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Generic Projections in the Semi-nice Dimensions

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Abstract. In his celebrated paper 'Generic Projections', Mather obtained the key result that a generic projection to an affine subspace of a smooth submanifold in Euclidean space is jet-transverse to any 'modular' submanifold of (multi-) jet space. He also gave an explicit stratification by modular submanifolds, and used it to conclude that the projection, if in the nice dimensions, is generically (C^∞-) stable. In this article, we extend the result to the semi-nice dimensions (where only C^0-stability is obtained), using the stratification given in our book. We first recall the definitions of the nice and semi-nice dimensions, review the main known results which involve them, and proceed to the statement of our main results. Next we discuss the condition of modularity, and present a number of methods for establishing modularity of particular strata. Finally, we show that all the strata needed for the main result are covered by these methods.

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Introduction

In a celebrated paper [11], Mather showed that if N^n is a (compact) smooth submanifold of a Euclidean space E, and π a generic linear projection of E onto a p-dimensional subspace Y^p such that (n, p) are nice dimensions, then $\pi \,|\, N\colon N \to P$ is a (C^∞-) stable map. The main objective of this paper is to obtain a corresponding result with 'nice' replaced by 'semi-nice' and 'C^∞-stable' by 'C^0-stable'. The key to this extension is the analysis in our book [6] of topological stability.

In the first section, we give the definitions of the nice and semi-nice dimensions and review the main known results which involve them. We then give the full statement of Mather's theorem on generic projections, and recall how it, and the recent extension by [2], follow from results about transversality and stratifications. We proceed to the statement of our main result, and reduce this to the problem of constructing enough modular submanifolds to cover a certain subset of jet space.

In Section 2 we define and discuss the condition of modularity, and present a number of methods for establishing modularity of particular strata; in particular, we establish modularity of all unimodal strata. We then show that enough boundary strata to give our main result are covered by these methods, treating the cases $n > p$ in Section 3, and the cases $n \leqslant p$ of finite maps in Section 4.

1. Statement of Results

1.1. NICE AND SEMI-NICE DIMENSIONS

We collect here a number of results which are somewhat scattered in the literature. The following key definition of [5] extends one of Mather [9, §7].

Define in turn $W^k_d(n,p)$ as the set of jets in $J^k(n,p)$ of $\mathcal{K}^k$-codimension at least d; ${}^m W^k_d(n,p)$ as the union of irreducible components of $W^k_d(n,p)$ of codimension at most $d-m$ in $J^k(n,p)$;

$${}^m W^k(n,p) := \bigcup_{d \geqslant 0} {}^m W^k_d(n,p);$$

${}^m\sigma^k(n,p)$ as the codimension in $J^k(n,p)$ of ${}^m W^k(n,p)$; and ${}^m\sigma(n,p) := \lim_{k\to\infty} {}^m\sigma^k(n,p)$.

This can be rephrased as follows. For given n,p and k, consider the classification of k-jets in $J^k(n,p)$ under the (algebraic) action of the contact group $\mathcal{K}^k(n,p)$. By general theory, there exists a finite partition of $J^k(n,p)$ into $\mathcal{K}^k(n,p)$-invariant submanifolds S_α on each of which the action has a smooth quotient space Q_α. For any m, write ${}^m B^k(n,p)$ for the union of those S_α with $\dim Q_\alpha \geqslant m$, and let ${}^m\sigma^k(n,p)$ be the codimension in $J^k(n,p)$ of ${}^m B^k(n,p)$. It is easy to see that ${}^m\sigma^k(n,p)$ decreases as k increases: write ${}^m\sigma(n,p)$ for its final constant value. Roughly speaking, ${}^m\sigma(n,p)$ is the codimension of the set of jets with $\mathcal{K}$-modality $\geqslant m$.

The value of ${}^1\sigma(n,p)$ was calculated by Mather [10]; the value of ${}^2\sigma(n,p)$ by Wall (see [12] for $n > p$ and [14] for $n \leqslant p$). The results may be tabulated as follows.

THEOREM 1.1. *Let ${}^1\tau$ and ${}^2\tau$ be given as follows:*

s	$\leqslant -4$	$\{-3,-2,-1\}$	0	1	2	$\{3,4,5,6,7\}$	$\geqslant 7$
${}^1\tau(s)$	$8-6s$	$9-6s$	9	9	8	$s+7$	$s+7$
${}^2\tau(s)$	$7-7s$	$16-4s$	13	11	13	$2s+4$	$s+11$

Then, in all cases,

$$n < {}^1\sigma(n,p) \Leftrightarrow n < {}^1\tau(n-p),$$

$$n \leqslant {}^2\sigma(n,p) \Leftrightarrow n \leqslant {}^2\tau(n-p), \qquad n < {}^2\sigma(n,p) \Leftrightarrow n < {}^2\tau(n-p).$$

Indeed, for $n \geqslant 4$, the functions

$${}^1\sigma(n,p) = {}^1\tau(n-p) \quad \text{and} \quad {}^2\sigma(n,p) = {}^2\tau(n-p)$$

depend only on $n-p$.

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The importance of these calculations is shown by the following results.

THEOREM 1.2 ([9]). *For N compact, C^∞-stable maps are dense in $C^\infty(N^n, P^p)$ if and only if $n < {}^1\sigma(n, p)$.*

THEOREM 1.3 ([6]). *If $n < {}^2\sigma(n, p)$ there is an explicitly given stratification $\mathcal{B}(n, p)$ of a subset of $J^k(n, p)$ (for large enough k) whose complement $X^k(n, p)$ is semi-algebraic and of codimension ${}^2\sigma(n, p)$, such that a map in $C^\infty(N^n, P^p)$ (N compact) is topologically stable if and only if its multijet extensions avoid $X^k(n, p)$ and are multi-transverse to $\mathcal{B}(n, p)$.*

THEOREM 1.4 ([4, 5]). *Map-germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are finitely C^∞-determined in general if and only if $n \leqslant {}^2\sigma(n, p)$.*

A few comments are in order. Mather's Theorem 1.2 is well known. Pairs (n, p) satisfying the condition $n < {}^1\sigma(n, p)$ are called *nice dimensions*. Geometrically the condition means that a generic map will have no germ whose jet requires any moduli for $\mathcal{K}$-equivalence.

The topological stability Theorem 1.3 is the main result of [6]. Although it may be possible to extend the range of dimensions in this result, numerous strata obstruct the extension of the calculations. Also, the above statement fails to mention one qualification, as follows.

Consider the stratum MFU_6 $(xz, xy + z^3, x^2 \pm y^2 + vz^3, yz^2)$. On [6, p. 599] we exhibit a deformation (when the + sign is chosen) which distinguishes the case $v = 0$ from the cases $v \neq 0$. However, this deformation does *not* fit the criteria which allow us to prove that the case $v = 0$—denoted $MFU_6(0)$—is ST-distinct from the rest, and hence that transversality to $MFU_6(0)$ is necessary for topological stability. (The codimension $7(p - n) + 12$ of MFU_6 exceeds ${}^2\sigma(n, p)$ if $p - n \geqslant 2$, so unless $p - n = 1$ a topologically stable map will avoid the whole of MFU_6 anyway). However, taking $MFU_6(0)$ as a separate stratum does give a stratification such that transversality to it is sufficient for topological stability. This suffices for the needs of this paper, so this case will not appear as exceptional below.

It seems natural to call the pairs with $n < {}^2\sigma(n, p)$ the *semi-nice* dimensions. They are those where a generic map will have no germ of $\mathcal{K}$-modality $\geqslant 2$.

For Theorem 1.4, the original statement was a little more complicated, and was clarified by the proof in [14] that the values of ${}^2\sigma$ calculated over $\mathbb{R}$ and $\mathbb{C}$ are the same. For finite determinacy it suffices to avoid germs of $\mathcal{K}$-modality $\geqslant 2$ in the complement of the origin, so there is a slight relaxation of the dimension condition. We will refer to pairs with $n \leqslant {}^2\sigma(n, p)$ as *weakly semi-nice*, to resolve the inconsistency in terminology between [3] and the older references. See Table 1.

Table I. The nice (N) and semi-nice (S) dimensions up to 40

40	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–
39	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	W	–	–	–	–	–
38	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–
37	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–
36	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–	–
35	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–
34	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–
33	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–
32	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–	–
31	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	W	–	–	–	–	–	–	–	–	–	–	–	–
30	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–	–	–	–	–	–	–
29	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–	–	–	–
28	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
27	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
26	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	W	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
25	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
24	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
23	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	S	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
22	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
21	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	W	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

20	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
19	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
18	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
17	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
16	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
15	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
14	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
13	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*W*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
12	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
11	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	–	*W*	–	–	–	–	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*	*W*
10	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	*W*	*S*	–	–	–	*W*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*
9	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	*S*	*S*	–	–	*W*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*
8	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	*S*	–	*W*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*
7	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	*W*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*	*S*
6	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*S*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
5	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
4	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
3	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
2	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
1	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*	*N*
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	33	32	33	34	35	36	37	38	39	40

The source dimension (n) is tabulated at the bottom and the target dimension (p) on the left. Nice dimensions are denoted N, semi-nice dimensions S and weakly semi-nice dimensions W.

1.2. GENERIC PROJECTIONS

The next result shows the relevance of nice dimensions to the topic of this paper: our main result will extend it to the semi-nice dimensions. Before stating the theorem, we introduce some notation which will be fixed for the rest of this section.

Let X^q and Y^p be real vector spaces with $q > p$; write $\mathcal{L}(X, Y)$ for the set of linear maps $X \to Y$. Let N^n be a compact smooth manifold, and $\phi : N \to X$ a smooth map.

THEOREM 1.5 ([11, Theorem 3]). *Suppose ϕ an embedding and (n, p) in the nice range. Then for almost all $\ell \in \mathcal{L}(X, Y)$, $\ell \circ \phi$ is stable.*

Theorem 1.5 is deduced from a more particular result referring to individual strata in jet space.

THEOREM 1.6 ([11, Theorem 1]). *Let ϕ be an embedding and W be a modular submanifold of some jet space ${}_rJ^k(N, Y)$. Then for almost all $\ell \in \mathcal{L}(X, Y)$, $\ell \circ \phi$ is transverse to W.*

Let F: $A \times N \to Y$ be a smooth map, W a smooth submanifold of Y. For $a \in A$, write F_a: $N \to Y$ for the map with $F_a(x) = F(a, x)$. For any smooth $f : N \to Y$ and $x \in N$, set $\delta(f, W, x) := \dim Y - \dim(TW_{fx} + df(TN_x))$ if $f(x) \in W$, and 0 if not. We will use 'almost all' in the sense of Lebesgue measure.

LEMMA 1.7 ([11, Lemma 2]). *Suppose that for all $(x, a) \in N \times A$ either $\delta(F_a, W, x) = 0$ or $\delta(F, W, (x, a)) < \delta(F_a, W, x)$. Then F_a is transverse to W for almost all $a \in A$.*

Mather uses this lemma, calculation of tangent spaces, and a bootstrap induction argument to prove Theorem 1.6. In [2, Theorem 3.4] a careful re-presentation of the proof of Lemma 1.7 is given, leading on to the following generalisations.

THEOREM 1.8 ([2, Theorem 2.2]). *Let ϕ be a C^∞-stable map and W be a modular submanifold of some jet space ${}_rJ^k(N, Y)$. Then for almost all $\ell \in \mathcal{L}(X, Y)$, $\ell \circ \phi$ is transverse to W.*

THEOREM 1.9 ([2, Theorem 2.3]). *Suppose ϕ a C^∞-stable map and (n, p) in the nice range. Then for almost all $\ell \in \mathcal{L}(X, Y)$, $\ell \circ \phi$ is stable.*

The proof of Theorem 1.5 or Theorem 1.9 involves four steps: the transversality Theorem 1.6 or 1.8, Mather's characterisation [9] of C^∞-stability by multi-transversality to strata in jet space, the existence of a finite stratification $\mathcal{A}(n, p)$ of jet space such that every stratum which is not a $\mathcal{K}$-orbit has codimension $\geqslant {}^1\sigma(n,p)$, and the proof that all the strata of $\mathcal{A}(n,p)$ are modular.

