INTEGER POLYGONS OF GIVEN PERIMETER

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Abstract

A classical result of Honsberger states that the number of incongruent triangles with integer sides and perimeter n is the nearest integer to $n^2/48$ (n even) or $(n + 3)^2/48$ (n odd). We solve the analogous problem for m-gons (for arbitrary but fixed $m \geq 3$) and for polygons (with arbitrary number of sides).

1. Introduction and statement of the main results

This article concerns *integer polygons*—polygons whose side lengths are all integers. The study of such polygons dates back at least a few millennia, as the Babylonians and Egyptians were interested in right-angled integer triangles. A comparatively recent 1904 result attributed to Whitworth and Biddle (see [4, page 199]) states that there are exactly five (incongruent) integer triangles with perimeter equal to area. Phelps and Fine [12] showed that there is only one with perimeter equal to twice the area, namely the $(3, 4, 5)$ triangle. Subbarao [14] and Marsden [11] considered analogous problems for other multiples. Related to these, the following question appears to have been first asked (and answered) by Jordan *et al.* [9] in 1979.

QUESTION 1.1. How many incongruent integer triangles have perimeter n?

Several formulations of the answer exist [1, 7, 9, 10, 13] and we believe the most elegant is the following.

THEOREM 1.2 (Honsberger [7]). *The number of incongruent integer triangles with perimeter n is $[n^2/48]$ if n is even or $[(n + 3)^2/48]$ if n is odd.*

Here, $[x]$ denotes the nearest integer to the real number x (if such exists). A number of proofs of Theorem 1.2 have been given (see [5–8]), most of which use recurrence relations and/or generating functions, sometimes ingeniously.

If the word 'triangles' is simply replaced with 'quadrilaterals' in Question 1.1, then the answer is not very interesting: for example, there are infinitely many incongruent
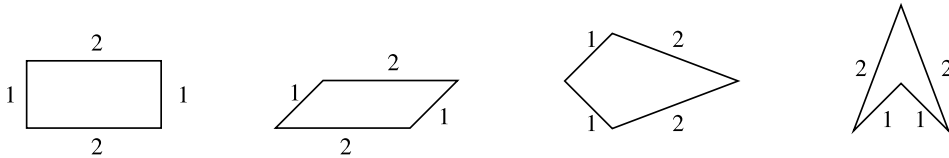
FIGURE 1. Several quadrilaterals with edge lengths $1, 1, 2, 2$.

rhombuses with edges $(1, 1, 1, 1)$. The same is true for pentagons, hexagons and so on. Thus, rather than congruence, we consider a different form of polygon equivalence, defined as follows. Let P and Q be m-gons for some $m \geq 3$, with side lengths $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, respectively, beginning from any side and reading clockwise or anti-clockwise. We say that P and Q are *equivalent* if we may obtain the m-tuple $(b_1, \ldots, b_m)$ from $(a_1, \ldots, a_m)$ by cyclically re-ordering and/or reversing the entries. In Figure 1, for example, the first and second quadrilaterals are equivalent, the third and fourth are equivalent but the first and third are inequivalent.

Thus, we would like to answer the following two questions.

QUESTION 1.3. How many inequivalent integer m-gons have perimeter n?

QUESTION 1.4. How many inequivalent integer polygons have perimeter n?

As far as we are aware, neither question has been answered previously, apart from the $m = 3$ case of Question 1.3 (Theorem 1.2 above); see also [2], which considers integer m-gons up to arbitrary re-orderings of the sides, a different problem for $m \geq 4$. The main purpose of the current article is to answer both Questions 1.3 and 1.4 and we now state the results that do so.

If d and m are integers, we write $d \mid m$ to indicate that d divides m. We write $\lfloor x \rfloor$ for the floor of the real number x, that is, the greatest integer not exceeding x. A binomial coefficient $\binom{n}{k}$ has its usual meaning if n, k are nonnegative integers with $k \leq n$ and is zero otherwise. Finally, φ is Euler's totient function; so, for a positive integer n, $\varphi(n)$ is the size of the set $\{d \in \{1, \ldots, n\} : \gcd(d, n) = 1\}$.

