V_j are $\mathcal{P}(G)$-matched.

Corollary 5.7. *Let G be a finite Oliver group. If G satisfies the G^{nil}-coset condition, then for any integer $k \geqslant 2$, there exist gap-non-negative $\mathcal{L}(G)$-free real G-modules $V_1, \dots, V_k$ such that V_i and V_j are $\mathcal{P}(G)$-matched for $1 \leqslant i, j \leqslant k$, and non-isomorphic when $i \neq j$.*

Proof. By Theorem 5.6, there exist two gap-non-negative $\mathcal{P}(G)$-matched and G/G^{nil}-matched real G-modules U and V that are not isomorphic to each other. As

$$\mathrm{PO}(G, G^{\mathrm{nil}}) \leqslant \mathrm{PLO}(G)$$

by Lemma 3.8, we may assume that U and V are $\mathcal{L}(G)$-free. For an integer $k \geqslant 2$, set

$$V_i = (k-i)U \oplus iV$$

for $i = 1, \dots, k$. Then V_i and V_j are $\mathcal{P}(G)$-matched, and non-isomorphic when $i \neq j$. Clearly, each V_i is gap-non-negative and $\mathcal{L}(G)$-free. □

Theorem 4.7 and Corollary 5.7 allow us to obtain the following theorem.

Theorem 5.8. *Let G be a finite Oliver group. If G satisfies the G^{nil}-coset condition, then for any integer $k \geqslant 2$ there exists a smooth action of G on a sphere S^n with exactly k fixed points at which the tangent G-modules are mutually non-isomorphic. Moreover, for any $P \in \mathcal{P}(G)$, the P-fixed point set $(S^n)^P$ is connected.*

Proof. By Corollary 5.7, for any $k \geqslant 2$, there exist gap-non-negative $\mathcal{L}(G)$-free real G-modules $V_1, \dots, V_k$ such that for $1 \leqslant i \neq j \leqslant k$, V_i and V_j are $\mathcal{P}(G)$-matched and non-isomorphic. Therefore, the required action of G exists by Theorem 4.7. □

Once we prove Theorem B, we obtain also the following theorem.

Theorem 5.9. *Let G be a finite non-solvable group not isomorphic to $\mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2, 27)$. If $r_G \geqslant 2$, then for any integer $k \geqslant 2$ there exists a smooth action of G on a sphere S^n with exactly k fixed points at which the tangent G-modules are mutually non-isomorphic. Moreover, for any $P \in \mathcal{P}(G)$, the P-fixed point set $(S^n)^P$ is connected.*

Proof. By Theorem B, the group G satisfies the G^{nil}-coset condition if and only if $r_G \geqslant 2$. So, if $r_G \geqslant 2$, the conclusion follows from Theorem 5.8. □

6. Proof of the G^{nil}-Coset Theorem

For a finite group G and $H \trianglelefteq G$, let $\mathrm{PO}(G, H)^{\mathrm{gap}}_{\geqslant 0}$ be the subgroup of $\mathrm{RO}(G)$ consisting of the differences of two gap-non-negative $\mathcal{P}(G)$-matched and G/H-matched real G-modules. Now, using the group $\mathrm{PO}(G, H)^{\mathrm{gap}}_{\geqslant 0}$, we restate Theorem 5.6.

Theorem 6.1 (the G^{nil}-Coset Theorem). *If G is a finite Oliver group satisfying the G^{nil}-coset condition, then $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0$.*

The rest of this section is devoted to proving Theorem 6.1. For any finite Oliver group G that satisfies the G^{nil}-coset condition, we shall construct two gap-non-negative $\mathcal{P}(G)$-matched and G/G^{nil}-matched real G-modules that are not isomorphic to each other.

For a finite Oliver group G and $K \leqslant G$, consider the induction homomorphism

$$\mathrm{Ind}_K^G \colon \mathrm{RO}(K) \to \mathrm{RO}(G).$$

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

If a real K-module V is $\mathcal{L}(K)$-free, then the induced real G-module $\mathrm{Ind}_K^G(V)$ is $\mathcal{L}(G)$-free. Moreover, if two real K-modules U and V are $\mathcal{P}(K)$-matched, then $\mathrm{Ind}_K^G(U)$ and $\mathrm{Ind}_K^G(V)$ are $\mathcal{P}(G)$-matched (see [**20**, Lemma 4.1]). Therefore, $\mathrm{Ind}_K^G(\mathrm{PLO}(K)) \leqslant \mathrm{PLO}(G)$.

We need the following two lemmas about the induction homomorphism Ind_K^G.

Lemma 6.2. *Let G be a finite Oliver group and let K be a gap subgroup of G. Then*

$$\mathrm{Ind}_K^G(\mathrm{PLO}(K)) \leqslant \mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0}.$$

Proof. As K is a gap group, any element of $\mathrm{PLO}(K)$ can be regarded as the difference $U - V$ of two $\mathcal{P}(K)$-matched $\mathcal{L}(K)$-free real K-modules U and V that are gap-positive (cf. Lemma 4.8), and thus, by [**35**, Lemma 1.7], $\mathrm{Ind}_K^G(U)$ and $\mathrm{Ind}_K^G(V)$ are both gap-non-negative. Therefore, the homomorphism $\mathrm{Ind}_K^G\colon \mathrm{RO}(K) \to \mathrm{RO}(G)$ maps $\mathrm{PLO}(K)$ into $\mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0}$. □

Lemma 6.3. *Let G be a finite Oliver group, let $H = G^{\mathrm{nil}}$ and let K be a subgroup of G not of prime power order. Suppose that there exists an $\mathcal{L}(K)$-free real K-module W which is gap-positive at all pairs $(P, L) \in \mathcal{PH}(K)$ with $[L : P] = 2$. Then*

$$\mathrm{Ind}_K^G(\mathrm{PO}(K, H \cap K)) \leqslant \mathrm{PO}(G, H)^{\mathrm{gap}}_{\geqslant 0}.$$

Proof. Any element of $\mathrm{PO}(K, H \cap K)$ is the difference $U - V$ for two $\mathcal{P}(K)$-matched and $K/(H \cap K)$-matched real K-modules U and V. Set $U' = U - U^{H\cap K}$ and $V' = V - V^{H\cap K}$.

Then $U - V = U' - V'$ and, by the assumption on the K-module W, for a sufficiently large integer $m \geqslant 0$, $U' \oplus mW$ and $V' \oplus mW$ are both gap-positive at all pairs $(P, L) \in \mathcal{PH}(K)$ with $[L : P] = 2$. So, for sufficiently large integers $m, n \geqslant 0$, the two real G-modules

$$\mathrm{Ind}_K^G(U' \oplus mW) \oplus nV(G) \quad \text{and} \quad \mathrm{Ind}_K^G(V' \oplus mW) \oplus nV(G)$$

are both gap-non-negative, and clearly both are $\mathcal{L}(G)$-free, completing the proof. □

We denote by $\mathrm{NPP}(G)$ the set of not of prime power order (NPP) elements of G. Set

$$\mathrm{RConj}(G) = \{(g)^{\pm} \mid g \in G\} \quad \text{and} \quad \mathrm{RConj}(\mathrm{NPP}(G)) = \{(g)^{\pm} \mid g \in \mathrm{NPP}(G)\}.$$

Lemma 6.4. *Let G be a finite group. Then the character map*

$$\chi\colon \mathrm{RO}(G) \to \mathrm{map}(\mathrm{RConj}(G), \mathbb{R})$$

induces an isomorphism $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{RO}(G) \to \mathrm{map}(\mathrm{RConj}(G), \mathbb{R})$ such that the the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{PO}(G) & \xrightarrow{\cong} & \mathrm{map}(\mathrm{RConj}(\mathrm{NPP}(G)), \mathbb{R}) \\ \downarrow & & \downarrow \\ \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{RO}(G) & \xrightarrow{\cong} & \mathrm{map}(\mathrm{RConj}(G), \mathbb{R}) \end{array}$$

where the second vertical map is determined by sending the maps defined on the set $\mathrm{RConj}(\mathrm{NPP}(G))$ *to their extensions on* $\mathrm{RConj}(G)$ *that vanish outside of* $\mathrm{RConj}(\mathrm{NPP}(G))$.