In [11], Mather sketches a proof that this modularity property holds in his case. More precisely he exhibits, for all nice dimensions (n,p) and large enough k, a collection of modular strata of $J^k(n,p)$, each of codimension $\geqslant {}^1\sigma(n,p)$, whose complement is a finite union of $J^k\mathcal{K}$-orbits of $\mathcal{K}$-sufficient jets.

Correspondingly, our main technical result is

THEOREM 1.10 (i) *If (n,p) are semi-nice dimensions, the strata of the canonical stratification $\mathcal{B}(n,p)$ are modular.*

(ii) *The complement $X^k(n,p)$ is covered by a finite union of modular submanifolds.*

THEOREM 1.11. *Suppose ϕ an embedding, or indeed any C^∞-stable map, and (n,p) in the semi-nice range. Then for almost all $\ell \in \mathcal{L}(X,Y)$, $\ell \circ \phi$ is topologically stable.*

Proof. The result follows at once follows from Theorems 1.3, 1.6 (or 1.8) and 1.10. □

There is no need to specify in the statement that ℓ must be surjective, though (unless p is large) only surjective ℓ can satisfy the conclusion. Since topological stability (like stability) is an open condition, the set of ℓ for which the conclusion holds is open, as well as dense.

It would be interesting to decide whether the hypothesis can be weakened to demand merely topological stability of ϕ, and whether the result can be extended outside the semi-nice range.

We defer the definition of modularity to Section 2.2. The proof of Theorem 1.10 will be given in Section 3 for the cases $n > p$ and in Section 4 for the cases $n \leqslant p$. It might be expected that invariance of W under $\mathcal{K}$-equivalence would suffice for Theorem 1.6. Mather observes that in the nice dimensions, it follows from Theorem 1.5 that it does. We now show that

THEOREM 1.12. *Let ϕ be C^∞-stable, and W be any $\mathcal{K}$-invariant submanifold of jet space ${}_rJ^k(N,Y)$. Suppose (n,p) are semi-nice dimensions. Then for almost all $\ell \in \mathcal{L}(X,Y)$, $\ell \circ \phi$ is transverse to W.*

Proof. By the previous results, we may assume that multitransversality already holds to all strata of $\mathcal{B}(n,p)$. In particular, the only strata of $\mathcal{B}(n,p)$ that are encountered are those of codimension $\leqslant n$. By the very definition of 'semi-nice', these strata are of modality at most 1.

Now the $\mathcal{K}$-invariant manifold W is itself stratified by its intersections with the strata of $\mathcal{B}(n,p)$, and any map transverse to these intersections is automatically transverse to W. But each such intersection is either itself a stratum of $\mathcal{B}(n,p)$, for which transversality already holds, or has codimension 1 in a stratum of modality 1, and hence is a finite union of $\mathcal{K}$-orbits. The proof is then completed by a further application of Theorem 1.8. □

In [11, Theorem 5] a negative result on genericity of projections is obtained under certain dimensional restrictions. However, it is made explicit in the course of the proof that what is really required is that finite C^∞-determinacy does not hold in general. Moreover, Mather's constructions give $\mathcal{K}$-invariant manifolds. We can thus restate his conclusion as

THEOREM 1.13 ([11]). *If the dimensions (n, p) are not weakly semi-nice, there exist a manifold N, a smooth embedding $\phi : N \to X$, a smooth $\mathcal{K}$-invariant immersed manifold W in $J^k(N, Y)$, and an open nonvoid set L_0 in $\mathcal{L}(X, Y)$ such that, for any $\ell \in L_0$, $\ell \circ \phi$ is not transversal to W.*

Mather gives precise details about the embedding ϕ, but makes clear that all that is needed is for ϕ to be 'sufficiently twisted'-kth order non-degenerate will suffice. This result shows that the conclusion of Theorem 1.12 fails outside the weakly semi-nice dimensions, so just leaves the boundary cases $n = {}^2\sigma(n, p)$ in doubt.

As in [11], the proofs carry over without essential change to the complex case.

2. Modularity

In this section we first introduce notation and some basic results for multi-jets, and then define modularity. We then obtain general criteria for establishing modularity, which will be applied later to the particular examples we need.

2.1. MULTI-JETS

Before giving the definition, we recall some facts about multi-jet paces. If N^n and P^p are smooth manifolds, we have the space $J^k(N, P)$ of k-jets from N to P, which fibres over $N \times P$. Denote by $N^{(R)}$ the configuration space of R-tuples of distinct points in N. Then [9, §1], the multi-jet space ${}_RJ^k(N, P)$ is defined to be the preimage of $N^{(R)}$ under the projection $(J^k(N, P))^R \to N^R$. It will be convenient here to consider R as a set, with cardinality r.

We also consider the projection to P^R. Let z have components $\{z_i \mid i \in R\}$, and denote by ρ the projection $J^k(N, P) \to P$. Denote by $\pi(z)$ the partition of R corresponding to the equivalence relation on R given by $i \sim j$: if $\rho(z_i) = \rho(z_j)$. For any partition π of R, set ${}_\pi J^k(N, P) := \{z \in_r J^k(N, P) \mid \pi(z) = \pi\}$. The main case of interest is the partition with a single part.

There is a natural action of $\mathcal{A} = \mathrm{Diff}(N) \times \mathrm{Diff}(P)$ on ${}_rJ^k(N, P)$; it preserves the subspaces ${}_\pi J^k(N, P)$.

Denote the fibre of ${}_rJ^k(X, Y) \to X^{(R)}$ over $\mathbf{x}$ by ${}_rJ^k(X, Y)_x$ and the fibre of ${}_rJ^k(X, Y) \to X^{(R)} \times Y^R$ over $(\mathbf{x}, \mathbf{y})$ by ${}_rJ^k(X, Y)_{x,y}$. Then there are natural identifications of tangent spaces at the multi-jet z of a map $f : N \to P$:

$$T({}_rJ^k(X, Y)_x)_z \cong J^k(f^*TY)_x \quad \text{and} \quad T({}_rJ^k(X, Y)_{x,y})_z \cong \mathfrak{m}_x J^k(f^*TY)_x.$$

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In each case the right hand side has a natural module structure over the ring J^kX_x of k-jets at $\mathbf{x}$ of smooth functions on X.

Two multi-jets z and z' are said to be *contact equivalent* [9, §4] if z_i and z_i' are contact equivalent for each $i \in R$, and $\pi(z) = \pi(z')$. Thus if W is a subset of ${}_\pi J^k(N, P)$ saturated for contact equivalence, then for each $i \in R$ the jets z_i with $z \in W$ form a subset $W_i \subset J^k(N, P)$ saturated for contact equivalence, and $W = \Pi_{i\in R} W_i \cap {}_\pi J^k(N, P)$. Further, W is a manifold if and only if each W_i is. Indeed, we have $T_zW_{x,y} = \oplus_{i\in R} T_{z_i}W_{i,x_i}$.

Note that according to [9, 4.5], the tangent space to the contact class containing the k-jet of the map f at the finite set S is

$$(tf(\mathfrak{m}_S\theta(N)_S) + \omega f(\theta(P)_{f(S)}) + (f^*\mathfrak{m}_{f(S)} + \mathfrak{m}_S^{k+1})\theta(f)_S)\big/\mathfrak{m}_S^{k+1}\theta(f)_S$$

If we restrict to the fibre of the contact class over $f(S) = y \in P$, this simplifies to

$$(tf(\mathfrak{m}_S\theta(N)_S) + (f^*\mathfrak{m}_y + \mathfrak{m}_S^{k+1})\theta(f)_S)/\mathfrak{m}_S^{k+1}\theta(f)_S$$

If further S is a single point x and we identify the source with $\mathbb{R}^n$ and the target with $\mathbb{R}^p$ using local coordinates at x and y respectively, it simplifies further.

We introduce the notation used for calculations. Write $\mathcal{E}_n$ for the ring of germs of smooth functions at the origin in $\mathbb{R}^n$; $\mathfrak{m}_n$ for the maximal ideal in it of functions vanishing at the origin. Denote by $\mathcal{E}_n^{\times p}$ the free module of rank p over $\mathcal{E}_n$: we can identify it with the module $\theta(f)$ of vector fields along the smooth map-germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, where the coordinate vectors represent the unit vectors ϵ_i parallel to the coordinate axes in the target $\mathbb{R}^p$. The basic vector fields in the source $\mathbb{R}^n$ will be denoted $\partial/\partial x_i$, or ∂_i for short. Then

$$T\mathcal{R}(f) = \sum_1^n \mathfrak{m}_n\partial_i f \quad \text{and} \quad T\mathcal{C}(f) = \sum_{i,j=1}^p \mathcal{E}_n f_i\epsilon_j;$$

$T\mathcal{K}(f)$ is their sum.

Now the tangent space to the contact class of the k-jet of the map f at 0 is

$$(tf(\mathfrak{m}_n^{\times n}) + (f^*\mathfrak{m}_p + \mathfrak{m}_n^{k+1}) \cdot \mathcal{E}_n^{\times p})/\mathfrak{m}_n^{k+1} \cdot \mathcal{E}_n^{\times p}.$$

2.2. DEFINITION OF MODULARITY

Mather [11] defined a smooth submanifold W of multi-jet space ${}_rJ^k(X, Y)$ to be *modular* if

(i) It is $\mathcal{A}$-invariant, and lies in ${}_\pi J^k(X, Y)$ for some partition π, and
(ii) For any $x \in X^{(r)}$ and smooth $f\colon X \to Y$ with $z = {}_rj^kf(x) \in W$, the subspace $E(f, x, W)$ of $J^k(f^*TY)_x$ corresponding to $(TW_{x,y})_z$ is a J^kX_x-submodule.

As to (i), the condition that $W \subset_\pi J^k(X, Y)$ for some partition π is automatic in the examples one wishes to take. The condition of $\mathcal{A}$-invariance is essential to the main result [11, Theorem 1], as Mather shows in [11, Theorem 4]. In fact the submanifolds

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W arising in the criterion of [6] for topological stability are invariant under contact equivalence, and this will be important for us.

The main point of the condition, however, and the reason for the name, is condition (ii). As Mather observes [11, p. 234], it follows from the formula for the tangent space to a $\mathcal{K}$-orbit ([7, 7.4], or [6, p. 23]) that any $\mathcal{K}$-orbit is modular. Mather also showed that Thom–Boardman strata are modular, but apart from the first order cases Σ^r, this is not of use to us.

Now suppose also that W is invariant under contact equivalence. Then it follows from the above calculation of tangent spaces that W is determined by submanifolds W_i of monojet space, and W is modular if and only if each W_i is. Thus in the study of modularity it suffices to work with monojets.

We re-state condition (ii) for modularity in this case. The tangent space to $J^k(X, Y)$ at a point z which is the k-jet of f at (x, y) can be identified with $\theta(f)/\mathfrak{m}_x^{k+1} \cdot \theta(f)$ or, in terms of local coordinates, $\mathcal{E}_n^{\times p}/\mathfrak{m}_n^{\times p}$; the tangent space to W is a subspace of this. Modularity requires that this subspace is a module over $J^k X_x$, i.e. $\mathcal{E}_x/\mathfrak{m}_x^{k+1}$. Equivalently, we can lift the tangent space of W by forgetting the denominator, and requiring that the resulting subspace of $\mathcal{E}_n^{\times p}$ is a module over $\mathcal{E}_n$.

2.3. GROUP ACTIONS

To verify modularity for a particular example we need to be able to calculate tangent spaces to submanifolds of jet space – and also to prove that our example defines a submanifold – using data convenient for defining an example, which may take a form such as: the $\mathcal{K}$-classes of the germs $f_t = f_0 + t\alpha : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. If the $\mathcal{K}$-orbits of the k-jets of the f_t all have the same codimension (if $n > p$ this is equivalent to constancy of the Tjurina numbers $\tau(f_t)$) we may use the following easy lemma, somewhat in the style of [8, 3.1].

LEMMA 2.1. *Suppose given a smooth action of a Lie group G on a manifold M and a smooth submanifold $V \subset M$ such that*

(a) $\dim G.x$ *is constant for* $x \in V$, *and*
(b) *for all* $x \in V$, $T_x V \cap T_x(G.x) = 0$. *Then*

  (i) *$G.V$ is a smooth submanifold, and*
  (ii) *we have* $T_x(G.V) = T_x V + T_x(G.x)$.

