SOME RESULTS RELATING THE BEHAVIOUR OF FOURIER TRANSFORMS NEAR THE ORIGIN AND AT INFINITY

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1. Introduction. It is known that under special conditions, Fourier sine transforms and Fourier cosine transforms behave asymptotically like a power of x, either as $x \to 0$ or as $x \to \infty$ or both. For example (**3**),

$$F_c(x) \sim \phi(+0)\left(\frac{2}{\pi}\right)^{1/2} \Gamma(1-\alpha) \sin\tfrac{1}{2}\pi\alpha\, x^{\alpha-1} \qquad (x \to \infty),$$
$$\sim \phi(+\infty)\left(\frac{2}{\pi}\right)^{1/2} \Gamma(1-\alpha) \sin\tfrac{1}{2}\pi\alpha\, x^{\alpha-1} \qquad (x \to 0),$$

where $f(x) = x^{-\alpha}\phi(x)$, $0 < \alpha < 1$, and $\phi(x)$ is of bounded variation in $(0, \infty)$ and $F_c(x)$ is the Fourier cosine transform of $f(x)$. This suggests that other results connecting the behaviour of a function at infinity with the behaviour of its Fourier or Watson transform near the origin might exist. In this paper we derive various such results. For example, a special case of these results is

$$f(+0) = \left(\frac{2}{\pi}\right)^{1/2} \lim_{T\to\infty} \frac{1}{T} \int_0^T xg(x)\, dx,$$

where $f(x)$ is the Fourier sine transform of $g(x)$. It should be noted that the Fourier inversion formula fails to give $f(+0)$ directly in this case. Some applications of these results to show the relationships between various forms of known summation formulae are given.

2. Definition. A function $S(N)$ is limitable by Riesz means (R, N, τ), to S as $N \to \infty$, if

$$\lim_{N\to\infty} \tau N^{-\tau} \int_0^N S(t)(N-t)^{\tau-1}\, dt = S$$

for a sufficiently large τ.

3. The main results.

THEOREM 1. *If $g(x) \in L(0, \infty)$ and has a $\pi J_{p/2-1}(2\pi x^{1/2})$-transform $f(x)$ and $x^{1/2-p/4}g(x)$ is of bounded variation near $x = 0$, then*

$$\lim_{T\to\infty} \frac{\pi^{p/2}}{\Gamma(p/2)} \int_0^T x^{p/4-1/2} f(x)\left(1 - \frac{x}{T}\right)^{\tau} dx = \lim_{x\to+0} x^{1/2-p/4} g(x),$$

where $\tau > (p-1)/2$ and p is a positive integer.

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Proof. Let, for $p > 0$,

$$L(x) = x^{p/4-1/2}, \quad 0 < x \leqq T,$$
$$= 0 \qquad x > T.$$

Its $\pi J_{p/2-1}(2\pi x^{1/2})$-transform is given by

$$H(x) = \pi \int_0^T t^{p/4-1/2} J_{p/2-1}(2\pi x^{1/2}t^{1/2})\,dt$$
$$= T^{p/4}x^{-1/2}J_{p/2}(2\pi x^{1/2}T^{1/2}).$$

Now

$$(1.1) \qquad \int_0^T f(x)x^{p/4-1/2}\,dx = T^{p/4}\int_0^\infty x^{-1/2}g(x)J_{p/2}(2\pi x^{1/2}t^{1/2})\,dx,$$

since both integrals are equal to the absolutely convergent double integral

$$\pi \int_0^\infty \int_0^T x^{p/4-1/2}g(t)J_{p/2-1}(2\pi x^{1/2}t^{1/2})\,dx\,dt.$$

Split the range of integration of the right-hand side of (1.1) into $(0, \Delta)$ and (Δ, ∞), $\Delta > 0$, and let

$$\lim_{x\to 0} x^{1/2-p/4}g(x) = k, \quad \text{a constant.}$$

Now the integral with the range $(0, \Delta)$ can be written as

$$(1.2) \quad kT^{p/4}\int_0^\Delta x^{p/4-1}J_{p/2}(2\pi x^{1/2}T^{1/2})\,dx$$
$$+ T^{p/4}\int_0^\Delta x^{p/4-1}J_{p/2}(2\pi x^{1/2}T^{1/2})\{x^{1/2-p/4}g(x) - k\}\,dx$$
$$= S_1(T) + S_2(T), \quad \text{say.}$$