Proof. Through the map $\mathbb{R}\otimes_{\mathbb{Z}}\mathrm{RO}(G) \to \mathrm{map}(\mathrm{RConj}(G), \mathbb{R})$ induced by the character map $X \mapsto \chi_X$ for $X \in \mathrm{RO}(G)$, $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{RO}(G)$ is the space of real valued functions f on G constant on the real conjugacy classes $(g)^{\pm}$, and $\mathbb{R} \otimes_{\mathbb{Z}} \mathrm{PO}(G)$ is the space of those f that vanish on $(g)^{\pm}$ when $g \in G$ is of prime power order (cf. [**53**, Theorem 25, p. 95 and Corollary 1, p. 96]). Here, $\mathrm{rk}\,\mathrm{RO}(G) = |\mathrm{RConj}(G)|$ and $\mathrm{rk}\,\mathrm{PO}(G) = r_G = |\mathrm{RConj}(\mathrm{NPP}(G))|$ (cf. [**24**, Lemma 2.1]). So, the horizontal maps are isomorphisms. As $\chi_X((g)^{\pm}) = f((g)^{\pm}) = 0$ for any $X \in \mathrm{PO}(G)$ and any $g \in G$ of prime power order, the diagram commutes. $\square$

Lemma 6.5. *Let G be a finite group. Let U and V be two $\mathcal{P}(G)$-matched real G-modules. Then $d_U(P,H) = d_V(P,H)$ for any pair $(P,H) \in \mathcal{PH}(G)$ such that H is a 2-group.*

Proof. As U and V are $\mathcal{P}(G)$-matched, U and V are isomorphic (in particular) as H-modules, because H is a 2-group. For any pair $(P,H) \in \mathcal{PH}(G)$, $P < H$ and clearly $\dim U^H = \dim V^H$ and $\dim U^P = \dim V^P$. Therefore, $d_U(P,H) = d_V(P,H)$. $\square$

The notion of gap function of a real G-module is introduced in Definition 4.1. This notion can also be defined naturally for a virtual real G-module of a finite group G.

Letting $X = U - V$ for two real G-modules U and V, set $d_X(P,H) = d_U(P,H) - d_V(P,H)$. Then, for a subgroup K of G,

$$d_{\mathrm{Ind}_K^G(X)}(P,H) = \sum_{PgK \in (P\backslash G/K)^{P\backslash H}} d_X(g^{-1}Pg \cap K, g^{-1}Hg \cap K)$$

for any pair $(P,H) \in \mathcal{PH}(G)$ with $[H : P] = 2$. Here, $P \backslash H$ acts canonically on $P \backslash G/K$, and the fixed point set $(P \backslash G/K)^{P\backslash H}$ is the set of double cosets PgK with $PgK = HgK$.

Definition 6.6. A virtual G-module $X = U - V$ is called *gap-positive* (respectively, *gap-non-negative*) at $(P,H) \in \mathcal{PH}(G)$ if $d_X(P,H) > 0$ (respectively, $d_X(P,H) \geqslant 0$). Moreover, X is called *gap-positive* (respectively, *gap-non-negative*) if $d_X(P,H) > 0$ (respectively, $d_X(P,H) \geqslant 0$) for all $(P,H) \in \mathcal{PH}(G)$.

The results of [**23**, Theorem 2.3] and [**35**, Lemma 0.6] yield the following lemma.

Lemma 6.7. *Let G be a finite group with $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$. For $K \leqslant G$, suppose U and V are two real K-modules such that the following two conditions hold:*

(i) *U and V are gap-non-negative at any (P,H) with $P < H \leqslant K$ and $[H : P] = 2$;*

(ii) *$\mathrm{Ind}_K^G(U)$ and $\mathrm{Ind}_K^G(V)$ are $\mathcal{L}(G)$-free.*

Then there exist two real gap-non-negative $\mathcal{L}(G)$-free G-modules U' and V' such that

$$U' - V' = \mathrm{Ind}_K^G(U) - \mathrm{Ind}_K^G(V).$$

Proof. Set $U' = \mathrm{Ind}_K^G(U) \oplus \ell V(G)$ and $V' = \mathrm{Ind}_K^G(V) \oplus \ell V(G)$ for some integer $\ell \geqslant 1$. Let $(P, H) \in \mathcal{PH}(G)$. If $V(G)$ is gap-positive at (P, H), i.e. $d_{V(G)}(P, H) > 0$, then U' and V' are $\mathcal{L}(G)$-free G-modules that are gap-positive at (P, H) for sufficiently large ℓ.

Recall that $V(G)$ is gap-non-negative. Suppose that $d_{V(G)}(P, H) = 0$. It holds that

$$d_{\mathrm{Ind}_K^G(U)}(P, H) = \sum_{PgK \in (P\backslash G/K)^{P\backslash H}} d_U(g^{-1}Pg \cap K, g^{-1}Hg \cap K).$$

Since $[g^{-1}Hg \cap K : g^{-1}Pg \cap K] = 2$ by the assumption, $\mathrm{Ind}_K^G(U)$ is gap-non-negative at (P, H), and thus U' is gap-non-negative at (P, H). Similarly, V' is gap-non-negative at (P, H). □

Proposition 6.8. *Let G be a finite group and let $H = G^{\mathrm{nil}}$. If G contains two* NPP *elements x and y lying in a gap subgroup of G, such that $xH = yH$ and $(x)^{\pm} \neq (y)^{\pm}$, then*

$$\mathrm{PLO}(G, H)^{\mathrm{gap}}_{\geqslant 0} \neq 0.$$

Proof. Let K be a gap subgroup of G containing the elements x and y. By the assumptions on x and y, it follows that $r_G^K > r_{(G,H)}^K$ (cf. Definition 3.5). According to Lemma 3.6 (ii),

$$\mathrm{rk}\,\mathrm{Ind}_K^G(\mathrm{PO}(K, H \cap K)) = r_G^K - r_{(G,H)}^K > 0.$$

Now, the conclusion that $\mathrm{PLO}(G, H)^{\mathrm{gap}}_{\geqslant 0} \neq 0$ follows easily from Lemma 6.3. □

According to the example of U and V in [**40**, p. 865], the following lemma holds.

Lemma 6.9. *Let G be a finite Oliver group. If the order of the nilpotent group G/G^{nil} has two (or more) odd prime divisors, then* $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} = \mathrm{PO}(G, G^{\mathrm{nil}}) \neq 0$.