Proof. It suffices to work locally at a point $x \in V$. From the group action we have an exact sequence $0 \to T_1(G_x) \to T_1 G \to T_x(G.x) \to 0$, where G_x denotes the isotropy group of x. Pick a germ H of submanifold at $1 \in G$ with $T_1 H$ a complementary subspace to $T_1 G_x$ in $T_1 G$. By (a), the dimension of $T_1(G_y)$ is constant for y near x, so this subspace varies continuously, so $T_1 H$ is also complementary to $T_1 G_y$ for y near x.

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Then T_1H maps isomorphically to $T_1(G.y)$, so H is locally diffeomorphic to $G.y$. Thus $H \times V$ locally maps onto $G.V$.

By (b) the differential of the map $H \times V \to M$ given by restricting the G-action is injective at $(1, y)$ for y near x. Thus the map from $H \times V$ to M is locally an immersion, with image $G.V$. Both conclusions now follow. $\square$

However, in many of the examples we wish to include, the hypothesis of constant dimension is not satisfied. A typical example is with $M = J^k(n, 1)$, V the (linear) space of jets of functions of weight $\geqslant d$ (with respect to some assignment of (positive) weights w_i to the variables x_i), and G the group of right equivalences. Here V satisfies (ii) of the definition of modularity, but is not G-invariant, and $G.V$ is usually not a manifold. The following covers what we need.

LEMMA 2.2. *Suppose given a smooth action of a Lie group G on a manifold M, a submanifold $N \subset M$ transverse at each point to the G-orbit through that point, and a smooth submanifold $V \subset N$ such that $G.V \cap N = V$. Then*

(i) *$G.V$ is a smooth submanifold, and*
(ii) *for each $x \in V$, $T_x(G.V) = T_xV + T_x(G.x)$.*

Proof. At each point $x \in V$, the composite $T_1G \to T_xG.x \subset T_xM \to T_xM/T_xN$ is surjective. Thus the dimension of the kernel K_x is independent of x, and K_x depends smoothly on x. Pick a germ H of submanifold at $1 \in G$ with T_1H a complementary subspace to K_x in T_1G: then for all y near x in V, T_1H is complementary to K_y in T_1G. Hence the map $H \times N \to M$ induced by the group action is a local diffeomorphism at $(1, y)$ for all such y.

Now the image of $H \times V$ is a submanifold near x, and is clearly contained in $G.V$. The result will thus follow if we can show that the opposite inclusion holds locally. But K_x is the tangent space to the local stabiliser of N, and hence, by hypothesis, also stabilises V, so locally $G.V$ is the image of $G \times V$, hence of $H \times K_x \times V$, hence of $H \times V$. $\square$

2.4. LINEARLY ADAPTED COORDINATES

We next describe reduction using linearly adapted coordinates. The coordinates $(\mathbf{x}, \mathbf{y})$ are said to be *linearly adapted* for the map-germ f if we have

$$y_i \circ f(\mathbf{x}) = x_i \quad (1 \leqslant i \leqslant r), \qquad j^1(y_i \circ f)\ (\mathbf{x}) = 0 \quad (r < i \leqslant n).$$

The concept is introduced by Mather early in [9] and discussed more fully in [9, §§9, 10]. We recall some of his results, in simplified notation. In particular, we work in J^k throughout, and suppress the k from the notation.

Define $Z_r(n, p)$ to be the subset of $J^k(n, p)$ of jets of rank r, and write $\Lambda_r(n, p)$ for the set of those which are linearly adapted. Also write $Z_r^0(n, p)$ for the intermediate set consisting of the jets in $Z_r(n,p)$ of germs f such that $f^*y_1,\dots,f^*y_r,x_{r+1},\dots,x_n$ form local coordinates in the source. Clearly $Z_r^0(n,p)$ is a Zariski-open subset of $Z_r(n,p)$ meeting all $\mathcal{K}$- (even all $\mathcal{L}$-) orbits. We have, in particular, $Z_0(n,p)=Z_0^0(n,p)=\Lambda_0(n,p)$.

For any $z\in Z_r^0(n,p)$ define coefficients $\alpha_{i,j}\in\mathbb{R}$ for $1\leqslant i\leqslant r$, $r<j\leqslant p$ by

$$\mathrm{d}(y_j\circ f)(0)=\sum_1^r\alpha_{i,j}\mathrm{d}(y_i\circ f)(0);$$

then define local diffeomorphisms h of $\mathbb{R}^n$, h' of $\mathbb{R}^p$ by

$$x_i\circ h=\begin{cases}y_i\circ f & (1\leqslant i\leqslant r),\\ x_i & (i>r),\end{cases}\qquad y_i\circ h'=\begin{cases}y_i & (1\leqslant i\leqslant r),\\ y_i-\sum_1^r\alpha_{j,i}y_j & (i>r).\end{cases}$$

Then $h'\circ f\circ h^{-1}$ is linearly adapted, and we have constructed a local section β of the group action $\mathcal{A}^k\times\Lambda_r(n,p)\xrightarrow{\alpha}Z_r(n,p)$; indeed, we have a subgroup $\mathcal{A}_0$ such that $\mathcal{A}_0\times\Lambda_r(n,p)\cong Z_r^0(n,p)$.

Write $I\colon\mathbb{R}^{n-r}\to\mathbb{R}^n$ for the inclusion, $J\colon\mathbb{R}^p\to\mathbb{R}^{p-r}$ for the projection, and define $\lambda(f):=J\circ f\circ I$. Thus $\lambda:\Lambda_r(n,p)\to Z_0(n-r,p-r)$. The $\mathcal{K}$-orbits X in $Z_r(n,p)$ correspond bijectively to $\mathcal{K}$-orbits X' in $Z_0(n-r,p-r)$: we will say, as in [6], that such orbits are $\mathcal{E}\mathcal{K}$-equivalent. Indeed, we have $X'=\lambda(X\cap\Lambda_r(n,p))$ and $X=\mathcal{K}\lambda^{-1}(X')$.

The construction also shows that $\mathcal{K}$-invariant submanifolds W of Z_r correspond bijectively, by the same formulae, to $\mathcal{K}$-invariant submanifolds W' of Z_0, and allows comparison of tangent spaces.

The next result will allow us to ignore superfluous coordinates in the calculations to follow.

PROPOSITION 2.3. *The $\mathcal{K}$-invariant submanifold W of $Z_r(n,p)$ is modular if and only if the corresponding submanifold W' of $Z_0(n-r,p-r)$ is.*

Proof. For this proof, we write $\mathbf{x}=(x_1,\dots,x_{n-r})$ for coordinates in $\mathbb{R}^{n-r}$ and similarly use $\mathbf{u}$ for coordinates in $\mathbb{R}^r$ and $\mathbf{y}$ for coordinates in $\mathbb{R}^{p-r}$, so that elements of $\Lambda_r(n,p)$ have the form $(\mathbf{x},\mathbf{u})\mapsto(\mathbf{g}(\mathbf{x},\mathbf{u}),\mathbf{u})$, and $I(\mathbf{x})=(\mathbf{x},\mathbf{u})$, $J(\mathbf{y},\mathbf{u})=\mathbf{y}$. The map induced by λ on tangent spaces to jet spaces is given (modulo $\mathfrak{m}^{k+1}$) by

$$T\lambda(\phi_1(\mathbf{x},\mathbf{u}),\dots,\phi_p(\mathbf{x},\mathbf{u}))=(\phi_1(\mathbf{x},\mathbf{0}),\dots,\phi_{p-r}(\mathbf{x},\mathbf{0})).$$

The tangent space to $\Lambda_r(n,p)$ is given (modulo $\mathfrak{m}^{k+1}$) by $\mathcal{E}_n^{p-r}\oplus 0$. It is now immediate that if z is a point of $W\cap\Lambda_r(n,p)$ such that T_zW is an $\mathcal{E}_{x,u}$-module, so is the tangent space $T_zW\cap T\Lambda_r(n,p)$ at z to $W\cap\Lambda_r(n,p)$. Now applying λ means ignoring the last r components and setting $\mathbf{u}=\mathbf{0}$ in the rest. This gives an $\mathcal{E}_x$-module. But this is the tangent space at $\lambda(w)$ to $W'=\lambda(W\cap\Lambda_r(n,p))$. Thus W' is modular if W is.

Now suppose W' modular. Let $z\in W\cap\Lambda_r(n,p)$ let $(\phi_1,\dots,\phi_p)\in T_wW$, and let $\psi\in\mathcal{E}_{x,u}$. We wish to show that $\psi.(\phi_1,\dots,\phi_p)\in T_wW$. Applying λ, we see it is

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sufficient to show that $\psi(\mathbf{x}, \mathbf{0}).(\phi_1(\mathbf{x}, \mathbf{0}), \dots, \phi_{p-r}(\mathbf{x}, \mathbf{0})) \in T_{\lambda(w)}W'$. But this follows since W' is modular and $(\phi_1(\mathbf{x}, \mathbf{0}), \dots, \phi_{p-r}(\mathbf{x}, \mathbf{0})) \in T_{\lambda(w)}W'$. Since W is the $\mathcal{K}$-saturation of $W \cap \Lambda_r(n, p)$, it follows that W is modular. □

2.5. THE SPLITTING THEOREM

The enumeration of strata is most conveniently accomplished using a number of techniques for reducing a problem to one previously resolved. In this subsection we establish modularity for strata obtained by Thom's splitting theorem. Let us recall it briefly.

The splitting theorem applies to (finitely determined) function-germs, and states that any such germ f is (right) equivalent to one of the form $g(x_1, \dots, x_r) + \sum_{r+1}^{n} \pm x_i^2$, where g has zero 2-jet. Moreover, the equivalence class of f determines that of g. The integer r is called the *corank* of f. We also say that f is an $(n-r)$-fold suspension of g.

LEMMA 2.4. *Let W be a modular submanifold of function-germs with zero 2-jets. Then the $\mathcal{K}$-saturation W_s of the set of function-germs obtained by s-fold suspension is also a modular submanifold.*

Proof. Since suspension (with fixed signs $\pm$) is an embedding of the space of maps, the s-fold suspension of W is a manifold whose tangent space is the image of that of W under the natural embedding $\mathcal{E}_r \to \mathcal{E}_{r+s}$.

Let $f = g(x_1, \dots, x_r) + \sum_{r+1}^{r+s} \pm x_i^2$ be an s-fold suspension of g. The tangent space $T\mathcal{K}f$ at f to the $\mathcal{K}$-orbit of f is the ideal in $\mathcal{E}_{r+s}$ generated by f and its first order partial derivatives; in particular, it includes the ideal I_s generated by $x_{r+1}, \dots, x_{r+s}$. Now we can identify $\mathcal{E}_{r+s}/I_s$ with $\mathcal{E}_r$, and the image of $T\mathcal{K}f$ in $\mathcal{E}_r$ is thus just $T\mathcal{K}g$, which is contained in $T_g W$.

Thus $T\mathcal{K}f + T_f W$ has the same codimension as W, thus is independent of the choice of $g \in W$. Hence we have a manifold, whose tangent space is the preimage of the ideal $T_g W$ under the projection $\mathcal{E}_{r+s} \to \mathcal{E}_{r+s}/I_s \cong \mathcal{E}_r$, and hence is an ideal in $\mathcal{E}_{r+s}$. □

By adapting the argument slightly, we see that the converse also holds.

2.6. LOW DEGREES

Consider the question of modularity of a $\mathcal{K}$-invariant submanifold W of J^k. If $k = 1$, W is necessarily a finite union of classes Σ^r, each of which is a $J^1\mathcal{K}$-orbit, so modularity is automatic. This proves (i) of

LEMMA 2.5. (i) *Any invariant submanifold of J^1 is modular.*

(ii) *If W is a modular submanifold of J^k and $K > k$, then the preimage W' of W under the natural projection $J^K \to J^k$ is modular.*

(iii) *An invariant submanifold of* J^k *which is the* $\mathcal{K}$*-saturation of a family of maps with the same* $(k-1)$*-jet is modular.*

(iv) *An invariant submanifold of* J^2 *contained in some* Σ^r *is modular.*

Proof. The tangent space to W' is the preimage of that of W under the projection $(\mathcal{E}_n/\mathfrak{m}^{K+1})^{\times p} \to (\mathcal{E}_n/\mathfrak{m}^{k+1})^{\times p}$, so if the latter is a $\mathcal{E}_n/\mathfrak{m}^{k+1}$-module, the former must be a $\mathcal{E}_n/\mathfrak{m}^{K+1}$-module.