We shall now show that $S_1(T) + S_2(T)$ is limitable by Riesz means (R, T, τ) to a finite limit as $T \to \infty$, for a large τ,

$$\lim_{T\to\infty} \tau T^{-\tau}\int_0^T S_1(t)(T-t)^{\tau-1}\,dt$$
$$= k\tau \lim_{T\to\infty} T^{-\tau}\int_0^T t^{p/4}(T-t)^{\tau-1}\,dt \int_0^\Delta x^{p/4-1}J_{p/2}(2\pi x^{1/2}t^{1/2})\,dx$$
$$= k\tau \lim_{T\to\infty} T^{-\tau}\int_0^\Delta x^{p/4-1}\,dx \int_0^T t^{p/4}(T-t)^{\tau-1}J_{p/2}(2\pi x^{1/2}t^{1/2})\,dt.$$

By Sonine's integral (**3**), the inner integral can be evaluated to yield:

$$k\Gamma(\tau+1)\pi^{-\tau}\lim_{T\to\infty} T^{p/4-\pi/2}\int_0^\Delta x^{p/4-\tau/2-1}J_{p/2+\tau}(2\pi x^{1/2}T^{1/2})\,dx$$
$$= k\Gamma(\tau+1)\pi^{-p/2}2^{\tau-p/2+1}\int_0^\infty t^{p/2-\tau-1}J_{p/2+\tau}(t)\,dt,$$

by putting $2\pi x^{1/2}T^{1/2} = t$ and making $T \to \infty$, the inner integral is equal to:

$$\pi^{-p/2}\Gamma(p/2) \lim_{x\to+0} x^{1/2-p/4}g(x), \tag{1.3}$$

where $\tau > (p-3)/2$.

Now consider

$$\lim_{T\to\infty} \tau T^{-\tau} \int_0^T S_2(t)(T-t)^{\tau-1}\,dt$$
$$= \lim_{T\to\infty} \tau T^{-\tau} \int_0^T t^{p/4}(T-t)^{\tau-1}\,dt \int_0^\Delta x^{p/4-1}J_{p/2}(2\pi x^{1/2}t^{1/2})\phi(x)\,dx,$$

where

$$\phi(x) = x^{1/2-p/4}g(x) - k$$
$$= \lim_{T\to\infty} \tau T^{-\tau} \int_0^\Delta x^{p/4-1}\phi(x)\,dx \int_0^T t^{p/4}(T-t)^{\tau-1}J_{p/2}(2\pi x^{1/2}t^{1/2})\,dt.$$

Again by Sonine's integral, the inner integral can be evaluated to yield:

$$\Gamma(\tau+1)\pi^{-\tau} \lim_{T\to\infty} T^{p/4-\tau/2} \int_0^\Delta x^{p/4-\tau/2-1}J_{p/2+\tau}(2\pi x^{1/2}T^{1/2})\phi(x)\,dx.$$

Choose Δ small enough so that $\phi(x)$ is of bounded variation in $(0, \Delta)$ and tends to zero with x. Therefore, we can write

$$\phi(x) = \phi_1(x) - \phi_2(x),$$

where ϕ_1 and ϕ_2 are positive non-decreasing bounded functions in $(0, \Delta)$ and tend to zero as $x \to 0$. The above integral can now be written as

$$\Gamma(\tau+1)\pi^{-\tau} \lim_{T\to\infty} T^{p/4-\tau/2} \int_0^\Delta x^{p/4-\tau/2-1}J_{p/2+\tau}(2\pi x^{1/2}T^{1/2}) \times \{\phi_1(x) - \phi_2(x)\}\,dx. \tag{1.4}$$

Given any positive ϵ, choose Δ so that $|\phi_1(\Delta)| < \epsilon$. Since $\phi_1(x)$ is a positive non-decreasing bounded function, we see, by the second Mean Value Theorem, that the first part of the above integral is

$$\phi_1(\Delta) \lim_{T\to\infty} T^{p/4-\tau/2} \int_\delta^\Delta x^{p/4-\tau/2-1}J_{p/2+\tau}(2\pi x^{1/2}T^{1/2})\,dx,$$

where, for all T, $0 \leqq \delta \leqq \Delta$. Since $|\phi_1(\Delta)| < \epsilon$ and the integral is bounded as shown above in (1.3), we see that the absolute value of the last expression is less than $A\epsilon$, where A is some constant. Hence, the first expression in (1.4) vanishes. Similarly, the second expression in (1.4) also vanishes.