Proof of Theorem 6.1. Let G be a finite Oliver group satisfying the G^{nil}-coset condition. So, according to Definition 5.1, G has a G^{nil}-coset containing two NPP elements x and y that are not real conjugate in G, and one of the following holds:

(i) the elements x and y are both contained in a gap subgroup of G, or

(ii) the orders of x and y are even and the involutions of $\langle x\rangle$ and $\langle y\rangle$ are conjugate in G, where $\langle x\rangle$ and $\langle y\rangle$ are the cyclic groups generated by x and y, respectively.

We shall prove that $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0$. If (i) holds, Proposition 6.8 completes the proof. So, in the proof, we may assume that (ii) holds, while (i) does not. We may also assume that the involutions of $\langle x\rangle$ and $\langle y\rangle$ coincide by exchanging, if necessary, y and its conjugate.

According to Lemma 6.9, it is sufficient to consider the case where the order of G/G^{nil} has at most one odd prime divisor. Let $N = O^2(G)$. Then $N\langle x\rangle = N\langle y\rangle$ and $O^2(N) = N$.

First, we suppose that $x \in N$. By the assumption, N is not a gap group. Since x is an NPP element, N is not of prime power order, and since N is not perfect and $O^2(N) = N$, $O^p(N) \neq N$ for some odd prime p. If $O^p(N)$ is not of prime power order, N is a gap group, which never happens. Thus, $|O^p(N)|$ is a power of a prime q with $q \neq p$. So, the G-module $V(N)$ is gap-positive at all pairs $(P, H) \in \mathcal{PH}_2(N)$. It follows from Lemma 6.3 that

$$\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0.$$

Now, we suppose that $x \notin N$. Recall that x and y are NPP elements of even orders and $O^2(N) = N$. Consider the two cases according to whether $\mathcal{P}(N) \cap \mathcal{L}(N)$ is empty or not. When $\mathcal{P}(N) \cap \mathcal{L}(N) \neq \varnothing$, we argue as follows. Since $O^p(N)$ has a 2-power order for some odd prime p, $V(N)$ is gap-positive at any $(P, H) \in \mathcal{PH}_2(N)$. Hence, by Lemma 6.3,

$$\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0.$$

So, it remains to consider the case where $\mathcal{P}(N) \cap \mathcal{L}(N) = \varnothing$, with $x \notin N$. By [**58**, Theorem B], $N \cap \langle x \rangle$ is of order a power of an odd prime (remember N is not a gap group). Therefore, $|\pi(\langle x \rangle)| = 2$. Hence, $\pi(\langle x \rangle) = \{2, p\}$ for some odd prime p. Similarly, it holds that $|\pi(\langle y \rangle)| = 2$. Furthermore, $\pi(\langle y \rangle) = \{2, p\}$ because $C_G(z)$ is not a gap group for the involution z of $\langle x \rangle$. Therefore, $|N \cap \langle x \rangle| = p^a$ and $|N \cap \langle y \rangle| = p^b$ for some integers $a, b \geqslant 0$. Set

$$N' = G^{\mathrm{nil}} \langle x \rangle = G^{\mathrm{nil}} \langle y \rangle.$$

Now, we construct a non-zero element of $\mathrm{PO}(G, G^{\mathrm{nil}})$ (see Lemma 6.11, below). We define a virtual real $\langle x \rangle$-module $V_{\mathbb{R}}(x)$ as follows. Let $x = x_2 x_p$, where x_2 is of 2-power order and x_p is of p-power order. Let $C = \langle x \rangle$, $C_r = \langle x_r \rangle$, and let ξ_r be the irreducible complex C_r-module whose character sends x_r^a to $\exp(2a\pi\sqrt{-1}/|C_r|)$ for $r \in \pi(C)$. Let $i \colon C \to C_2 \times C_p$ be the isomorphism sending x to (x_2, x_p). For a complex C_r-module σ_r with $r = 2$ or p, we denote by $i^*(\sigma_2 \otimes \sigma_p)$ the complex C-module satisfying the condition that

$$\chi_{i^*(\sigma_2 \otimes \sigma_p)}(x) = \chi_{\sigma_2}(x_2) \chi_{\sigma_p}(x_p).$$

We denote by $U_{\mathbb{C}}(x)$ the virtual complex C-module $i^*((\mathbb{C} - \xi_2) \otimes (\mathbb{C} - \xi_p))$ and by $U_{\mathbb{R}}(x)$ its realification. Set $U(N', x) = \mathrm{Ind}_C^{N'} U_{\mathbb{R}}(x)$. The epimorphism $\tau_p \colon C_p \to C_p/(C_p \cap G^{\mathrm{nil}})$ induces a homomorphism $\tau_p^* \colon R(C_p/(C_p \cap G^{\mathrm{nil}})) \to R(C_p)$. Let η_p be the irreducible complex $C_p/(C_p \cap G^{\mathrm{nil}})$-module whose character sends $\tau_p(x_p)^a$ to $\exp(2a\pi\sqrt{-1}/|C_p/C_p \cap G^{\mathrm{nil}}|)$. Set

$$V_{\mathbb{C}}(x) = i^*((\mathbb{C} - \xi_2) \otimes \tau_p^*(\mathbb{C} - \eta_p)) \quad \text{and} \quad V(N', x) = \mathrm{Ind}_C^{N'} V_{\mathbb{R}}(x),$$

where $V_{\mathbb{R}}(x)$ is the realification of $V_{\mathbb{C}}(x)$.

Lemma 6.10. *For any element $x^c \in C = \langle x \rangle$,*

$$\chi_{U(N',x)^{G^{\mathrm{nil}}}}(x^c) = \begin{cases} 2\,\mathrm{Re}(1 - \chi_{\xi_2}(x_2^c)), & C \cap G^{\mathrm{nil}} \neq \{e\}, \\ 2\,\mathrm{Re}((1 - \chi_{\xi_2}(x_2^c))(1 - \chi_{\xi_p}(x_p^c))), & C \cap G^{\mathrm{nil}} = \{e\}, \end{cases}$$

and

$$\chi_{V(N',x)^{G^{\mathrm{nil}}}}(x^c) = 2\,\mathrm{Re}((1 - \chi_{\xi_2}(x_2^c))(1 - \chi_{\eta_p}(\tau_p(x_p^c)))).$$