This proves (ii): now (iii) is a special case of (ii) since a $J^k\mathcal{K}$-orbit is modular. As to (iv), let $k = 2$ and suppose that $W \subseteq \Sigma^r$ for some r. Then W is the saturation of a partial unfolding of some f by terms of degree exactly 2, and if X is the $\mathcal{K}$-orbit of f, $T_fW = T_fX + A$ for some $A \subset \mathfrak{m}_n^2.\theta(f)$. Now modularity follows since T_fX is a $\mathcal{E}_x$-module, X being an orbit, and $\mathfrak{m}$ annihilates A modulo $\mathfrak{m}_n^3.\theta(f)$. □

We will make frequent use of this, and recall the notation introduced by Mather [10, p. 237]. Suppose $x_1, \dots, x_r$ are indeterminates, $f_1, \dots, f_q$ a set of power series (in practice, always polynomials) in $x_1, \dots, x_r$. Define

$$Q(x;f) = Q(x_1, \dots, x_r; f_1, \dots, f_q) := \mathbb{R}[[x_1, \dots, x_r]]/\langle f_1, \dots, f_q\rangle$$

and, if k is a positive integer or ∞,

$$Q_k(x;f) := \mathbb{R}[[x_1, \dots . x_r]]/(\langle f_1, \dots, f_q\rangle + \mathfrak{m}_r^{k+1}).$$

If z is the k-jet of the map f with components the f_i, $Q(z)$ is defined to be $Q_k(x;f)$. Now define

$$V_k(x_1, \dots, x_r; f_1, \dots, f_q)_{n,p} := \{z \in J^k \mid Q(z) \cong Q_k(x_1, \dots, x_r; f_1, \dots, f_q)\}.$$

In practice, we abbreviate this notation: n and p will be omitted if clear from the context, and the list of variables will be omitted if it coincides with the list of all variables occurring in any of $f_1, \dots, f_q$. We thus write simply $V_k(f_1, \dots, f_q)$ for the $\mathcal{K}^k$-orbit just defined, or for its preimage in J^K with $K > k$. By the lemma, these varieties are all modular.

2.7. CANONICAL STRATA

For the proof of Theorem 1.5, Mather had to establish modularity of the 'boundary strata' *ad hoc*, but the rest are modular in virtue of being single $\mathcal{K}$-orbits. In this section we deal with the canonical strata for the present problem. This argument was indeed the genesis of the whole paper.

To explain the idea, consider a 1-parameter family $f_t := f_0(\mathbf{x}) + t\phi(\mathbf{x})$, with the understanding that we seek the union W of the $\mathcal{K}$- (or rather $\mathcal{EK}$-) orbits of members of the family. As noted above, the value $\tau(f_t)$ may be independent of t or may jump at certain values of t, which may or may not be included in the stratum.

LEMMA 2.6. *Suppose* ϕ *does not belong to the tangent space* M *to the* $\mathcal{K}$*-orbit of* f_0 *(where* $M = tf_0(\mathfrak{m}_n\theta_n) + f_0^*\mathfrak{m}_p\theta_{f_0}$*). Then the stratum is modular if and only if* $\mathfrak{m}_x\phi \subseteq M$.

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Proof. It follows from Lemma 2.2 that the tangent space to W at f_0 is $M + \mathbb{R}\phi$. Then W is modular if and only if this is a $\mathcal{E}_x$-module. We already know that M is a $\mathcal{E}_x$-module, so the condition is that $\mathcal{E}_x\phi \subseteq M + \mathbb{R}\phi$, or equivalently, that $\mathfrak{m}_x\phi \subseteq M + \mathbb{R}\phi$. Suppose for some $g \in \mathfrak{m}_x$ that $g\phi = m + c\phi$ with $m \in M$ and $c \neq 0$. Then $c - g$ is a unit in $\mathcal{E}_x$, so $\phi = -(c-g)^{-1}m$ belongs to the $\mathcal{E}_x$-module M, contrary to hypothesis. Thus $\mathfrak{m}_x\phi \subseteq M$. Conversely, this condition certainly implies modularity. □

EXAMPLE Take $f_0(x, y) = x^3 + y^9$. Then we can take M to have basis consisting of all monomials except $\{y^i, xy^i \mid 0 \leqslant i \leqslant 7\}$, x^2 and y^8. If we set $\phi = xy^7$, the condition is satisfied, and we obtain a modular stratum $E_0^3(0)$. If, however, $\phi = xy^6$, then ϕ and $y\phi$ are independent modulo M, and the union $E_0^3(wh)$ of $\mathcal{K}$-orbits of weighted homogeneous germs in E_0^3 is not modular.

We now consider the 'canonical strata' of [6] in general. A crucial requirement for a stratum to be admitted to the list was that it be civilised. We recall that in all cases where a stratum (not consisting of just a single $\mathcal{K}$-orbit) was shown to be civilised, the proof was obtained by reduction to [6, Theorem 9.6.6]. The statement of this result is complicated, and we do not need to recall all of it. However, the hypotheses did include the following:

Let F: $M \times U \to N \times U$ be homogeneous, U-level-preserving, with weights which are positive on M and N and nonpositive on U, and $U = V \oplus V'$ with all weights on V strictly greater than those on V'. Let X be an open subset of V' such that

(i) *the germ of F at each point of $\mathbf{0} \times X$ is C^∞-stable*;
(ii) *there is a neighbourhood W of $\mathbf{0} \times X$ in $M \times U$ such that, for any $\mathbf{x} \in X$, the local algebra Q_x of F_x at 0 is presented in W only at points of $\mathbf{0} \times X$.*

The stratum T is defined to be the $\mathcal{K}$-saturation of the set of k-jets of the sections F_u (where $F(m, u) = (F_u(m), u)$) at points of X.

PROPOSITION 2.7. *In the above situation, the stratum T is smooth and modular. Thus* (*i*) *of Theorem* 1.10 *holds.*

Proof. We may suppose without loss of generality that F is a miniversal unfolding of the germ of F_u at a point $x = (m, u)$ of X. Since F is level preserving, we may define a restricted jet map J_1^kF which takes each point $(m, u) \in M \times U$ to the k-jet at m of the map F_u. Since F is stable, the image by TJ_1^kF of the tangent space to U at x is transverse at x to the $J^k\mathcal{K}$-orbit of $J_1^kF_x$; since F is ministable, the tangent space to U maps isomorphically to a complement of the tangent to the orbit.

We now apply Lemma 2.2, taking the group action to be that of $J^k\mathcal{K}$ on $J^k(M, N)$, and the smooth submanifold to be the (germ of the) image of U. We have just established the transversality hypothesis of the lemma, and the other hypothesis is given by (ii) above. Hence the lemma applies, and tells us that T is indeed smooth, and its tangent space is the sum of the image V'' of the tangent space V' of X with the tangent space KO to the $\mathcal{K}$-orbit.

The quotient of the tangent space to $J^k(M, N)$ by the tangent space to the orbit of $J^k\mathcal{K}$ is the image of U, and acquires a natural $\mathbb{C}^\times$-action. Since weights are positive on M, multiplying by any element of $\mathfrak{m}_m$ will strictly increase the weight. By hypothesis, all weights in V are strictly greater than those in V', so the weights of the unfolding monomials in V' are strictly greater than those in V. But the tangent space to T is the span of these monomials, so is invariant under multiplication by $\mathfrak{m}_m$. The result now follows by the argument of Lemma 2.6. □

The hypothesis of the proposition is much weaker than that of [6, Theorem 9.6.6], and is easy to verify for a wide range of strata.

3. The Case $n > p$

We now begin the construction of the stratifications required for (ii) of Theorem 1.10. Recall that we need to show, for (n, p) semi-nice dimensions and k large enough, that $X^k(n, p)$ is covered by a finite union of modular submanifolds. The explicit definition of $X^k(n, p)$ when $n > p$ is given in [6, Theorem 12.3.1], though this refers to the closures of strata and makes no reference to the integer k; thus some extra care is necessary below.

The required classifications of strata were given in [6, Chapter 7] amplifying (and in some cases correcting) the treatment in [12]. Here we retain the same pattern. We will construct the manifolds explicitly in turn, so write $\mathcal{L}_k = \mathcal{L}_k(n, p)$ for the list of them. The discussion of each subcase will then conclude by assigning certain explicit modular submanifolds to $\mathcal{L}_k$ (we fix a sufficiently large k and keep it throughout).

Using linear reduction, and Proposition 2.3, it suffices to consider germs with zero 1-jet. If $p \geqslant 3$ or if $p = 2, n \geqslant 5$ there are no unimodal germs: in these cases it suffices to add the modular stratum $V_1(0)_{n,p}$ to $\mathcal{L}_k$.

In the hypersurface case $p = 1$ we next reduce using Thom's splitting theorem and Lemma 2.4 to the case when the 2-jet is 0. If also $n \geqslant 4$ there are no unimodals, and we add $V_2(0)_{n,1}$ to $\mathcal{L}_k$.

We consider in turn the remaining cases, when (n, p) takes the respective values (2,1), (3,1), (4,2) and (3,2).

3.1. PLANE CURVE SINGULARITIES

First suppose $n = 2$ and $p = 1$. We have reduced to the case of zero 2-jet, so first consider the 3-jet. This is equivalent to one of $x^3 \pm xy^2$, x^2y, x^3 and 0. The 3-jets $x^3 \pm xy^2$ are sufficient, each giving a single $\mathcal{K}$-orbit.

A function with 3-jet x^2y is either $\mathcal{K}$-equivalent to $x^2y \pm y^r$ for some r or has ∞-jet equivalent to x^2y. Each $\mathcal{K}$-orbit is modular, and $V_k(x^2y)$ has codimension tending to ∞ with k and, hence, $\geqslant {}^2\sigma(n, p)$ for k large enough. We thus add $V_k(x^2y)$ to the list $\mathcal{L}_k$.

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We will use a similar argument in numerous cases where we have an infinite collection of modular canonical strata which exhaust the space of ∞-jets with a given 2-jet up to a subset of infinite codimension.

Similarly if the 3-jet is x^3, only one modulus appears, so it suffices to add a suitable $V_k(x^3)$ to $\mathcal{L}_k$. Of these strata, the *E-series*, all are single $\mathcal{K}$-orbits save those labelled E_0^p (which includes, e.g., x^3+y^{3p}). Although Proposition 2.7 applies to all of these, only for $p \leqslant 3$ were they explicitly covered by the calculations of [6, Chapter 11]. We must thus assign the (finitely many) E-strata in $X^k(n,p)$ meeting the complement of $V_k(x^3)$ to $\mathcal{L}_k$. (There is also a more efficient argument, assigning weights $(4,1)$ to (x,y) and dealing with all functions of weight $\geqslant 12$ and nonzero coefficient of x^3, similar to Proposition 3.1.)

If the 3-jet is 0 and the 4-jet not a fourth power, we have at most one modulus, and add $V_k(x^4 \pm x^2y^2)$, $V_k(x^2 \pm y^2)^2$ and $V_k(x^3y)$ to $\mathcal{L}_k$. Here again the $X^k(n,p)$ of [6] contains further strata in the Z-series, and we argue as with the E-series above. We put the whole modular stratum $V_4(0)$ in $\mathcal{L}_k$, so may from now suppose the 4-jet non-zero, and equal to y^4.

This gives Arnold's W-series. The nonunimodal strata are all contained in the set S_W of germs such that all terms have degree at least 8 with respect to the assignment of weights 1, 2 to the variables x, y. This case will be typical of numerous ones to follow, and we discuss it in some detail as a model.

We first show how to reduce any germ with 4-jet $f_0 := y^4$ to a 'pre-normal' form (using a 'complete transversal' in the terminology of [1]). The Jacobian ideal of f_0 itself is generated by y^3. This ideal contains all monomials of a given degree k except $x^k, x^{k-1}y$ and $x^{k-2}y^2$. Thus if $f \in S_W$ is formed from f_0 by adding higher terms, each monomial other than those listed is the sum of an element of Jf and higher degree terms. Now if b is homogeneous of degree d the substitution $x' = x, y' = y + b$ will replace f by $f - b\partial f/\partial y$ added to terms of degree at least $d+3$. We may thus proceed inductively to remove terms other than $x^k, x^{k-1}y$ and $x^{k-2}y^2$ from f. We do not need to discuss the convergence of this procedure, since we are effectively working in jet space, so will simply *define* the pre-normal form of such functions f to be $y^4 + y^2a(x) + yb(x) + c(x)$. Now write S_W^0 for the subset of S_W of germs with 4-jet y^4, and P_W for the subset of those in pre-normal form.