The integral with the range (Δ, ∞) in the right-hand side of (1.1) yields:

$$\lim_{T\to\infty} \tau T^{-\tau} \int_0^T t^{p/4}(T-t)^{\tau-1}\,dt \int_\Delta^\infty x^{-1/2}g(x)J_{p/2}(2\pi x^{1/2}t^{1/2})\,dx$$
$$= \lim_{T\to\infty} \tau T^{-\tau} \int_\Delta^\infty x^{-1/2}g(x)\,dx \int_0^T t^{p/4}(T-t)^{\tau-1}J_{p/2}(2\pi x^{1/2}t^{1/2})\,dt.$$

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By Sonine's first integral, we obtain

$$\lim_{T\to\infty} O\left(T^{p/4-\tau/2-1/4}\int_\Delta^\infty x^{-(\tau/2+3/4)}g(x)\,dx\right) = \lim_{T\to\infty} O(T^{p/4-\tau/2-1/4}) = 0,$$

for $\tau > (p-1)/2$. Thus, the right-hand side of (1.1) is limitable (R, N, τ) by Riesz means to

$$\pi^{-p/2}\Gamma(p/2)\lim_{x\to+0} x^{1/2-p/4}g(x),$$

as $N\to\infty$ for $\tau > (p-1)/2$. The left-hand side of (1.1) yields:

$$\lim_{T\to\infty}\tau T^{-\tau}\int_0^T (T-t)^{\tau-1}\,dt\int_0^t f(x)x^{p/4-1/2}\,dx$$
$$= \lim_{T\to\infty}\int_0^T f(x)x^{p/4-1/2}\left(1-\frac{x}{T}\right)^\tau dx.$$

Hence, (1.1) reduces to

$$\lim_{T\to\infty}\frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^T x^{p/4-1/2}f(x)\left(1-\frac{x}{T}\right)^\tau dx = \lim_{x\to+0} x^{1/2-p/4}g(x),$$

as required, for $\tau > (p-1)/2$.

Note 1. Consider the formulae (**1**),

$$\lim_{N\to\infty}\left\{\sum_{n=1}^N r_p(n)n^{1/2-p/4}f(n)\left(1-\frac{n}{N}\right)^\tau - \frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^N x^{p/4-1/2}f(x)\left(1-\frac{x}{N}\right)^\tau dx\right\},$$

$$\lim_{N\to\infty}\left\{\sum_{n=1}^N r_p(n)n^{1/2-p/4}g(n)\left(1-\frac{n}{N}\right)^\tau - \frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^N x^{p/4-1/2}g(x)\left(1-\frac{x}{N}\right)^\tau dx\right\},$$

where $r_p(n)$ is the number of ways of expressing n as the sum of squares of p integers and $g(x)$ is the $\pi J_{p/2-1}(2\pi x^{1/2})$-transform, $f(x), f'(x), \ldots, f^{(2\tau-3)}(x)$ are integrals, $f(x), xf'(x), x^2f''(x), \ldots, x^{2\tau-2}f^{(2\tau-2)}(x) \in L^2(0,\infty)$, $\tau > (p-1)/2$. By Theorem 1, with appropriate conditions, the summation formula can be written, formally, as

$$(1.5)\qquad \sum_{n=1}^\infty r_p(n)n^{1/2-p/4}f(n) - \lim_{x\to+0} x^{1/2-p/4}g(x)$$
$$= \sum_{n=1}^\infty r_p(n)n^{1/2-p/4}g(n) - \lim_{x\to+0} x^{1/2-p/4}f(x).$$

Put $p = 1$. Then

$$g(x) = \pi\int_0^\infty f(t)J_{-1/2}(2\pi x^{1/2}t^{1/2})\,dt.$$

Let $(2\pi x)^{1/2} = u$ and $(2\pi t)^{1/2} = v$. Then we obtain

$$u^{1/2}g\left(\frac{u^2}{2\pi}\right) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty v^{1/2}g\left(\frac{v^2}{2\pi}\right)\cos uv\,dv.$$

Similarly,

$$u^{1/2}f\left(\frac{u^2}{2\pi}\right) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty v^{1/2}g\left(\frac{v^2}{2\pi}\right)\cos uv\,dv.$$