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

Proof. For $k, h \in N'$, we denote by $\beta(k,h)$ the number of elements (a,b) of $G^{\mathrm{nil}} \times N'$ with $a = b^{-1}k^{-1}bh$. Since N'/G^{nil} is cyclic, we see that if $k = x^c$, then $b^{-1}k^{-1}bz \in G^{\mathrm{nil}}$ for any $z \in x^c(C \cap G^{\mathrm{nil}})$ and any $b \in N'$. Thus, $\beta(x^c, z) = |N'|$ for $z \in x^c(C \cap G^{\mathrm{nil}})$ and $\beta(x^c, z) = 0$ for $z \in C \smallsetminus x^c(C \cap G^{\mathrm{nil}})$. Then the character of $U(N', x)^{G^{\mathrm{nil}}}$ sends x^c to

$$\begin{aligned}
\chi_{U(N',x)^{G^{\mathrm{nil}}}}(x^c) &= \frac{1}{|G^{\mathrm{nil}}|} \sum_{x^c \in G^{\mathrm{nil}}} \chi_{U(N',x)}(x^c a) \\
&= \frac{1}{|G^{\mathrm{nil}}|\,|C|} \sum_{a \in G^{\mathrm{nil}}} \sum_{\substack{b \in N', \\ b^{-1}x^c ba \in C}} \chi_{U_{\mathbb{R}}(x)}(b^{-1}x^c ba) \\
&= \sum_{z \in C} \frac{\beta(x^c, z)}{|G^{\mathrm{nil}}|\,|C|} \chi_{U_{\mathbb{R}}(x)}(z) \\
&= \frac{|N'|}{|G^{\mathrm{nil}}|\,|C|} \sum_{a \in x^c(C \cap G^{\mathrm{nil}})} \chi_{U_{\mathbb{R}}(x)}(a) \\
&= \frac{1}{|G^{\mathrm{nil}} \cap C|} \sum_{a \in x^c(C \cap G^{\mathrm{nil}})} \chi_{U_{\mathbb{R}}(x)}(a) \\
&= \frac{2}{|G^{\mathrm{nil}} \cap C|} \operatorname{Re}\left((1 - \chi_{\xi_2}(x_2^c)) \sum_{b \in C \cap G^{\mathrm{nil}}} (1 - \chi_{\xi_p}(x_p^c)\chi_{\xi_p}(b)) \right) \\
&= 2\operatorname{Re}((1 - \chi_{\xi_2}(x_2^c))(1 - \chi_{\xi_p}(x_p^c)\dim(\xi_p^{C \cap G^{\mathrm{nil}}}))) \\
&= \begin{cases} 2\operatorname{Re}(1 - \chi_{\xi_2}(x_2^c)), & C \cap G^{\mathrm{nil}} \neq \{e\}, \\ 2\operatorname{Re}((1 - \chi_{\xi_2}(x_2^c))(1 - \chi_{\xi_p}(x_p^c))), & C \cap G^{\mathrm{nil}} = \{e\}. \end{cases}
\end{aligned}$$

Similarly, we obtain that

$$\chi_{V(N',x)^{G^{\mathrm{nil}}}}(x^c) = 2\operatorname{Re}((1 - \chi_{\xi_2}(x_2^c))(1 - \chi_{\eta_p}(\tau_p(x_p^c))))$$

completing the proof. □

Lemma 6.11. *The following two conclusions hold:*

(i) *if $\langle x \rangle \cap G^{\mathrm{nil}} = \langle y \rangle \cap G^{\mathrm{nil}} = \{e\}$ or $|\langle x \rangle \cap G^{\mathrm{nil}}|$ and $|\langle y \rangle \cap G^{\mathrm{nil}}|$ are both divisible by p, then $U(G,x) - U(G,y) \neq 0$ and the difference lies in $\mathrm{PO}(G, G^{\mathrm{nil}})$;*

(ii) *if $\langle x \rangle \cap G^{\mathrm{nil}} = \{e\}$ and $|\langle y \rangle \cap G^{\mathrm{nil}}|$ is divisible by p, then $U(G,x) - V(G,y) \neq 0$ and the difference lies in $\mathrm{PO}(G, G^{\mathrm{nil}})$.*

Proof. Suppose that $\langle x \rangle \cap G^{\mathrm{nil}} = \langle y \rangle \cap G^{\mathrm{nil}} = \{e\}$ or $|\langle x \rangle \cap G^{\mathrm{nil}}|$ and $|\langle y \rangle \cap G^{\mathrm{nil}}|$ are both divisible by p. Then the characters of $U(N', x)^{G^{\mathrm{nil}}}$ and $U(N', y)^{G^{\mathrm{nil}}}$ coincide, and so

$$0 \neq U(N', x) - U(N', y) \in \mathrm{PO}(N', G^{\mathrm{nil}}).$$

Hence, $U(G,x) - U(G,y) \in \mathrm{PO}(G, G^{\mathrm{nil}})$. If $\langle x \rangle \cap G^{\mathrm{nil}} = \{e\}$ and $|\langle y \rangle \cap G^{\mathrm{nil}}|$ is divisible by p, then $|x| < |y|$ and the characters of $U(N', x)^{G^{\mathrm{nil}}}$ and $V(N', y)^{G^{\mathrm{nil}}}$ coincide, and so

$$U(N', x) - V(N', y) \in \mathrm{PO}(N', G^{\mathrm{nil}}).$$

Therefore, $0 \neq U(G,x) - V(G,y) \in \mathrm{PO}(G, G^{\mathrm{nil}})$, completing the proof. □

Now, we show that $U(G,x) - U(G,y)$ and $U(G,x) - V(G,y)$ are gap-non-negative.

Lemma 6.12. *Let z be the involution of $\langle x \rangle$. Let r and s be the orders of Sylow 2-subgroups of $C_G(z)$ and C, respectively. Then $(P \setminus G/C)^{P \setminus H} = P \setminus PC_G(z)/C$ and $|(P \setminus G/C)^{P \setminus H}| = r/s$.*

Proof. To prove the result, let $PaC \in (P \backslash G/C)^H$. Then $z \in PaCa^{-1}$, say $z = paca^{-1}$ for $p \in P$ and $c \in C$. Take a positive odd integer k such that $|p^{-1}z|/k = 2$. As $(p^{-1}z)^k$ is conjugate to z in P, ac^ka^{-1} is an involution equal to aza^{-1}. Thus, $p'^{-1}zp' = aza^{-1}$ for some $p' \in P$, and so $a \in PC_G(z)$.

Therefore, $(P \setminus G/C)^H = P \setminus PC_G(z)/C$. The bijection from $O^2(C_G(z)) \setminus C_G(z)$ to $P \setminus PC_G(z)$ sending $O^2(C_G(z))a$ to Pa induces a bijection from $C_G(z)/O^2(C_G(z))C$ to $P \setminus PC_G(z)/C$ and thus $|(P \setminus G/C)^H| = |C_G(z)/O^2(C_G(z))C| = r/s$.

Hence, we obtain the following commutative diagram:

$$\begin{array}{ccc}
P \setminus PC_G(z) & \longrightarrow & P \setminus PC_G(z)/C \\
\cong \uparrow & & \cong \uparrow \\
O^2(C_G(z)) \setminus C_G(z) & \longrightarrow & O^2(C_G(z)) \setminus C_G(z)/C \\
 & & \cong \downarrow \\
 & & C_G(z)/O^2(C_G(z))C
\end{array}$$

which completes the proof. □

Set $U(G,x,y) := U(G,x) - U(G,y)$ and consider the integer

$$m = -\min(\{d_{U(G,x,y)}(P,H) + 1 \mid (P,H) \in \mathcal{PH}(G)\} \cup \{0\}).$$

Choose an integer ℓ so that $U(G,x,y) + \ell V(G)$ is a real (non-virtual) G-module.

According to [**58**, Lemma 4.3], there exists a real $\mathcal{L}(G)$-free G-module $W(z)$ satisfying the following three properties:

(i) $d_{W(z)}(P,H) \geqslant 0$ for any $(P,H) \in \mathcal{PH}(G)$;

(ii) $d_{W(z)}(P,H) > 0$ for any $(P,H) \in \mathcal{PH}(G) \smallsetminus \mathcal{PH}_2(G)$;

(iii) for $(P,H) \in \mathcal{PH}_2(G)$ with $z \in H$, $d_{W(z)}(P,H) = 0$ if and only if $P \geqslant O^2(C_G(z))$.