PROPOSITION 3.1. (i) *For any $f \in S_W^0$ and any k, j^kf is equivalent to the k-jet of an element of P_W.*

(ii) *For any $f \in P_W$ we have*

$$T\mathcal{K}(f) + TP_W = S_W + \mathbb{C}\{xf_y\}. \tag{1}$$

(iii) *The saturation $\mathcal{K}.P_W$ of P_W is a smooth submanifold of jet space containing S_W^0.*

(iv) *$\mathcal{K}.P_W$ is modular.*

Proof. (i) This was proved in the discussion preceding the proposition.

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(ii) First we show that the left hand side of (1) is contained in the right. Clearly $TP_W \subset S_W$, and the only basis element of $T\mathcal{K}(f)$ of degree 8 is $x\partial f/\partial y$, which we write as xf_y for short.

Now we must show that the right hand side of (1) is contained in the left. This is clear for the second term. As to the first, the argument establishing the pre-normal form showed that any homogeneous element of S_W is congruent modulo $T\mathcal{R}f$ to the sum of an element of higher degree and a linear combination of monomials of the form $x^k, x^{k-1}y$ and $x^{k-2}y^2$ which belong, by definition, to TP_W.

(iii) In view of our calculation of tangent spaces, it will suffice to show that the right hand side of (1) determines a subspace of constant dimension in jet space as f varies. But xf_y contains the monomial xy^3 of weight 7 which does not appear elsewhere.

(iv) This follows since the right hand side of (1) is visibly a $\mathcal{E}_{x,y}$-module. $\square$

We now add $\mathcal{K}.P_W$ to $\mathcal{L}_k$.

The essential features of this modularity argument are as follows. We define weights W_j for the variables x_j and let S_g be the subspace of k-jets (for k large) with f_i of weight at least d_i for each i. We will say 'the weights are $(W_1, \ldots, W_n/d_1, \ldots, d_p)$'. If D is the least order of any term in S_g (in our later examples, D is 3 or 2), the set of D-jets of elements of S_g contains an open orbit. Write S_g^0 for the subset of S_g with D-jet in this orbit, and f_0 for a polynomial of degree D representing the orbit.

Let G be the group of $\mathcal{K}$-equivalences preserving the weight, G_+ the subgroup acting trivially on each term of the associated filtered group. Choose a linear subspace P_g of S_g such that $LG_+.f_0 + P_g = S_g$ and P_g is a direct sum of homogeneous subspaces, and all elements of P_g have zero D-jet.

Then an easy induction on weight shows that $G_+.(f_0 + P_g)$ contains all elements of S_g with the same D-jet as f_0, so $G.(f_0 + P_g) = S_g^0$: our 'pre-normal form' is $f_0 + p$ with $p \in P_g$. An induction on weight with a subsidiary induction on degree shows that for any $f \in (f_0 + P_g)$, $LG_+.f + P_g = LG_+.f_0 + P_g = S_g$.

Now LG has finite codimension in the tangent space to $\mathcal{K}$, and we can choose as basis of the complement the 'tangent vectors of negative weight' viz. the $m_i\partial/\partial x_j$ with m_i a non-constant monomial in the x_j of weight $< W_j$ and the $m_i f_j \epsilon_k$ with $\deg m_i > 0$, $wt(m_i) + d_j < d_k$. We apply each of these to any $f = f_0 + p$ with $p \in P_g$, and evaluate them modulo S_g. We find, in each case in this section, that the resulting vectors are linearly independent—again, it suffices to calculate the terms arising from f_0 as these are just those of least degree, and they are already independent. We conclude, as above, that $\mathcal{K}.S_g^0$ is a modular submanifold.

3.2. SURFACES IN $\mathbb{R}^3$

If $n = 3$ we must distinguish cases according to the 3-jet, which corresponds to a cubic curve in P^2. For curves with (at worst) nodes, no moduli appear, and it suffices

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to cut off at a high enough jet: add $V_k(x^r + y^s + xyz)$ for all $3 \leqslant r \leqslant s \leqslant k$ to $\mathcal{L}_k$. For a cuspidal cubic, all strata are 1-modal; add $V_k(x^3 + yz^2)$, but again there are further strata in $X^k(n, p)$ and we argue as for the E-series.

For a conic and tangent (S-series) the non-unimodal strata are all contained in the set S_S of germs with weights $(1, 2, 3/7)$. This case is very similar to the W series. We can reduce the 3-jet to $f_0 := xz^2 - y^2z$. The Jacobian ideal of f_0 is generated by z^2, yz and $2xz - y^2$, and so contains all monomials of degree k except $x^k, x^{k-1}y, x^{k-1}z$ and $x^{k-2}y^2$—and the latter two may be reduced to each other modulo this ideal. Now define P_S to consist of the germs $y^2a(x) + yb(x) + c(x)$ in S_S.

All the above conditions hold here, and the tangent vectors of negative weight produce the subspace $\mathbb{C}\{xf_y, x^2f_y, xf_z, yf_z\}$ (modulo S_S); these four terms include the respective monomials xz^2, xz^3, y^3, xyz, and in each case the monomial listed has weight $\leqslant 6$ and does not occur in any subsequent one of the 4 listed elements. So the space has constant dimension 4 and $\mathcal{K}.S_S^0$ is a modular submanifold. We thus add $\mathcal{K}.P_S$ to $\mathcal{L}_k$.

The case of 3 concurrent lines (U-series) is very similar. The non-unimodal strata are all contained in the set S_U of germs with weights $(1, 2, 2/6)$. We can take $f_0 = y^2z \pm z^3$ and P_U as the set of functions in S_U of the form $y^2a(x) + yb(x) + zc(x) + d(x)$. Again all the conditions hold; the tangent vectors of negative weight give $\mathbb{C}\{xf_y, xf_z\}$, and we see by considering the coefficients of xy^2 and xyz that these span a 2 dimensional space modulo S_U. Thus $\mathcal{K}.S_U^0$ is the union of 2 modular submanifolds. We now add $\mathcal{K}.P_U$ to $\mathcal{L}_k$.

The remaining cases contain no unimodals, so it suffices to add the modular strata $V_3(x^2y)$, $V_3(x^3)$ and $V_3(0)$ to $\mathcal{L}_k$.

3.3. SURFACES IN $\mathbb{R}^4$

We next consider the cases when $n = 4$ and $p = 2$. These behave similarly to the above. We consider the possible 2-jets of the map f. We may consider these as defining pencils of quadrics in projective space P^4. It is convenient to classify them in terms of the geometry of the base locus Γ (over $\mathbb{C}$) of the pencil into four types: Γ is smooth (of degree 4), or nodal, or has isolated singularity only, or is either non-reduced or of higher dimension. (A fuller discussion of the geometry is given in the complex case in [13]; the real case is also discussed in [6].)

When Γ is smooth we have the open stratum $\tilde{D}_5$, which is 2-determined, and hence modular by Lemma 2.5. Outside this we have a finite union of $J^2\mathcal{K}$-orbits.

Those of the fourth type (over $\mathbb{C}$ there are 16 cases) contain no functions of modality less than 2, so we add the corresponding modular submanifolds $V_2(f_1, f_2)$ to $\mathcal{L}_k$.

For the cases when Γ has nodes as its only singularities, no moduli arise in the classification of germs with this 2-jet. This classification was given in [15]: the normal forms are

$$(2x_2y_2 \pm x_1^{a_1} \pm y_1^{b_1}, 2x_1y_1 \pm x_2^{a_2} \pm y_2^{b_2}),$$

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$$(x_2^2 + y_2^2 + \Re(x_1 + iy_1)^{a_1}, x_1^2 + y_1^2 + \Re(x_2 + iy_2)^{a_2}),$$

$$(x_2^2 + y_2^2 \pm x_1^{a_1} \pm y_1^{b_1}, 2x_1y_1 + \Re(x_2 + iy_2)^{a_2}),$$

$$(\Re(g), \Im(g)), \text{ where } g = (x_1 + iy_1)^{a_1} + (x_2 + iy_2)^{a_2} + 2(x_1 - iy_1)(x_2 - iy_2),$$

where the parameters a_1, b_1, a_2, b_2 run from 2 to ∞.

We thus add to $\mathcal{L}_k$ the corresponding submanifolds V_k where the parameters run from 2 to $k+1$ and at least one is $k+1$.

The remaining pencils are those of the types labelled I, J', K', L and M in [12]. The situation for the J'-series is the same as for the E-series, and we add $V_k(wz - x^2, xz - y^2)$ and a finite list of further modular strata to $\mathcal{L}_k$. In each of the other series, the stratum to be excluded consists of all functions with the given 2-jet and all terms of weight above a certain value, with respect to given weights for the variables, as in the table below, where the variables (w, x, y, z) are arranged in order of increasing weight. In Table II, the pre-normal form is listed as '$+\phi$': this signifies $f_0 + \phi$.

Table II.

Series	$2-$jet	Weights	Degree	Pre-normal form
I	$(x(y-z), y(x-z))$	1,2,2,2	4,4	$+(0, xya + xb + yc + zd + e)$
K'	$(wz - x^2, y^2 \pm z^2)$	1,2,3,3	4,6	$+(0, x^2a + xb + c)$
L	$(wz + xy, y^2 + xz)$	1,2,3,4	5,6	$+(0, x^2a + xb + c)$
M	$(2wz + x^2 \pm y^2, 2xz)$	1,2,2,3	4,5	$+(0, x^2a + xb + yc + d)$

where a, b, c, d and e depend only on w.

In fact in the I case we need 3 distinct normal forms f_0 corresponding to generic pencils of plane conics with 4, 2 or 0 real base points (one could e.g. take $(y^2 - xz, y(x \pm z))$ and $(y^2 - xz, (x+z)(2x+z))$: the details will be a little more complicated. In all these cases we can apply the technique given after Proposition 3.1 to show that the stratum is modular. We obtain modular strata $\mathcal{K}.(f_0 + P_c)$ to add to $\mathcal{L}_k$.

3.4. CURVES IN $\mathbb{R}^3$

Finally we treat the cases with $n = 3$ and $p = 2$. The 2-jet here defines a pencil of plane conics except in the degenerate cases where the two components are dependent. The degenerate cases contain no unimodal strata, so we add $V_2(x^2 + y^2 \pm z^2)$, $V_2(x^2 \pm y^2)$, $V_2(x^2)$ and $V_2(0)$ to $\mathcal{L}_k$.

Otherwise, in each case when the pencil contains a smooth conic there is a complete list of strata, all unimodal. It thus suffices to add $V_k(xy, x^j \pm z^2)$ (for all j with $2 \leqslant k \leqslant k+1$), $V_k(xz - y^2, xy)$ and $V_k(xz - y^2, x^2)$ to $\mathcal{L}_k$.

This leaves three cases, denoted F, G and H in [12]. We treat them in turn.

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For the F series, the 2-jet may be taken as (xy, xz). Here we find the pre-normal form $(xy + a(z), xz + b(y, z))$. The $\mathcal{K}$-classification corresponds to that of the function $yb(y, z) - za(z)$. The relevant part of $X^k(n, p)$ consists of the union of the closures of the strata FN_{16}, $FW_{1,0}$ and $FZ_{1,0}$. Define weights and degrees as follows:

Stratum	FN_{16}	$FW_{1,0}$	$FZ_{1,0}$
Weights of (x, y, z)	3,1,1	7,3,2	4,2,1
Degrees of (a, b)	4,4	10,9	6,5

We use the same argument as for Proposition 3.1 to establish modularity in each case of the $\mathcal{K}$-closure of the set of maps of the above type with each term of a, b having at least the specified degree. For the FN_{16} stratum, the basis elements of $T\mathcal{K}f$ of lower weight are $\{y, z, y^2, yz, z^2\}$ multiplied by $\partial_x f$; these are independent modulo terms of weight 4. For the $FW_{1,0}$ stratum we may require the coefficient of y^3 in $\psi(x, y)$ to be 1 (cases where this coefficient vanishes lie already in the FN_{16} stratum). Then the basis elements of lower weight are $\{y, z, y^2, yz, z^2, z^3\}.\partial_x f$, $z\partial_y f$ and $f_2\epsilon_1$. These are independent modulo terms of the specified weight: the key point to check turns out to be independence of the coefficients of $xz\epsilon_1$, $y^3\epsilon_1$ and $y^2z\epsilon_2$ in $y^2\partial_x f$, $z\partial_y f$ and $f_2\epsilon_1$. Similarly for the $FZ_{1,0}$ stratum we require the coefficient of y^2z in $\psi(x, y)$ to be 1 and then the argument works. We add these three strata to $\mathcal{L}_k$.