Let $F(x) = x^{1/2}f(x^2/2\pi)$ and $G(x) = x^{1/2}g(x^2/2\pi)$. Thus, $F(x)$ and $G(x)$ are Fourier cosine transforms and (1.5) becomes

$$\sum_{n=0}^{\infty} F(\sqrt{(2\pi n)}) = \sum_{n=0}^{\infty} G(\sqrt{(2\pi n)}),$$

which is the Poisson summation formula, where the terms $n = 0$ should be divided equally. Putting $p = 2$ in (1.5) yields:

$$\sum_{n=0}^{\infty} r(n)f(n) = \sum_{n=0}^{\infty} r(n)g(n),$$

where $f(x)$ and $g(x)$ are $\pi J_0(2\pi x^{1/2})$-transforms, and $r(n)$ is the number of solutions of the Diophantine equation $x^2 + y^2 = n$. This is the Hardy-Landau summation formula.

THEOREM 2. *If $g(x) \in L(0, \infty)$ and has a $\pi J_\nu(2\pi x^{1/2})$-transform $f(x)$ and $x^{1/4}g(x)$ is of bounded variation near $x = 0$, then*

$$\lim_{T\to\infty} T^{-(\nu+1/2)/2} \int_0^T x^{\nu/2}f(x)\,dx = \frac{\pi^{-1/2}\Gamma(\nu/2 + 3/4)}{\Gamma(\nu/2 + 5/4)} \lim_{x\to+0} x^{1/4}g(x),$$

where $\nu > -1$.

Proof. Let

$$L(x) = x^{\nu/2}, \quad 0 < x < T,$$
$$= 0, \quad x > T.$$

Its $\pi J_\nu(2\pi x^{1/2})$-transform is given by

$$H(x) = \pi \int_0^T t^{\nu/2} J_\nu(2\pi x^{1/2}t^{1/2})\,dt$$
$$= (2\pi)^{-(\nu+1)}x^{-(\nu+2)/2} \int_0^{2\pi x^{1/2}t^{1/2}} u^{\nu+1}J_\nu(u)\,du$$
$$= T^{(\nu+1)/2}x^{-1/2}J_{\nu+1}(2\pi x^{1/2}T^{1/2}), \qquad \nu > -1.$$

Applying Fubini's theorem,

$$\text{(2.1)} \quad T^{-(\nu+1/2)/2} \int_0^T x^{\nu/2}f(x)\,dx = T^{1/4} \int_0^\infty x^{-1/2}J_{\nu+1}(2\pi x^{1/2}T^{1/2})g(x)\,dx.$$

Split the range of integration on the right-hand side of (2.1) into $(0, \Delta)$ and (Δ, ∞), and let $\lim_{x\to+0} x^{1/4}g(x) = k$, where k is some constant. Then the integral with the range $(0, \Delta)$ can be written as

$$kT^{1/4} \int_0^\Delta x^{-3/4}J_{\nu+1}(2\pi x^{1/2}T^{1/2})\,dx$$
$$+ T^{1/4} \int_0^\Delta x^{-3/4}J_{\nu+1}(2\pi x^{1/2}T^{1/2})(x^{1/4}g(x) - k)\,dx.$$

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The first integral in the above expression yields

$$k\pi^{-1/2}\frac{\Gamma(\nu/2+3/4)}{\Gamma(\nu/2+5/4)} \quad \text{as } T\to\infty,$$

where $\nu > -3/2$. The second integral is

$$T^{1/4}\int_0^{\Delta} x^{-3/4}J_{\nu+1}(2\pi x^{1/2}T^{1/2})\phi(x)\,dx, \tag{2.2}$$

where $\phi(x) = x^{1/4}g(x) - k$.