Lemma 6.13. *$U := (U(G,x,y) + \ell V(G)) \oplus m(W(z) \oplus V(G))$ is gap-non-negative.*

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Proof. Let $(P, H) \in \mathcal{PH}(G)$. If $d_{W(z)\oplus V(G)}(P, H) > 0$, then

$$d_U(P, H) \geqslant d_{U(G,x,y)}(P, H) + m > 0.$$

Suppose that $d_{W(z)\oplus V(G)}(P, H) = 0$. Then $d_U(P, H) = d_{U(G,x,y)}(P, H)$ and $(P, H) \in \mathcal{PH}_2(G)$. By Lemma 6.5, we may further suppose that H is not a 2-group. We shall analyse the number $d_{U(G,x)}(P, H)$. Recall that

$$d_{U(G,x)}(P, H) = \sum_{PgC \in (P\backslash G/C)^{P\backslash H}} d_{U_{\mathbb{R}}(x)}(g^{-1}Pg \cap C, g^{-1}Hg \cap C).$$

If the real conjugacy class $(z)^{\pm}$ does not intersect with H, $(P \setminus G/C)^H$ is empty and therefore $d_{U(G,x)}(P, H) = 0$. Similarly, $d_{U(G,y)}(P, H) = 0$.

Suppose that $(z)^{\pm}$ intersects with H. Take an element $a \in G$ such that $z \in a^{-1}Ha$. Set $P' = a^{-1}Pa$ and $H' = a^{-1}Ha$. Then

$$\begin{aligned} d_{U(G,x)}(P, H) &= d_{U(G,x)}(P', H') \\ &= 2 \sum_{P'gC \in (P'\backslash G/C)^{P'\backslash H'}} d_{\mathbb{C}-\xi_2}(\{e\}, \{e, z\}) \\ &= -4|(P' \setminus G/C)^{P'\setminus H'}|. \end{aligned}$$

Therefore, $d_{U(G,x)}(P, H) = -4r/s$, where r and s are the integers considered in Lemma 6.12. Since $|C \cap G^{\mathrm{nil}}|$ and $|\langle y\rangle \cap G^{\mathrm{nil}}|$ are odd and $xG^{\mathrm{nil}} = yG^{\mathrm{nil}}$, s is the order of a Sylow 2-subgroup of $\langle y\rangle$. Hence, $d_{U(G,y)}(P, H) = -4r/s$ by the argument above, and thus $d_{U(G,x,y)}(P, H) = 0$. Putting all together, U is gap-non-negative. □

It follows that $U(G, x, y) + mW(z) + \ell V(G)$ and $mW(z) \oplus \ell V(G)$ are gap-non-negative $\mathcal{L}(G)$-free real G-modules, when ℓ is sufficiently large, which yields (by Lemma 6.11) that

$$U(G, x) - U(G, y) \in \mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0}.$$

By similar arguments, we can show that, for a sufficiently large integer ℓ,

$$U := U(G, x) - V(G, y) + mW(z) + \ell V(G)$$

is also a gap-non-negative $\mathcal{L}(G)$-free real G-module, where

$$m = -\min(\{d_{U(G,x)}(P, H) - d_{V(G,y)}(P, H) + 1 \mid (P, H) \in \mathcal{PH}(G)\} \cup \{0\}).$$

Set $V := mW(z) \oplus \ell V(G)$. Clearly, V is a gap-non-negative $\mathcal{L}(G)$-free real G-module and

$$U - V = U(G, x) - V(G, y) \neq 0$$

lies in $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0}$, completing the proof of Theorem 6.1. □

7. The Non-solvable Group Theorem

In this section, we prove our second key algebraic result, which reads as follows.

Theorem 7.1 (the Non-solvable Group Theorem). *Except for* $G = \mathrm{Aut}(A_6)$ *or* $\mathrm{P\Sigma L}(2, 27)$, *any finite non-solvable group* G *with* $r_G \geqslant 2$ *satisfies the* G^{sol}*-coset condition.*

In order to prove Theorem 7.1, we argue by contradiction using an approach similar to that in [**40**, Proposition 3.1]. Instead of assuming that $r_G = r_{(G,G^{\mathrm{sol}})}$ as is done in the proof of [**40**, Proposition 3.1], we suppose that G does not satisfy the G^{sol}-coset condition and check the corresponding statements in [**40**, §§ 2 and 3] to obtain a contradiction.

Proposition 7.2. *Let* G *be a finite group with* $H \trianglelefteq G$ *such that* $|\pi(H)| \geqslant 3$. *If* $Z(G) \neq 1$, *then* G *satisfies the* H*-coset condition.*

Proof. Take an element $x \in Z(G)$ of order r for a prime r. By the assumption, there exist two distinct primes p and q and two elements y and z of H such that $p \neq r$, $q \neq r$, $|y| = p$ and $|z| = q$. Therefore, xy and xz are NPP elements of the gap group $H\langle x\rangle$. Hence, G satisfies the H-coset condition. □

By Burnside's theorem, it holds that $|\pi(L)| \geqslant 3$ for a non-solvable group L. Therefore, we see that the following corollary is true.

Corollary 7.3. *If a finite non-solvable group* G *does not satisfy the* G^{sol}*-coset condition, then the centre of* G *is trivial.*

Lemma 7.4. *Let* G *be a finite group and let* $H \trianglelefteq G$. *Suppose either that* G *is a gap group or that* $O^2(G)$ *is a gap group and each element of* $G \smallsetminus O^2(G)$ *has prime power order. Then* G *satisfies the* H*-coset condition if and only if* $r_G > r_{(G,H)}$.

Proof. By Lemmas 3.4 and 5.2, $r_G > r_{(G,H)}$ if and only if there exists an H-coset of G containing two NPP elements x and y which are not real conjugate in G. If G is a gap group, Definition 5.1 (i) completes the proof. As x and y are NPP elements, they both lie in $O^2(G)$ if each element of $G \smallsetminus O^2(G)$ has prime power order. So, if in addition $O^2(G)$ is a gap group, again Definition 5.1 (i) completes the proof. □

The next lemma immediately follows from Definition 5.1 and is very useful to see whether G satisfies the H-coset condition.

Lemma 7.5. *Let* G *be a finite gap group and let* $H \trianglelefteq G$. *Let* x *and* y *be two* NPP *elements of* G *such that* $xH = yH$ *and for some* $N \trianglelefteq G$, *the cosets* xN *and* yN *have distinct orders in* G/N. *Then* G *satisfies the* H*-coset condition.*

By [**40**, Lemma 2.7], if a finite non-abelian simple group L is without NPP elements or all NPP elements have the same order, then L is isomorphic to one of the following groups:

(i) $\mathrm{PSL}(2, q)$ with $q \equiv \pm 3 \pmod 8$; or

(ii) $\mathrm{PSL}(2, q)$ with $q = 9$ or q a Fermat or Mersenne prime; or

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(iii) $\mathrm{PSL}(2, 2^n)$ or $\mathrm{Sz}(2^n)$, $n \geqslant 3$; or

(iv) $\mathrm{PSL}(3, 3)$, $\mathrm{PSL}(3, 4)$, A_7, M_{11} or M_{22}.