THEOREM 1.5. *If $3 \leq m \leq n$, then the number $p_{m,n}$ of inequivalent integer m-gons with perimeter n is given by*

$$p_{m,n} = \sum_{d \mid \gcd(m,n)} \frac{\varphi(d)}{2n} \binom{\frac{n}{d}}{\frac{m}{d}} + \frac{1}{2} \left(\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n-m}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \right) - \binom{\lfloor \frac{n}{2} \rfloor}{m-1} - \binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}} \right).$$

THEOREM 1.6. *If $n \geq 3$, then the number p_n of inequivalent integer polygons with perimeter n is given by*

$$p_n = \sum_{d \mid n} \frac{\varphi(d) \cdot 2^{n/d-1}}{n} + 2^{\lfloor (n-3)/2 \rfloor} - \begin{cases} 3 \cdot 2^{\lfloor (n-4)/4 \rfloor} & \text{if } n \equiv 0 \text{ or } 1 \ (\text{mod } 4), \\ 2^{\lfloor (n+2)/4 \rfloor} & \text{if } n \equiv 2 \text{ or } 3 \ (\text{mod } 4). \end{cases}$$

Although Theorems 1.5 and 1.6 are not as striking as Honsberger's $n^2/48$ result, we may nevertheless use these theorems to obtain elegant asymptotic formulae.

THEOREM 1.7. *The number of inequivalent integer polygons with perimeter n is asymptotic to $2^{n-1}/n$.*

THEOREM 1.8. *For fixed $m \geq 3$, the number of inequivalent integer m-gons with perimeter n is asymptotic to*

$$\frac{2^{m-1} - m}{2^m m!} n^{m-1}.$$

From Theorem 1.5, we may deduce Honsberger's Theorem 1.2 as a special case. As an additional application, we also give an analogous 'nearest integer formula' for quadrilaterals; again, we are not aware of any previous proof of such a formula.

THEOREM 1.9. *For any positive integer n, the number of inequivalent integer quadrilaterals with perimeter n is $[(n^3 - 3n^2 + 20n)/96]$ if n is even or $[(n^3 - 7n)/96]$ if n is odd.*

Similar formulae could be obtained for pentagons, hexagons and so on, although these quickly become unwieldy. However, Theorem 1.8 easily leads to the asymptotic formulae: $p_{5,n} \sim 11n^4/3840$, $p_{6,n} \sim 13n^5/23040$, $p_{7,n} \sim 19n^6/215040$ and so on.

The proofs of Theorems 1.5 and 1.6 follow the same pattern. In both cases, we (i) identify an action of the dihedral group $\mathcal{D}_n$ (defined below) on a certain set of n-tuples, (ii) show that the polygons in question are in one–one correspondence with the orbits of the action and (iii) enumerate the orbits using Burnside's lemma (stated below). We carry out the first two tasks in Section 2, where we also show how the third reduces to the calculation of certain parameters associated to the elements of $\mathcal{D}_n$. We calculate these parameters (the bulk of the work) in Section 3 and then complete the proofs in Section 4. Section 5 gives the above-mentioned applications to triangles and quadrilaterals, and Section 6 contains tables of calculated values, a discussion of relevant entries in the Online Encyclopedia of Integer Sequences [15] and some concluding remarks.

2. Dihedral group actions

An action of a group G on a set X is a map $G \times X \to X : (g, x) \mapsto g \cdot x$ such that $\mathrm{id}_G \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $x \in X$ and $g, h \in G$. An action induces an equivalence relation on X, namely $\{(x, y) \in X \times X : x = g \cdot y \, (\exists g \in G)\}$, the equivalence classes of which are the *orbits* of the action. The set of orbits is denoted by X/G and the number of orbits is given by Burnside's lemma (see [3, page 246] for a proof).

LEMMA 2.1 (Burnside's lemma). *If a finite group G acts on a set X, then the number of orbits of the action is given by*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} \mathrm{fix}_X(g),$$

where $\mathrm{fix}_X(g)$ is the cardinality of the set $\mathrm{Fix}_X(g) = \{x \in X : g \cdot x = x\}$ for $g \in G$.
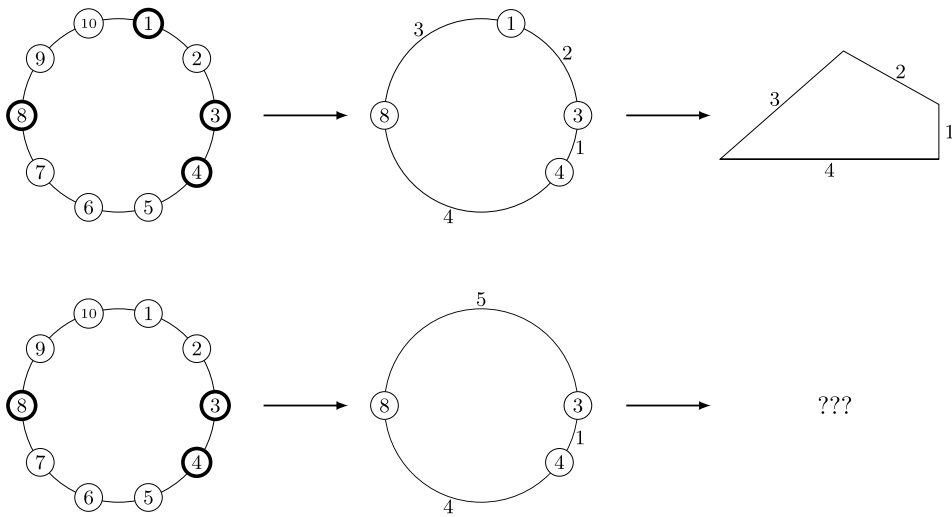
FIGURE 2. Top: the subset $A = \{1, 3, 4, 8\}$ of $\{1, \ldots, 10\}$ corresponds to a quadrilateral with side lengths 1, 2, 3, 4. Bottom: the subset $B = \{3, 4, 8\}$ of $\{1, \ldots, 10\}$ does not correspond to a polygon.