Choose Δ small enough so that $\phi(x)$ is of bounded variation over $(0, \Delta)$, and therefore can be expressed as a difference of two non-decreasing bounded functions $\phi_1(x)$ and $\phi_2(x)$, say. Then by applying the second Mean Value Theorem, as before, the absolute value of expression (2.2) can be made less than ϵ. Hence,

$$\lim_{T\to\infty} T^{1/4}\int_0^{\Delta} x^{-1/2}J_{\nu+1}(2\pi x^{1/2}T^{1/2})g(x)\,dx = \frac{\pi^{-1/2}\Gamma(\nu/2+3/4)}{\Gamma(\nu/2+5/4)}\lim_{x\to+0} x^{1/4}g(x). \tag{2.3}$$

Next, consider the integral with the range (Δ, ∞). By the asymptotic expansion (**4**) of the Bessel function $J_{\nu+1}$,

$$\begin{aligned} &T^{1/4}\int_{\Delta}^{\infty} x^{-1/2}J_{\nu+1}(2\pi x^{1/2}T^{1/2})g(x)\,dx \\ &= \frac{1}{\pi}\int_{\Delta}^{\infty} x^{-3/4}g(x)\cos(2\pi x^{1/2}T^{1/2} - \nu\pi/2 - 3\pi/4)\,dx + \int_{\Delta}^{\infty} O(T^{-1/2}x^{-5/4}g(x)\,dx \\ &= \frac{1}{\pi}\int_{\Delta}^{\infty} x^{-3/4}g(x)\cos(2\pi\sqrt{(xT)} - \theta)\,dx + O\left(T^{-1/2}\int_0^{\infty} x^{-5/4}g(x)\,dx\right). \end{aligned}$$

Since $g(x) \in L(0, \infty)$, we see that $x^{-3/4}g(x)$, $x^{-5/4}g(x)$, and $x^{-7/4}g(x)$ belong to $L(\Delta, \infty)$, $\Delta > 0$. Hence, all the integrals above tend to zero as $T\to\infty$, the first and the third by virtue of Riemann-Lebesgue theorem (**2**, p. 11), that is,

$$\lim_{T\to\infty} T^{1/4}\int_{\Delta}^{\infty} x^{-1/2}J_{\nu+1}(2\pi\sqrt{(xT)})g(x)\,dx = 0. \tag{2.4}$$

Combining (2.4) and (2.3), we obtain, from (2.1),

$$\lim_{T\to\infty} T^{-(\nu+1/2)/2}\int_0^{T} x^{\nu/2}f(x)\,dx = \frac{\pi^{-1/2}\Gamma(\nu/2+3/4)}{\Gamma(\nu/2+5/4)}\lim_{x\to+0} x^{1/4}g(x),$$

when $\nu > -1$, as required.

Note 1. Letting $u = (2\pi x)^{1/2}$ and $v = (2\pi t)^{1/2}$, the conditions of Theorem 2 become

$$u^{1/2}g\left(\frac{u^2}{2\pi}\right) = \int_0^{\infty} v^{1/2}f\left(\frac{v^2}{2\pi}\right)J_\nu(uv)(uv)^{1/2}\,dv,$$

or,

$$G(u) = \int_0^\infty F(v)(uv)^{1/2} J_\nu(uv)\, dv,$$

where $x^{1/2}g(x^2/2\pi) = G(x)$ and $x^{1/2}f(x^2/2\pi) = F(x)$. Similarly,

$$F(u) = \int_0^\infty G(v)(uv)^{1/2} J_\nu(uv)\, dv.$$

Making the same substitutions in the main result we obtain:

$$\lim_{N\to\infty} N^{-(\nu+1/2)} \int_0^N t^{\nu+1/2} F(t)\, dt = \frac{\Gamma(\nu/2 + 3/4)}{2^{1/2}\Gamma(\nu/2 + 5/4)} \lim_{x\to+0} G(x), \tag{2.5}$$

where $F(x)$ and $G(x)$ are $x^{1/2}J_\nu(x)$-transforms.

Let $\nu = 1/2$. Then (2.5) becomes

$$\lim_{N\to\infty} N^{-1} \int_0^N tF(t)\, dt = \left(\frac{2}{\pi}\right)^{1/2} G(+0),$$

where $F(x)$ and $G(x)$ are Fourier sine transforms.

Example. Let $G(x) = e^{-x}$. Then

$$F(x) = \left(\frac{2}{\pi}\right)^{1/2} \frac{x}{1 + x^2}$$

and $G(0) = 1$.