By using [**9**], we can check that if $L = \mathrm{PSL}(3, 3)$, A_7, M_{11} or M_{22}, then K is a gap group for $L \leqslant K \leqslant \mathrm{Aut}(L)$. Moreover, if $L = \mathrm{PSL}(3, 4)$, then $r_K = 1$ or K satisfies the L-coset condition for any subgroup K with $L \leqslant K \leqslant \mathrm{Aut}(L)$. Also, $\mathrm{PGL}(2, q)$ is a gap group for any prime power $q \neq 2, 3, 5, 7, 9, 17$ (see [**58**, Corollary 3.5]).

For the reader's convenience, we recall some notions from group theory. Let G be a finite group. The Fitting subgroup $F(G)$ of G is the largest normal nilpotent subgroup of G. In the proof, the Fitting subgroup of G plays an important role. The group G is called *quasisimple* if G is perfect and $G/Z(G)$ is simple. Moreover, G is said to be *semisimple* if G is the product of quasisimple groups G_i, $1 \leqslant i \leqslant m$, such that $[G_i, G_j] = 1$ for all $i \neq j$. Also, $E(G)$ denotes the largest normal semisimple subgroup of G, and $F^*(G) = E(G)F(G)$ is the generalized Fitting subgroup of G. Bender's Theorem says that if N is a normal subgroup of G such that $C_G(N) \leqslant N$, then $E(G) \leqslant N$. If $E(G) \neq 1$, then the uniquely determined quasisimple factors of G are called the *components* of G. Let $\mathrm{PSL}(3, 4)^* = \mathrm{PSL}(3, 4) \rtimes \langle u \rangle$ be the extension of $\mathrm{PSL}(3, 4)$ by an involutory graph-field automorphism u of order 2.

The following lemma shows that $F^*(G)$ is not a finite non-abelian simple group.

Lemma 7.6 (Pawałowski and Solomon [40, Lemma 2.8]). *Let G be a finite non-solvable group. Assume that, for $H = G^{\mathrm{sol}}$, G does not satisfy the H-coset condition and $F^*(G)$ is a finite non-abelian simple group. Then G is isomorphic to one of the groups listed below:*

(i) $\mathrm{PSL}(2, q)$, $q \in \{5, 7, 8, 9, 11, 13, 17\}$;

(ii) $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, A_7, $\mathrm{PSL}(3, 3)$, $\mathrm{PSL}(3, 4)$, M_{11}, M_{22};

(iii) $\mathrm{PGL}(2, 5)$, $\mathrm{PGL}(2, 7)$, $\mathrm{P\Sigma L}(2, 8)$, M_{10}, $\mathrm{PSL}(3, 4)^*$;

(iv) $\mathrm{Aut}(A_6)$ *or* $\mathrm{P\Sigma L}(2, 27)$.

Moreover, $r_G = 1$ for the groups listed in (i)–(iii), and $r_G = 2$ for the groups listed in (iv).

Proof. Set $L = F^*(G)$. Then L is one of the groups listed in (i)–(iv). If $L = G$, then $r_G = r_{(G, G^{\mathrm{sol}})}$, because G is a gap group. Thus, by arguing as in the proof of [**40**, Lemma 2.8], we complete the proof. □

A subgroup K of a finite group L is called a *subnormal* subgroup of L if there is a chain of subgroups of L, each one normal in the next, beginning at K and ending at L. For a prime p, the *p-rank* $m_p(L)$ of a finite group L is the largest non-negative integer n such that L contains an elementary abelian subgroup of order p^n. The largest normal solvable subgroup of L is called the *solvable radical* of L.

Proof of Theorem 7.1. Until the end of the proof, we assume that G is a finite non-solvable group such that $r_G \geqslant 2$ and G is not isomorphic to $\mathrm{Aut}(A_6)$ or $\mathrm{P\Sigma L}(2,27)$.

We set $H = G^{\mathrm{sol}}$ and denote by S the solvable radical of G. Contrary to the conclusion of Theorem 7.1, suppose that G does not satisfy the H-coset condition.

Lemma 7.7 (Pawałowski and Solomon [40, Lemma 3.2]). *The solvable radical of H is equal to S, and G/S is isomorphic to* $\mathrm{PGL}(2,5)$, $\mathrm{PGL}(2,7)$, $\mathrm{P\Sigma L}(2,8)$, M_{10}, $\mathrm{PSL}(3,4)^*$, $\mathrm{Aut}(A_6)$ *or* $\mathrm{P\Sigma L}(2,27)$.

Proof. Let S_0 be the solvable radical of H. Set $\bar{G} = G/S_0$ and $\bar{H} = H/S_0$ and note that G has a subnormal non-abelian simple subgroup $\bar{L} \leqslant \bar{H}$.

Supposing that $C_{\bar{G}}(\bar{L}) \neq 1$, we obtain a contradiction. Take an element c of $C_{\bar{G}}(\bar{L})$ of prime order. Then the group $\langle c\rangle\bar{L}$ is a gap group, as c commutes with elements of $\bar{L}$ and $|\pi(\bar{L})| \geqslant 3$ by [**58**, Theorem B]. There exist two elements x and y of $\bar{L}$ of distinct prime orders, coprime to the order of c. Then cx and cy are two NPP elements of distinct orders. So, in particular, cx and cy are not real conjugate.

Set $K = f^{-1}(\langle c\rangle\bar{L})$, where $f\colon G \to \bar{G}$ is the natural homomorphism. By [**57**, Theorem 1.2], K is also a gap group and K contains two elements that are not real conjugate in G, which means that G satisfies the H-coset condition: a contradiction.

Therefore, $C_{\bar{G}}(\bar{L}) = 1$, and thus $\bar{L} = F^*(\bar{G}) \neq \bar{G}$. Now, Lemma 7.6 describes the possible cases for $\bar{G}$. As the solvable radical $\bar{S}$ of $\bar{G}$ is trivial, we see that $S_0 = S$. □

Lemma 7.8. *$S \neq 1$ and G/S is not isomorphic to* $\mathrm{P\Sigma L}(2,8)$, $\mathrm{P\Sigma L}(2,27)$, *or* M_{10}.

Proof. By Lemma 7.7, $S \neq 1$ and according to Lemma 7.4 and [**40**, Proposition 3.1], G/S is not isomorphic to $\mathrm{P\Sigma L}(2,8)$, $\mathrm{P\Sigma L}(2,27)$ or M_{10}. □

In the remaining part of the proof of Theorem 7.1, we shall frequently use the following theorem, which goes back to [**17**, Proposition 8.3] or [**40**, Theorem 2.3].