For the G series we have 2-jet (x^2, y^2), with a variant form over $\mathbb{R}$ called G^* with 2-jet $(xy, x^2 - y^2)$. For G we have the pre-normal form $(x^2 + yb(z) + a(z), y^2 + xc(z) + d(z))$. If a or d has order 3 we either have a germ $\mathcal{K}$-equivalent to a normal form G_s for some s or an element of $V_k(x^2 + z^3, y^2)$. For G^* we have the pre-normal form $(xy + a(z), x^2 - y^2 + xb(z) + yc(z) + d(z))$. If a or d has order 3, we have the simple germ G_9^* (the discussion in [6, pp 234–5] is incomplete on this point).

In both cases, if each of a, d has order $\geqslant 4$ we have the stratum with weights $(2, 2, 1/4, 4)$. As usual, this is a modular stratum, and we add both variants to $\mathcal{L}_k$.

3.5. THE H SERIES

The H series is defined by the 2-jet (x^2, xy). Here matters are more complicated. We gave in [13] the pre-normal form $f(x, y, z) = (x^2 + 2xb(z) + c(y, z), xy + a(z))$, and observed that eliminating x gives $F(y, z) = a(z)^2 - 2ya(z)b(z) + y^2c(y, z)$. Although the relation between the classifications of germs f and germs F is less close in this case, we have [13, Lemma 5.7] $\mu(F) = \mu(f) + 2\operatorname{ord} a$, so that a μ-constant stratum of germs f will uniquely determine a μ-constant stratum of F.

According to [6, 7.6.8], if the coefficients of z^3 in both $a(z)$ and $c(y, z)$ vanish, the modality is at least 2. Indeed, if we assign weights 2,2,1 to x, y, z respectively, each component has weight $\geqslant 4$. This is in the closure of the modular stratum just discussed, and is itself modular, by essentially the same argument (there are slight differences, as here the orbit of f_0 is not open in the set of 2-jets of the given weight, so

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we need to require that the coefficient of y^2 in $c(y, z)$ vanishes, but this does not affect the main argument). Otherwise [6, p. 236] we may reduce to have $a(z) = z^3$, and then reduce further to have $b(z) \equiv 0$, so that $f(x, y, z) = (x^2 + c(y, z), xy + z^3)$ and $F(y, z) = z^6 + y^2 c(y, z)$.

First suppose z^3 does not appear in $c(y, z)$. Then cases depend on the highest power of y dividing the 3-jet c_3 of c as follows.

Name	c_3	$NP(F)$	Name of F
HB_{12+r}	yz^2 in c	z^6, y^3z^2, y^{5+r}	$NB^r_{(-1)}$
HD_{13}	y^2z in c	z^6, y^4z	NC_{19}
HD_{14}	y^3 in c	z^6, y^5	NF_{20}
HE_{19}	$j^3c = 0$	z^6, y^6	

In the table, the column $NP(F)$ lists the monomials defining the Newton polygon of F; the last column names the stratum in the terminology of [16].

Each of the cases HB_{12+r} $(0 \leqslant r < \infty)$, HD_{13}, HD_{14} is a single $\mathcal{K}$-orbit; we just add $V_k(x^2 + yz^2, xy + z^3)$ to $\mathcal{L}_k$. The boundary stratum HE_{19} requires several moduli; here we assign weights $(2, 1, 1/4, 3)$. As usual, this defines a modular stratum, and we add it to $\mathcal{L}_k$.

Now suppose the coefficient of z^3 in c nonzero. Here we have cases as follows.

Name	c_3	$NP(F)$	Name of F
HA_{11}	*distinct roots*	z^6, y^2z^3, y^5	$NA_{1,0}$
HA_{11+r}	*repeated root*	$z^6, y^2z^3, y^3z^2, y^{5+r}$	$NA_{1,r}$
HC_{13}	z^3	z^6, y^2z^3, y^6	$NB^1_{(-1)}$
HC_{14}	z^3	z^6, y^2z^3, y^5z	$NB^1_{(0)}$
HC_{15}	z^3	z^6, y^2z^3, y^7	$NB^1_{(1)}$
HE_{17}	z^3	$z^6, y^2z^3, y^4z^2, y^6z, y^8$	

Each case except the last is a single $\mathcal{K}$-orbit; we must add $V_k(x^2 + z^3 + yz^2, xy + z^3)$ to $\mathcal{L}_k$. The final case is more troublesome: we have found it as given by requiring all terms in c to have weight $\geqslant 6$ when y, z have weights 1, 2 respectively. We again adopt our usual argument, but since here things are not weighted homogeneous we give some details.

LEMMA 3.2. *Define the weights of* x, y, z *to be* 3, 1, 2 *respectively; write* S_H *for the space of pairs* (g_1, g_2) *where all terms in either component have weight* $\geqslant 6$. *Let* P_H *denote the set of pre-normal forms* $(x^2 + c(y, z), xy + z^3)$ *with* c *of weight* $\geqslant 6$ *and the coefficient of* z^3 *in* c *nonzero. Then*

(i) *For any* $f \in P_H$ *we have*

$$T\mathcal{K}(f) + TP_H = S_H + \mathbb{C}\{(2xy, y^2); (0, y^3), (2xz, yz); (0, y^4), (0, y^2z), (0, xy); (2xy^2, 0), (y\partial c/\partial z, 0), (0, y^5), (0, y^3z), (0, yz^2), (0, xy^2), (0, xz)\}.$$

(ii) *The saturation* $\mathcal{K}.P_H$ *is a smooth submanifold of jet space and is modular.*

Proof. (i) The only term in the pre-normal form itself of weight <6 is the xy in the second component. Thus the terms in $T\mathcal{K}(f) + TP_H$ not in S_H are

$$(y(xy+z^3), 0),\ (0, y(xy+z^3)),$$
$$\{y, y^2, z, y^3, yz, x, y^4, y^2z, xy, z^2\}\partial f/\partial x, \qquad \{y, y^2, x\}\partial f/\partial y \quad \text{and} \quad y\partial f/\partial z.$$

Expanding these, and removing any terms of weight $\geqslant 6$, gives

$$(xy^2, 0), (0, xy^2), (2xy, y^2), (2xy^2, y^3), (2xz, yz), (0, y^4), (0, y^2z), (0, xy), (0, y^5),$$
$$(0, y^3z), (0, xy^2), (0, yz^2), (0, xy), (0, xy^2), (0, xz), (y\partial c_6/\partial z, 3yz^2),$$

where c_6 denotes the sum of terms of weight 6 in c. These are independent except as follows: the term $(0, xy)$ appears twice, and $(0, xy^2)$ appears thrice (note that as z^3 appears in c, yz^2 appears in $y\partial c_6/\partial z$).

We now show that S_H is contained in $T\mathcal{K}(f) + TP_H$. As to the second component, $T\mathcal{R}(xy+z^3) = \langle x, y, z\rangle\langle x, y, z^2\rangle$ indeed contains all monomials of weight $\geqslant 6$. For the first component, by the preparation theorem, any g may be written as $(x^2 + c(y,z))Q(x,y,z) + xR_1(y,z) + R_2(y,z)$, and if g has weight at least 6, so have xR_1 and R_2. Reducing further modulo $xy+z^3$ we may suppose R_1 independent of y. Now $3z^2\partial f/\partial x - y\partial f/\partial z = (6xz^2 - y\partial c/\partial z, 0)$; subtracting a suitable multiple of this will reduce R_1 to zero: it may introduce a term $(y\partial c/\partial z, 0)$ of weight 5, but we have just seen that this belongs to the left hand side. Since $R_2 \in TP_H$, we are done.

(ii) It follows as usual from (i) that $\mathcal{K}.P_H$ is a smooth submanifold of jet space, with tangent space given by the lemma. Modularity follows by simply checking that multiplying any of our list of elements by a monomial produces another in the list, or something of weight at least 6. □

We conclude by adding $\mathcal{K} \cdot P_H$ to $\mathcal{L}_k$.

4. The Case $n \leqslant p$

For this section, by [6, Theorem 12.4.1], $X^k(n,p)$ is the union of the closures of the strata listed in [6, Table 12.1] if $n = p$, and in [6, Table 12.2] if $n < p$ (the list is not quite complete, and is corrected below). Detailed listing of strata is given in [6, Chapter 8], following the less complete treatment in [14].

We will write e for $p - n$. As in Section 3, it suffices in view of Proposition 2.3 to consider maps with zero 1-jet. Note that in this paper by 'codimension' we mean codimension in the jet space, or equivalently, in the source of a stable map. The target codimensions used in [6] are obtained from these by adding e.

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The second intrinsic derivative of f is a symmetric bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, inducing a homomorphism $\Delta\colon \bigcirc^2(\mathbb{R}^n) \to \mathbb{R}^p$, where $\bigcirc^2(\mathbb{R}^n)$ is the symmetric square of $\mathbb{R}^n$. Following Mather [10], we say that f belongs to $\Sigma^{n(a)}$ if a is the dimension of the kernel of Δ: computationally, $J^2 f$ has $\binom{n}{2} - a$ independent components.

Each $\Sigma^{n(a)}$ is a smooth submanifold of $\Sigma^n \subset J^2$; by (ii) of Lemma 2.5, these submanifolds are all modular. It follows from the calculations in [14, §1] that the codimensions of the non-empty $\Sigma^{n(a)}$ are no less than the codimension ${}^2\sigma(n,p)$ except in the cases when (n,a) is one of the following:

$$(1,a);\ (n,0);\ (n,1);\ (2,2);\ (2,3);\ (3,2);\ (3,3);\ (4,2).$$

We assign all the other $\Sigma^{n(a)}$ to $\mathcal{L}_k$, and treat the listed cases in turn.

4.1. THE CASE $n=1$

The set of k-jets in Σ^1 decomposes as the union of the strata A_j which are the $\mathcal{K}$-orbits corresponding to the local algebras $\mathbb{C}[x]/\langle x^j\rangle$ for $2 \leqslant j \leqslant k$ and the set $V_k(x;0)$. We add $V_k(0)$ to $\mathcal{L}_k$.

4.2. THE CASES $a \leqslant 1$

For $a=0$ we have $\Sigma^{n(0)}$ which consists of a single $\mathcal{K}$-orbit. Next, it was shown by Damon [3] that $\Sigma^{n(1)}$ consists of a countable union of $\mathcal{K}$-orbits $X(n,\ell,k)$ (the list is repeated on [6, p. 292]) and a subset of infinite codimension.

Consider $V_k(x_1,\dots,x_n; f_1,\dots,f_q)$, where the f_t are the monomials $x_i x_j$ for all $1 \leqslant i \leqslant j \leqslant n$ except $i=j=1$. Its codimension tends to infinity with k, and its complement is a finite union of $\mathcal{K}$-orbits, so we add it to $\mathcal{L}_k$.

4.3. THE CASE $n=a=2$

We next consider Σ^2: here we may also partition into the Boardman strata $\Sigma^{2,0}$, $\Sigma^{2,1}$ and $\Sigma^{2,2} = \Sigma^{2(3)}$. As was shown by Mather [10], $\Sigma^{2,0}$ is the union of countably many $\mathcal{K}$-orbits and the subset $V_\infty(xy) \cup V_\infty(x^2+y^2)$ of infinite codimension, so we add $V_k(xy) \cup V_k(x^2+y^2)$ to $\mathcal{L}_k$.

We saw in [14] that $\Sigma^{2,1}$ is also the union of countably many 1-modal strata and a subset of infinite codimension. Moreover, Proposition 2.7 applies to show that all these strata are modular. We could now conclude as for the E-series, but will give a more detailed treatment which serves as a model later on.