From (2.5), we obtain

$$\lim_{N\to\infty} T^{-1} \int_0^T \frac{t}{1 + t^2}\, dt = 1.$$

THEOREM 3. *If* $g(x) \in L(0, \infty)$ *and has a sine transform* $f(x)$ *and* $x^{\alpha-1}g(x)$ *is of bounded variation near* $x = 0$, *where* $1 \leqq \alpha < 3$, *then*

$$\lim_{T\to\infty} T^{-\alpha} \int_0^T xf(x)\, dx = \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(2 - \alpha)}{\alpha} \sin\tfrac{1}{2}\alpha\pi \lim_{x\to+0} x^{\alpha-1}g(x).$$

Proof. Let

$$L(x) = x, \quad 0 < x < T,$$
$$= 0, \quad x > T.$$

Then its sine transform

$$H(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^T t \sin xt\, dt$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin xT - xT\cos xT}{x^2}.$$

By Fubini's theorem, as before,

$$T^{-\alpha} \int_0^T xf(x)\, dx = T^{-\alpha}\left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \frac{\sin xT - xT\cos xT}{x^2} g(x)\, dx. \tag{3.1}$$

Split the range of integration on the right-hand side of (3.1) into $(0, \Delta)$ and (Δ, ∞).

Let $\lim_{x\to+0} x^{\alpha-1}g(x) = k$, say. Then the integral with the range $(0, \Delta)$ is:

$$(3.2)\quad \left(\frac{2}{\pi}\right)^{1/2} T^{-\alpha}k \int_0^{\Delta} x^{-(\alpha+1)}(\sin xT - xT\cos xT)\,dx$$
$$+\left(\frac{2}{\pi}\right)^{1/2} T^{-\alpha} \int_0^{\Delta} x^{-(\alpha+1)}(\sin xT - xT\cos xT)(x^{\alpha-1}g(x) - k)\,dx$$
$$= I_1 + I_2, \quad \text{say.}$$

$$\lim_{T\to\infty} I_1 = \left(\frac{2}{\pi}\right)^{1/2} k \lim_{T\to\infty} \int_0^{\Delta T} u^{-(\alpha+1)}(\sin u - u\cos u)\,du.$$

Integrating by parts, note that the integrated terms vanish when $1 < \alpha < 3$, and we obtain (**4**, p. 260),

$$\lim_{T\to\infty} I_1 = \left(\frac{2}{\pi}\right)^{1/2} \frac{k}{\alpha} \int_0^{\infty} \sin u\, u^{1-\alpha}\,du$$
$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(2-\alpha)}{\alpha} \sin\tfrac{1}{2}\alpha\pi \lim_{x\to+0} x^{\alpha-1}g(x).$$

It can be easily seen that when $\alpha = 1$, the above result holds. Hence, the value of I_1 is valid for $1 \leqq \alpha < 3$.

Now choose Δ small, so that $x^{\alpha-1}g(x) - k$ is of bounded variation in $(0, \Delta)$.

By the second Mean Value Theorem, the absolute value of I_2 in (3.2) can be made less than ϵ.

Write the integral, with the range (Δ, ∞), as

$$\left(\frac{2}{\pi}\right)^{1/2} T^{-\alpha}\left\{ \int_0^{\infty} \sin xT \frac{g(x)}{x^2}\,dx - T\int_0^{\infty} \cos xT \frac{g(x)}{x}\,dx\right\}.$$

Since $g(x) \in L(0, \infty)$, we see that $g(x)/x^2$ and $g(x)/x$ belong to $L(\Delta, \infty)$, $\Delta > 0$. By the Riemann-Lebesgue theorem, both the integrals in the last expression vanish as $T \to \infty$ and $\alpha \geqq 1$. Thus, (3.2) reduces to the value

$$\left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(2-\alpha)}{\alpha} \sin\tfrac{1}{2}\alpha\pi \cdot k,$$

as $T \to \infty$.

Hence,

$$\lim_{T\to\infty} T^{-\alpha}\int_0^{T} xf(x)\,dx = \left(\frac{2}{\pi}\right)^{1/2} \frac{\Gamma(2-\alpha)}{\alpha} \sin\tfrac{1}{2}\alpha\pi \lim_{x\to+0} x^{\alpha-1}g(x),$$

where $1 \leqq \alpha < 3$.

Note 2. Put $\alpha = 1$ in the result of the previous theorem and we obtain the following result.

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THEOREM 4. *If* (i) $g(x) \in L(0, \infty)$ *and has a Fourier transform* $f(x)$ *and* (ii) $g(x)$ *is of bounded variation in some neighbourhood of* $x = 0$, *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T xf(x)\, dx = \left(\frac{2}{\pi}\right)^{1/2} g(+0).$$

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