Theorem 7.9 (Gorenstein *et al.* [17]; Pawałowski and Solomon [40]). *If $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$ acts on an abelian q-group B for two distinct primes p and q, then $B = \langle C_B(a) \mid a \in A \setminus \{0\}\rangle$, where $C_B(a) = \{b \in B \mid ab = ba\}$.*

Lemma 7.10. *$F(H) = F(G)$ and $F(G)$ is a p-group for some prime p.*

Proof. Since $F(G) \leqslant S \leqslant H$ by Lemma 7.7, it holds that $F(H) = F(G)$. Furthermore, since G does not satisfy the H-coset condition, H is an EP group, which means that all elements of $\mathrm{NPP}(H)$ have the same order, and if $K \trianglelefteq H$ and $K \cap \mathrm{NPP}(H) \neq \varnothing$, then $\mathrm{NPP}(H) \subset K$. Therefore, by [**40**, Lemma 2.9], $F(H)$ is a p-group for some prime p. □

Lemma 7.11 (Pawałowski and Solomon [40, Lemma 3.3]). *S is a p-group for some prime p.*

Proof. Note that the perfect group H is an EP group and, by Lemma 7.7, H/S is isomorphic to $\mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,9)$ or $\mathrm{PSL}(3,4)$. Now, the proof of [**40**, Lemma 3.3] can be applied for H, instead of G. Then we see that the solvable radical of H, which is just S by Lemma 7.7, is a p-group. □

Lemma 7.12 (Pawałowski and Solomon [40, Lemma 3.4]). *S is either a 2-group or an elementary abelian p-group for some odd prime p, and in the latter case, every* NPP *element of H has order $2p$.*

Proof. Suppose that S is a p-group for an odd prime p. First, we show that S is an elementary abelian p-group. Let K be a commutator subgroup of S. Since K is a characteristic group, S acts on K and on S/K. Note that H contains $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $K \neq \{1\}$, then there exist two NPP elements x and y of H such that $O^2(\langle x\rangle) \leqslant K$ and $O^2(\langle y\rangle) \nleqslant K$ by Theorem 7.9. Therefore, G satisfies the H-coset condition: a contradiction.

Thus, S is an abelian group. Let L be a subgroup of S generated by all elements of S of order greater than p. Since L is also a characteristic group, an argument similar to that above yields $L = \{1\}$. Therefore, S is an elementary abelian p-group. Again, by Theorem 7.9, there exists an NPP element of H of order $2p$. If there exists an NPP element of order different from $2p$, then G satisfies the H-coset condition: a contradiction. □

Following [**40**, p. 879], set $\bar{G} = G/S$ and $\bar{H} = H/S$. By Lemmas 7.7 and 7.8, $S \neq 1$ and

- $\bar{G} \cong \mathrm{PGL}(2,5)$, $\mathrm{PGL}(2,7)$, $\mathrm{Aut}(A_6)$ or $\mathrm{PSL}(3,4)^*$, and
- $\bar{H} \cong \mathrm{PSL}(2,5)$, $\mathrm{PSL}(2,7)$, A_6 or $\mathrm{PSL}(3,4)$.

To complete the proof of Theorem 7.1, recall that, contrary to the conclusion in Theorem 7.1, we have assumed that G does not satisfy the H-coset condition for $H = G^{\mathrm{sol}}$. In particular, H does not contain two NPP elements that are not real conjugate in G.

The arguments of [**40**, Lemmas 3.6–3.11] lead to a contradiction by showing that there are two NPP elements in a coset xH of G that are not real conjugate in G. In [**40**], Pawałowski and Solomon do not claim that $\langle x\rangle H$ is a gap group (since it is not necessary), but we can check this is true always when xH contains such NPP elements. As the existence of such xH implies that G satisfies the H-coset condition (a contradiction), we shall assume that each xH contains at most one real conjugacy class of NPP elements of G. Now, Theorem 7.9 and Lemmas 7.10–7.12 allow us to argue under the hypotheses of Lemma 7.4 to obtain $r_G = r_{(G,H)}$. Consequently, the same arguments as in [**40**, Lemmas 3.6–3.11] show that $\bar{G}$ cannot be as in (i): a contradiction. □

8. Proofs of Theorems A, B, and C

First, for a finite Oliver group G, we focus on the c-primary Smith set $\mathrm{PSm}^{\mathrm{c}}(G)$. By setting $k = 2$ in Theorem 5.8, we see that if G satisfies the G^{nil}-coset condition, then $\mathrm{PSm}^{\mathrm{c}}(G) \neq 0$. As $\mathrm{PSm}^{\mathrm{c}}(G) \subseteq \mathrm{LSm}(G)$, Theorem A is true. Now, by pointing out two groups in $\mathrm{PSm}^{\mathrm{c}}(G)$, we give a more complete description of the c-primary Smith set $\mathrm{PSm}^{\mathrm{c}}(G)$.

Corollary 8.1. *If a finite Oliver group G satisfies the G^{nil}-coset condition, then*

$$0 \neq \mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \leqslant \mathrm{PLO}(G)^{\mathrm{gap}}_{\geqslant 0} \subseteq \mathrm{PSm}^{\mathrm{c}}(G).$$

Proof. By Lemma 3.8 and Theorem 4.9 (the Smith Equivalence Theorem), the inclusions both hold, and $\mathrm{PO}(G, G^{\mathrm{nil}})^{\mathrm{gap}}_{\geqslant 0} \neq 0$ by Theorem 5.6 (Theorem 6.1, the G^{nil}-Coset Theorem). □

Corollary 8.2. *Let G be a finite non-solvable group not isomorphic to* $\mathrm{Aut}(A_6)$ *or* $\mathrm{P\Sigma L}(2, 27)$. *Then the following five claims are equivalent:*

(i) *G satisfies the G^{sol}-condition;*

(i′) *G satisfies the G^{nil}-condition;*

(ii) *there are two or more real conjugacy classes of NPP elements of G, i.e. $r_G \geqslant 2$;*

(iii) $\mathrm{PO}(G, G^{\mathrm{sol}}) \neq 0$, *i.e.* $r_G > r_{(G,G^{\mathrm{sol}})}$;

(iii′) $\mathrm{PO}(G, G^{\mathrm{nil}}) \neq 0$, *i.e.* $r_G > r_{(G,G^{\mathrm{nil}})}$.

Proof. As $G^{\mathrm{sol}} \leqslant G^{\mathrm{nil}}$, (i) implies (i′) by Definition 5.1. Clearly, (i) and (i′) both imply (ii). By Theorem 7.1 (the Non-solvable Group Theorem), (ii) implies (i), proving that (i), (ii′) and (ii) are equivalent. In turn, by Lemma 3.4, (iii) and (iii′) both imply (ii), and, by Lemma 5.3, (i) implies (iii), and (i′) implies (iii′), proving that (i), (i′), (ii), (iii) and (iii′) are equivalent. □

As in Corollary 8.2, the claims (i′) and (ii) are equivalent, Theorem B is true.

Corollary 8.3. *Let G be a finite non-solvable group. Then the following four claims are equivalent:*

(i) *the c-primary Smith set* $\mathrm{PSm}^{\mathrm{c}}(G) \neq 0$*;*

(ii) *the Laitinen–Smith set* $\mathrm{LSm}(G) \neq 0$*;*

(iii) *the primary Smith set* $\mathrm{PSm}(G) \neq 0$*;*

(iv) *the number $r_G \geqslant 2$ and $G \ncong \mathrm{Aut}(A_6)$.*

Proof. As $\mathrm{PSm}^{\mathrm{c}}(G) \subseteq \mathrm{LSm}(G) \subseteq \mathrm{PSm}(G)$, (i) implies (ii), and (ii) implies (iii). If (iii) holds, then $r_G \geqslant 2$ and $G \ncong \mathrm{Aut}(A_6)$ by Lemma 2.3 and [**29**], respectively, proving (iv). Moreover, (iv) implies (i) by Theorem 5.9, except for $G = \mathrm{P\Sigma L}(2, 27)$, the case covered by [**30**]. □

As in Corollary 8.3 the claims (ii) and (iv) are equivalent, Theorem C is true.