We use the notation of [6, Chapter 8]. We can write maps in the pre-normal form $f(x,y) = (x^2 + a(y), xb_i(y) + c_i(y))$, where i runs from 2 to $e+2$. Then the invariant κ is defined by $\kappa(f) := \min(\mathrm{ord}\ a, 2\mathrm{ord}\ b_i, \mathrm{ord}\ c_i)$. We next show that the set of maps in $\Sigma^{2,1}$ with $\kappa(f) \geqslant k$ is a modular submanifold. To simplify notation, we present the details for the case $e=1$.

Assign weights $\mathrm{wt}(x) = k/2$, $\mathrm{wt}(y) = 1$ for some $k \geqslant 3$. Let S_k denote the set of all maps $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that each term in the Taylor expansion of each component has weight $\geqslant k$; and let S_k^0 be the subset where x^2 appears with nonzero coefficient in at least one component, so that $S_k^0 \subset \Sigma^{2,1}$. Write P for the set of all maps f in the 'prenormal form' $f(x, y) = (x^2 + a(y), xb_1(y) + c_1(y), xb_2(y) + c_2(y))$. Let $P_k := P \cap S_k$.

LEMMA 4.1. (i) *For all* $f \in P_k$, $T\mathcal{K}(f) + TP_k = S_k + \{y, y^2, \dots, y^l\}\partial_x f$, *where* $l := [(k-1)/2]$.

(ii) For $f \in P_k$, *the image of* $T\mathcal{K}(f) + TP_k$ *in* TJ^k *has dimension independent of* f. *Hence the saturation* $\mathcal{K}.P_k$ *is a manifold* Λ_k. *Moreover,* Λ_k *is modular.*

Proof. Since each term in f has weight $\geqslant k$, so has any element of $f^*\mathfrak{m}_p$, and hence $T\mathcal{C}f \subset S_k$. Also each term of $x\partial_x f$, $x\partial_y f$, $y\partial_y f$ and $y^r\partial_x f$ for $r \geqslant k/2$ has weight $\geqslant k$, so these all belong to S_k, while $y^r\partial_x f$ for $1 \leqslant r < k/2$ belongs explicitly to the right hand side. Thus $T\mathcal{R}f$ is contained in the right-hand side, so the inclusion $\subseteq$ holds.

For the converse inclusion, the terms $\{y, y^2, \dots, y^l\}\partial_x f$ are clearly in $T\mathcal{R}f$. Now $T\mathcal{K}f$ contains $(\mathcal{E}_n.f^*\mathfrak{m}_p)^{\times 3}$, and in particular $(x^2 + a(y), 0, 0)$, $(0, x^2 + a(y), 0)$ and $(0, 0, x^2 + a(y))$. By the preparation theorem, any element of $\mathcal{E}_{x,y}$ can be written in the form $Q(x, y)(x^2 + a(y)) + xR_1(y) + R_2(y)$, so it suffices to consider elements of S_k with each component linear in x.

Any such element with first component not involving x is already in TP_k. And $T\mathcal{R}f$ contains any $R(y)\partial_x f = R(y)(2x, b_1(y), b_2(y))$ with $R(0) = 0$. Hence the right hand side is contained in the left, proving (i).

The first assertion of (ii) holds since the terms $y^r\partial_x f$ with $1 \leqslant r \leqslant l$ are always independent modulo S_k, as we see by considering the coefficients of the terms $(xy^r, 0, 0)$. The second claim follows, using the group action of $J^k\mathcal{K}$ on k-jet space.

Finally we need to show that the tangent space $S_k + \{y, y^2, \dots, y^l\}\partial_x f$ to the manifold is an $\mathcal{E}_{x,y}$-module. Now S_k is certainly a module, so it suffices to show that if the remaining terms are multiplied by elements of $\mathfrak{m}_{x,y}$ they remain in the tangent space. It suffices to check for multiplication by x and by y. But the terms $x.y^r\partial_x f$ are in S_k; and $y.y^r\partial_x f$ is already in the list if $r < l$, while for $r = l$ it lies in S_k. □

COROLLARY 4.1.1. *The codimension of* Λ_k *in the space of jets with fixed target is* $(k + [(k-1)/2])(e+2) - [(k-1)/2]$.

For the monomials of weight strictly between 0 and k are $\{y^i | 1 \leqslant i < k\}$, $\{xy^i | 0 \leqslant i < k/2\}$, so are $k + [(k-1)/2]$ in number. There are $(e+2)$ target coordinates, and we see from the lemma that we must subtract a further $[(k-1)/2]$ to get the codimension of Λ_k.

The maps with $\kappa(f) = 3$ fall into a countable set of $\mathcal{K}$-equivalence classes, together with $V_\infty(x^2 + y^3)$, so add $V_k(x^2 + y^3)$ to $\mathcal{L}_k$.

For the maps with $\kappa(f) = 4$, the above corollary gives $6e + 9$ as the codimension of Λ_4. We have [6, p.279] a countable list of $\mathcal{K}$-orbits, $V_\infty(x^2 + y^4)$, $V_\infty(x^2 - y^4)$, and the strata given by the saturations of the normal forms

$$D_{\ell,\ell+2}\colon (x^2 \pm y^4, xy^\ell + cy^{\ell+2}, 0, \dots, 0);\ \bar{D}_{\ell,\ell+2}\colon (x^2 \pm y^4, xy^\ell + cy^{\ell+2}, y^{\ell+3}, 0, \dots, 0).$$

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Moreover [6, p. 529] at least for $3 \leqslant \ell \leqslant 9$ these are canonical strata: they are modular by Proposition 2.7. The rest have high enough codimension to ignore. More precisely, if $n = p$ ($e = 0$) we add to $\mathcal{L}_k$ the closure of $D_{4,6}$, which is defined by having 4-jet $(x^2 \pm y^4, 0)$ and no terms of weight less than (4,6) where x, y have weights 2,1 respectively: as usual, this is modular. If $e \geqslant 1$, add to $\mathcal{L}_k$ the stratum defined by weights at least $(4, 5, 6, \ldots, 6)$ and 4-jet $(x^2 \pm y^4, axy^3, 0 \ldots, 0)$ or, more simply, we can just add $V_4(x^2 \pm y^4)$.

Next, Λ_5 has codimension $8e + 12$. This exceeds $\tau(-e)$ if $e \geqslant 2$, but is 1 below it for $e = 0, 1$. For $e \geqslant 2$, we add Λ_5 to $\mathcal{L}_k$. In general we can add $V_4(x^2)$ to $\mathcal{L}_k$, as it has codimension $11e + 16$; there are only finitely many $\mathcal{K}$-orbits in Λ_5 not contained in this—they are those with a term xy^3 and, hence, have normal form $(x^2 + y^5, xy^3 + y^m, y^n)$ with $m = 5, 6$ and (if $e = 1$) $n = m, m + 1$ or $m + 2$.

Now Λ_6 has codimension $9e + 14$, so we can add this, and also Λ_7 to $\mathcal{L}_k$; higher levels are already included in $V_4(x^2)$.

4.4. THE CASE $n = 2, a = 3$

We turn to $\Sigma^{2,2}$. The 3-jet gives a symmetric trilinear map, inducing a homomorphism $\Delta\colon \bigcirc^3(\mathbb{R}^2) \to \mathbb{R}^p$; say that $f \in \Sigma^{2,2^{\text{'}}(a)}$ if $a = \dim Ker\Delta$.

The case $a = 0$ gives a single $\mathcal{K}$-orbit; for $a = 1$ we have a countable union of orbits and a set of infinite codimension; it suffices to add $V_k(x^3, x^2y, xy^2)$ to $\mathcal{L}_k$.

If $a = 2$ we have a pencil of binary cubics. The space of such pencils has an open subset (where the discriminant of the pencil has 4 distinct roots) which requires a modulus for classification. Over $\mathbb{R}$ this gives 4 strata. Each of these lifts to 2 strata in jet space, which are modular by Proposition 2.7, as are the modular strata which are unions $FA_3 \cup FC_0$ and $\overline{FA_3} \cup \overline{FC_0}$ (see [6, 8.5.2]).

The remaining pencils fall into finitely many isomorphism types. Of these $(x^3, x^2y \pm y^3)$ and (x^3, y^3) each give only 2 $\mathcal{K}$-orbits; $(x^3 \pm xy^2, x^2y)$, (x^3, xy^2), (x^2y, xy^2), $(x^3 + xy^2, x^2y + y^3)$ and (x^3, xy^2) give countably many, so we add $V_k(x^3 + xy^2, x^2y + y^3)$, $V_k(x^3, xy^2)$ and, for $k \geqslant q \geqslant 3$, $V_k(x^2y, xy^2, x^q)$, $V_k(x^2y, xy^2 \pm x^q)$ and $V_k(x^2y, xy^2 \pm x^q, x^{q+1})$ to $\mathcal{L}_k$.

Since $V_3(x^3, x^2y)$ is modular, and has codimension greater than ${}^2\sigma(n, p)$, we add it to $\mathcal{L}_k$; similarly the cases $a \geqslant 3$: add $V_3(x^3 \pm y^3)$, $V_3(x^2y)$, $V_3(x^3)$ and $V_3(0)$. However if $n = p$, $X^k(n, p)$ excludes the stratum FeC_1. We thus replace $V_3(x^3, x^2y)$ by the $\mathcal{K}$-closure of the set of germs $(x^3 + a(x, y), x^2y + b(x, y))$, where a, b have weights at least 9,8 with respect to $wt(x, y) = (3, 2)$. We obtain a prenormal form by restricting a and b to be linear in x. The only elements of $T\mathcal{K}f$ of lower weight are $y\partial_x f$ and $f_2\epsilon_1$, and the usual proof of modularity goes through.

4.5. THE CASE $n = 3, a = 2$

The case $\Sigma^{3(2)}$ is the one which caused Mather the most difficulty. We can re-phrase the preliminary reductions as follows. For any r, the 2-jet of a $\Sigma^{r(2)}$ singularity is given by a linear system of quadrics whose apolar system is a pencil. If the (complexified) pencil does not contain a perfect square, the 2-jet is sufficient, so defines a modular stratum. The cases when the pencil does contain a square are a_s: $(y_1^2, \sum_2^s \pm y_j^2)$, c_s: $(y_1^2, 2y_1y_2 + \sum_3^s \pm y_j^2)$ $(2 \leqslant s \leqslant r)$, and a variant form a_2^*: $(y_1^2 - y_2^2, 2y_1y_2)$ of a_2.

When $r = 3$, each of the cases other than c_2 is the union of countably many $\mathcal{K}$-orbits. For any k, we thus have (see [6, p. 312] as well as [10]) the union of finitely many orbits with (a_3 case) $V_k(x^2, y^2, xz, yz)$, (c_3 case) $V_k(x^2 + yz, xy, xz, y^2)$, ($a_2^*$ case) $V_k(x^2 + y^2, xz, yz, z^2)$, and ($a_2$ case) the union of $V_k(xy, xz, yz, z^2)$ and

$$\bigcup_{3 \leqslant m < k} (V_k(xy, xz, yz, z^2 \pm x^m) \cup V_k(xy, xz + y^m, yz, z^2) \cup V_k(xy, xz, yz, z^2, x^m),$$

so we add all these to $\mathcal{L}_k$.

In the remaining (c_2) case, the 2-jet is equivalent to (x^2, xz, yz, z^2), and we put the germ in pre-normal form: write P_D for the set of germs of the form

$$(x^2 + a(y), xz + xb(y) + c(y), yz, z^2 + xd(y) + e(y), xf_i(y) + g_i(y)).$$

Following [6, pp. 311–313], we define strata as follows. For the 'Damon stratum' Δ_δ, let x, y, z have weights 2, 1, 2 and let S_δ be the set of maps f with all components of weight $\geqslant 4$, and those after the fourth of weight $\geqslant 5$. Then by [14] the stratum containing $P_\delta := P_D \cap (S_\delta + yz\epsilon_3)$ is bimodal.

Next assign weights (1, 1, 2) to x, y, z and let S_ϵ be the set of maps with all components of weight $\geqslant 4$; then if $f \in P_\epsilon := P_D \cap (S_\epsilon + x^2\epsilon_1 + xz\epsilon_2 + yz\epsilon_3)$, the deformation $f + tz\epsilon_1$ is equivalent to the stabilisation of a 2-variable germ with 3-jet (x^3, x^2y), so again f is bimodal. The stratum Δ_ϵ corresponding to P_ϵ was omitted in error from [6, Table 12.2].