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

Appendix A. Oliver and Solomon groups

The notion of the Oliver group was introduced by Laitinen and Morimoto [**23**]. A finite group G is called an *Oliver group* if G is not of prime power order and the Oliver number $n_G = 1$.

According to Oliver [**37**], for a finite group G not of prime power order, $n_G = 1$ if and only if G *does not* contain a series of normal subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is a p-group and G/H is a q-group for some primes p and q, possibly $p = q$, and H/P is cyclic.

Example A 1. Examples of finite Oliver groups include

(i) nilpotent (e.g. abelian) groups with three or more non-cyclic Sylow subgroups,

(ii) the three solvable groups $\mathbb{Z}_3 \times S_4$, $S_3 \times A_4$, $(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ of order 72,

(iii) all non-solvable (e.g. perfect, in particular, non-abelian simple) groups.

Theorem A 2 (Oliver [37]). *A finite group G has a smooth fixed point free action on a disc if and only if G is not of prime power order and $n_G = 1$ (i.e. G is an Oliver group).*

Theorem A 3 (Laitinen *et al.* [25]; Laitinen and Morimoto [23]). *Any finite non-solvable group has a smooth action on a sphere with exactly one fixed point. A finite group G has a smooth action on a sphere with exactly one fixed point if and only if G is an Oliver group.*

In [**40**, Classification Theorem, p. 847], by using the *classification* of finite simple groups, Ronald Solomon has classified finite Oliver groups G with $r_G = 0$ or 1.

Here, a finite group G is called a *Solomon group* if G is not of prime power order and $r_G = 0$ or 1.

According to [**40**, Theorem C1, p. 851], there are just fourteen finite simple Solomon groups, including nine projective special linear groups $\mathrm{PSL}(n, q)$, the alternating group A_7, the Suzuki groups $\mathrm{Sz}(8)$ and $\mathrm{Sz}(32)$, and the Mathieu groups M_{11} and M_{22}. More precisely, a part of the result in [**40**, Theorem C1, p. 851] can be restated as follows.

Theorem A 4 (Pawałowski and Solomon [40]). *Let G be a finite simple Solomon group. Then the Smith set $\mathrm{Sm}(G) = 0$ and G satisfies the 8-condition. Moreover, the following two conclusions hold:*

(i) *if $r_G = 0$, then $G \cong \mathrm{PSL}(3, 4)$, $\mathrm{Sz}(8)$, $\mathrm{Sz}(32)$, or $\mathrm{PSL}(2, q)$ for $q = 5, 7, 8, 9$ or 17;*

(ii) *if $r_G = 1$, then $G \cong \mathrm{PSL}(3, 3)$, $\mathrm{PSL}(2, 11)$, $\mathrm{PSL}(2, 13)$, A_7, M_{11} or M_{22}.*

Also, among the groups listed in the conclusions (i) and (ii), only the following four groups contain elements of order 8: $\mathrm{PSL}(2, 17)$, $\mathrm{PSL}(3, 3)$, M_{11} and M_{22}.

By Lemma 2.3, the primary Smith set $\mathrm{PSm}(G) = 0$ for any finite group G with $r_G \leqslant 1$, but the Smith set $\mathrm{Sm}(G)$ may not be trivial even if $r_G = 0$. For example, $r_G = 0$ for

https://doi.org/10.1017/S0013091512000223 Published online by Cambridge University Press

$G = \mathbb{Z}_{2^a}$ with $a \geqslant 1$, and due to Cappell and Shaneson [**4–6**], $\mathrm{Sm}(\mathbb{Z}_{4q}) \neq 0$ for any integer $q \geqslant 2$.

Theorem A 4 opposes Dovermann's expectation that $\mathrm{Sm}(G) \neq 0$ for any finite Oliver group G (see [**52**, Comment (2), p. 547]). Moreover, it contradicts the conjecture posed by Dovermann and Suh [**14**, p. 44] asserting that once $\mathrm{Sm}(G) = 0$ for a finite group G, then also $\mathrm{Sm}(H) = 0$ for any subgroup H of G. In fact, $\mathrm{Sm}(G) = 0$ for $G = \mathrm{PSL}(2, 17)$, $\mathrm{PSL}(3, 3)$, M_{11} or M_{22}, and by [**4–6**], $\mathrm{Sm}(H) \neq 0$ for any cyclic subgroup H of G of order 8.

Example A 5. Similar phenomena occur for the two Solomon groups $\mathrm{GL}(2, 3)$ and $\mathrm{PGL}(2, 7)$ and their cyclic subgroups H of order 8: the Smith sets of $\mathrm{GL}(2, 3)$ and $\mathrm{PGL}(2, 7)$ are trivial by Propositions 2.7 and 2.8, respectively, and $\mathrm{Sm}(H) \neq 0$ by [**4–6**]. However, as the group H is of prime power order, $\mathrm{PSm}(H) = 0$ by Lemma 2.3.

Example A 6. For $H < G$, it may be that $\mathrm{Sm}(G) = 0$ and $\mathrm{PSm}(H) \neq 0$. In fact, due to the work of Morimoto [**29**], $\mathrm{Sm}(G) = 0$ for $G = \mathrm{Aut}(A_6)$. On the other hand, by Theorem 2.10, $\mathrm{PSm}(H) \neq 0$ for any subgroup H of G isomorphic to S_6 or $\mathrm{PGL}(2, 9)$.

By [**43**, Propositions 5.1–5.3], the following proposition holds.

Proposition A 7 (Pawałowski and Sumi [43]). *The three groups* $\mathbb{Z}_3 \times S_4$, $S_3 \times A_4$ *and* $(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ *of order* 72 *are solvable Oliver groups without elements of order* 8*. Moreover,*

(i) $r_G = 1$ *and* $\mathrm{Sm}(G) = 0$ *for* $G = (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$,

(ii) $r_G = 2$ *and* $\mathrm{Sm}(G) = 0$ *for* $G = S_3 \times A_4$, *and*

(iii) $r_G = 3$ *and* $\mathrm{Sm}(G) \cong \mathbb{Z}$ *for* $G = \mathbb{Z}_3 \times S_4$.

In particular, $G = (\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ is a solvable Oliver–Solomon group with $\mathrm{Sm}(G) = 0$, and $G = \mathbb{Z}_3 \times S_4$ is a Laitinen group with $\mathrm{PSm}(G) = \mathrm{Sm}(G) \cong \mathbb{Z}$ (see Proposition 1.6).

According to Theorem A 4, $\mathrm{Sm}(G) = 0$ for any finite simple Solomon group G. Therefore, one may pose the following problem.

Problem A 8. *Is* $\mathrm{Sm}(G) = 0$ *for any finite non-solvable Solomon group* G*?*

According to [**40**, Classification Theorem, p. 847, Conclusion (9)], there exists a finite solvable Oliver–Solomon group G with quotient $\mathbb{Z}_8$, and so, by [**4–6**], with $\mathrm{Sm}(G) \neq 0$.

Acknowledgements. We express our sincere thanks to the referee for careful reading of the previous drafts of the paper and giving us detailed remarks, comments and suggestions which led to many improvements in the presentation of the material.

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