We claim that in each of these cases the stratum is modular.

LEMMA 4.2. (i) *For all $f \in P_\delta$, we have*

$$T\mathcal{K}f + T(P_\delta) = S_\delta + yz.\{\epsilon_r\} + \{y\partial_x f, y\partial_z f\}.$$

(ii) *For $f \in P_\delta$, the image of $T\mathcal{K}(f) + T(P_\delta)$ in TJ^k has dimension independent of f. Hence the saturation $\mathcal{K}.P_\delta$ is a manifold. Moreover, this manifold is modular.*

Proof. (i) First we show $\subseteq$. Clearly $T(P_\delta) \subset S_\delta$. The only basis elements of $T\mathcal{R}f$ with weight less than that of f are $y\partial_x f$ and $y\partial_z f$, so $T\mathcal{R}f$ is contained in the right hand side. As to $T\mathcal{C}f$, the only basis element of $f^*\mathfrak{m}_p$ of weight <4 is the third component yz of f, and we have specifically allowed for that.

Now we show $\supseteq$. The 7 final terms clearly belong to the left hand side. By the preparation theorem, we can write any (C^∞-) function of x, y and z of weight is at least 3 as the sum of a linear combination of $(x^2 + a(y)$, $xz + xb(y) + c(y)$, $z^2 + xd(y) + e(y)$ and an expression of the form $x\alpha(y) + \beta(y)$. It thus suffices to show that an expression $x\alpha(y) + \beta(y)$ of weight $\geqslant 4$, appearing in any coordinate position, belongs to the

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left hand side. Now those in the second, fourth or later positions belong to $T(P_\delta)$. We reduce the other cases to these. For the third place, we use $y\partial_z f = (0, xy, y^2, 2yz, 0)$ (since the terms we need to deal with are divisible by y^2). Finally for the first position $\beta(y)$ is in $T(P_\delta)$ and we have $y^2\partial_x f = (2xy^2, y^2(z+b), 0, y^2 d, y^2 f)$, which reduces to $(2xy^2, 0, 0, 0, 0)$ modulo terms already dealt with.

(ii) The first assertion holds since the 7 final terms are always independent modulo S, as we see by considering the coefficients of the terms yz in all 5 positions and xy in the first two. The second claim follows, using the group action of $J^k\mathcal{K}$ on k-jet space.

Finally we show that the tangent space to the manifold is an $\mathcal{E}_{x,y,z}$-module. Now S is certainly a module. Thus it suffices to show that if the remaining terms are multiplied by an element of $\mathfrak{m}_{x,y,z}$ they remain in the tangent space. Indeed, it suffices to check for multiplication by x, y and z. But all the terms this gives have weight $\geqslant 4$. □

LEMMA 4.3. (i) *For all $f \in P_\epsilon$, we have $T\mathcal{K}f + T(P_\epsilon) = S_\epsilon + \mathbb{C}.F$, where F is the following list:*

$$\{x^2, x^3, x^2y, xz, yz\}.\{\epsilon_r\},\ \ xy\epsilon_1, xy^2\epsilon_1, xy\epsilon_2 + y^2\epsilon_3, xy^2\epsilon_2 + y^3\epsilon_3, xy\epsilon_3, xy^2\epsilon_3.$$

(ii) *For $f \in P_\epsilon$, the image of $T\mathcal{K}(f) + T(P_\epsilon)$ in TJ^k has dimension independent of f. Hence $\mathcal{K}.P_\epsilon$ is a manifold; this manifold is modular.*

Proof. (i) First we show $\subseteq$. Clearly $T(P_\epsilon) \subset S_\epsilon$. The basis elements of $f^*\mathfrak{m}_p$ of weight < 4 are the first 3 components. Thus any element of $f^*\mathfrak{m}_p.\mathcal{E}_n$ is congruent modulo elements of weight $\geqslant 4$ to a linear combination of $\{x^2, x^3, x^2y, xz, yz\}$. Thus $T\mathcal{C}f$ is contained in the right hand side. Now since $\partial_x f$ and $\partial_z f$ each contain a term of weight 1, and the lowest term of $\partial_y f$ has weight 2, the only basis elements of $T\mathcal{R}f$ with weight less than 4 are $\{x, y\}\partial_y f$, $\{x, y, x^2, xy, y^2, z\}.\{\partial_x f, \partial_z f\}$. Expanding these terms, and cancelling out any element of weight $\geqslant 4$, and also any linear combination of the terms $\{x^2, x^3, x^2y, xz, yz\}\epsilon_r$ previously listed, all vanish save for six, which reduce to $xy\epsilon_1, xy^2\epsilon_1, xy\epsilon_2 + y^2\epsilon_3, xy^2\epsilon_2 + y^3\epsilon_3, xy\epsilon_3, xy^2\epsilon_3$.

Now we show $\supseteq$. The above argument shows that it is sufficient to prove that S_ϵ is contained in the left hand side. As before we see, using the preparation theorem, that it suffices to show that an expression $x\alpha(y) + \beta(y)$ of weight $\geqslant 4$, appearing in any coordinate position, belongs to the left-hand side. Those not in the first position already belong to $T(P_\epsilon)$, as does $\beta(y)\epsilon_1$. Now subtracting $\frac{1}{2}\alpha(y)\partial_x f$ removes this term, and replaces it by others of the types just discussed.

The proof of (ii) is essentially the same as in the preceding case. □

The complement in the manifold of jets of (c_2) type of $\mathcal{K}.P_\delta \cup \mathcal{K}.P_\epsilon$ consists of the countable family of good $\mathcal{K}$-orbits listed on [6, p. 312] and the germs with ∞-jet $b_* : (x^2, xz, yz, z^2, xy^2)$. It thus suffices to add to $\mathcal{L}_k$ the modular strata $\mathcal{K}.P_\delta$, $\mathcal{K}.P_\epsilon$ and $V_k(x^2, xz, yz, z^2, xy^2)$.

This completes the discussion of $\Sigma^{3(2)}$.

4.6. THE CASE $n = a = 3$

We next turn to $\Sigma^{3(3)}$. The 2-jets here correspond to nets of conics. We gave the classification of these in [14] and [6, § 8.7], and retain the notation of those references. In the cases (labelled A, B, D, E) where the net has no base points, the 2-jet is sufficient and we have a canonical stratum which is modular in the A cases by Proposition 2.7 and in the rest since it is a $\mathcal{K}$-orbit. In the cases labelled B^*, C, D^*, E^*, F^*, the net has simple base points, and we have the union of a countable family of $\mathcal{K}$-orbits and a set of infinite codimension. To deal with cases B^*, C it suffices [6, p. 315] to add $V_k(2xz + y^2, 2yz, x^2 + 2\eta xz)$, for $\eta = -1, 0, +1$, to $\mathcal{L}_k$.

The cases D^* and E^* with more than one base point are rather more complicated, but using the tables on [6, p. 316] we see that it suffices to add to $\mathcal{L}_k$ the modular strata $V_k(yz, zx, xy, x^p, y^q)$, $V_k(yz, zx, xy, x^p \pm y^q)$, $V_k(yz \pm x^{p-1}, zx, xy, y^q)$ and $V_k(yz \pm x^{p-1}, zx \pm y^{q-1}, xy)$ for $3 \leqslant p, q \leqslant k$.

The strata with 2-jet of type F, F^*, G, G^* are called the MF-, LF-, MG- and LG-series respectively. All strata in the LF-series are enumerated; by Proposition 2.7, are all modular. We conclude as for the E-series: add $V_k(x^2, x(y+z), yz, z^q)$, $V_k(x^2 \pm z^q, x(y+z), yz)$, $V_k(x^2 \pm z^q, x(y+z), yz, z^{q+1})$ $(3 \leqslant q \leqslant k)$, $V_k(x^2, xy, y^2 + z^2)$ and finitely many other strata to $\mathcal{L}_k$.

For the MF- and MG-series we defined respective prenormal forms

$$f(x, y, z) = (x^2 \pm y^2 + a(z), xy + xb(z) + c(z), yz, xb_i(z) + c_i(z)),$$

$$f(x, y, z) = (x^2 + a(z), y^2 + xb(z) + c(z), yz, xb_i(z) + c_i(z));$$

setting $y = 0$ in either case yields a map Af in the prenormal form for singularities of type $\Sigma^{2,1}$, and the level $\kappa(f)$ is defined to be $\kappa(Af)$. Thus the subcases where $\kappa \geqslant 4$ are of codimension $e+2$ greater than the strata as a whole and, hence, $\geqslant {}^2\sigma(n, p)$. These subcases also form modular strata, by the same argument as for the $\Sigma^{2,1}$ case—here we take weights (2,2,1/4,4,3,4) and define the subspace S to consist of maps of admissible weights with each component of order $\geqslant 2$—and we add them to $\mathcal{L}_k$.

The strata of level 3 have all been enumerated and it follows from Proposition 2.7 that all are modular. It thus suffices to add to $\mathcal{L}_k$ the modular strata $V_k(x^2 \pm y^2 - z^3, xy, yz)$, $V_k(x^2 - z^3, y^2, yz)$, and a finite list of further strata. The case MFU_6 is the one causing difficulties in [6] and discussed above.

We proceed differently for the remaining cases. Consider a 2-jet Z with zero 1-jet. Write I_Z for the (homogeneous) ideal in $\mathcal{E}_r$ generated by the components of Z, and $\pi_Z\colon S^3 \to Q^3$ for the projection of the space S^3 of homogeneous functions of degree 3 in $x_1, \ldots, x_r$ to its quotient $S^3/(S^3 \cap I_Z)$. Then if, as we may suppose, the first t components of Z are linearly independent and the rest vanish, then if $j^2 f = Z$, the dimension of the local algebra $(f^* \mathfrak{m}_p.\mathcal{E}_r + \mathfrak{m}_r^4)/\mathfrak{m}_r^4$ is determined by the dimension of the span of $\pi(f_{t+1}), \ldots, \pi(f_r)$.

The set of 3-jets for which this dimension takes a given value form a $\mathcal{K}$-invariant, algebraic set. We may choose a $\mathcal{K}$-invariant stratification; then since by (iii) of Lemma 2.5, an invariant submanifold of J^3 which is the $\mathcal{K}$-saturation of a family of maps with the same 2-jet is modular, these strata are modular. If the local algebra has dimension 7, we have a single $\mathcal{K}$-orbit. We have just found modular strata covering its complement: these we must add to $\mathcal{L}_k$. This complement has codimension $7e + \gamma$, where $\gamma = 12, 11, 12, 14, 13$ for 2-jet of type G, G^*, H, I, I^* respectively. This is at least as large as ${}^2\sigma(n, p)$ *except* in the cases $e = 1$ for G, G^*.

It remains to consider maps with $e = 1$ and 2-jet of type G^*, viz. $(x^2, xy, yz, 0)$. Here we have a prenormal form $(x^2 + a(y), xy + b(z), yz, c(y) + xd(z) + e(z))$. If a, c and e have nonzero 3-jets, it is not difficult to reduce to $(x^2 + y^3, xy, yz, y^3 + z^3)$, which is thus 3-determined, and gives a $\mathcal{K}$-orbit. The complement of this in the set of 3-jets over jets equivalent to $(x^2, xy, yz, 0)$ is a proper algebraic subvariety of codimension ${}^2\sigma(n, p)$; it now suffices to choose a $\mathcal{K}$-invariant stratification, and apply Lemma 2.5 to prove modularity.

4.7. THE CASE $n = 4, a = 2$

The final case is $\Sigma^{4(2)}$. The 2-jet is given by a linear system (of rank 8) of quadratics in 4 variables. The apolar system has rank 2, so is a pencil, which is more convenient to describe. The generic pencil involves a modulus; the others fall into finitely many isomorphism classes, so each defines a modular submanifold. For the cases when the pencil does *not* include a perfect square, the 2-jet is again sufficient and we have a modular stratum.

For each of the others, we observed in [14] that there is a single $\mathcal{K}$-class of germs f with $f^*\mathfrak{m}_p.\mathcal{E}_n \supset \mathfrak{m}_n^3$. This is again equivalent to requiring the dimension δ of the algebra to be minimal. The complement of this class is an algebraic subset of J^3, of codimension $\geqslant {}^2\sigma(n, p)$. Arguing as in the $\Sigma^{3(3)}$ case, we see that this complement is a union of modular submanifolds.

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