In what follows, n generally denotes the perimeter of an integer polygon. Since a polygon has at least three sides, we will always assume that $n \geq 3$. We denote by $\mathcal{S}_n$ the *symmetric group* on the set $\{1, \ldots, n\}$, which consists of all permutations of this set. We call a permutation $\sigma \in \mathcal{S}_n$ a *rotation* if there exists $q \in \{0, 1, \ldots, n-1\}$ such that $\sigma(x) \equiv q + x \pmod{n}$ for all x; when $q = 0$, we obtain the identity map, id_n. Similarly, σ is a *reflection* if there exists q such that $\sigma(x) \equiv q - x \pmod{n}$ for all x. We write $\mathcal{D}_n$ for the set of all rotations and reflections and C_n for the set of all rotations. These are subgroups of $\mathcal{S}_n$: the *dihedral group* of order $2n$ and the *cyclic group* of order n, respectively. Note that $\mathcal{D}_n$ contains n rotations (including the identity) and n reflections. When n is odd, all n reflections fix a single point from $\{1, \ldots, n\}$; when n is even, $n/2$ of the reflections have two fixed points and $n/2$ have none.

Now consider a circular rope of length n units, with n equally spaced points labelled $1, \ldots, n$, read clockwise. For any subset A of $\{1, \ldots, n\}$, we may attempt to create a polygon out of the rope, with corners at the points from A, by pulling the strings taut between these selected points. With $n = 10$, for example, this is possible for $A = \{1, 3, 4, 8\}$, but not for $B = \{3, 4, 8\}$ (see Figure 2). Obviously, to create a polygon for such a subset A of $\{1, \ldots, n\}$, we would need $|A| \geq 3$, but there is an additional restriction coming from the fact that the sides of a polygon must all be less than half the perimeter. Namely, if $A = \{x_1, \ldots, x_m\}$, where $m \geq 3$ and $x_1 < \cdots < x_m$, then we require

$$\max\{x_2 - x_1, x_3 - x_2, \ldots, x_m - x_{m-1}, n + x_1 - x_m\} < n/2$$

to ensure that the length of any side is less than the combined lengths of the other $m - 1$ sides.

The above ideas are easier to work with when formulated in slightly different terms. We define the set $T = \{0, 1\}$ and denote by T_n the set of all n-tuples over T. We identify a subset A of $\{1, \ldots, n\}$ with an n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in T_n$ in the usual way: we have $a_x = 1 \Leftrightarrow x \in A$. Thus, given $\mathbf{a} \in T_n$, we may attempt to form a polygon of perimeter n as above. It turns out that the n-tuples corresponding to polygons in this way may be described very easily.

By a *block of 0's of length l* of an n-tuple $\mathbf{a} \in T_n$, we mean a sequence of l consecutive entries of $\mathbf{a}$ (possibly 'wrapping around n'), all of which are 0, and not contained in any larger such sequence of consecutive 0's. Thus, for example, the subsets A and B of $\{1, \ldots, 10\}$ defined above (see Figure 2) correspond to the 10-tuples $\mathbf{a} = (1, 0, 1, 1, 0, 0, 0, 1, 0, 0)$ and $\mathbf{b} = (0, 0, 1, 1, 0, 0, 0, 1, 0, 0)$; here, $\mathbf{a}$ has three blocks of 0's, of lengths 1, 2 and 3, while $\mathbf{b}$ has blocks of lengths 3 and 4; one block of $\mathbf{b}$ wraps around.

Writing $k = \lfloor n/2 \rfloor$, so that $n = 2k$ or $2k + 1$, we will call a block of 0's of an n-tuple $\mathbf{a} \in T_n$ *bad* if its length is at least $k - 1$ (n even) or k (n odd). If an n-tuple $\mathbf{a} \in T_n$ had a bad block of 0's, then it would be impossible to form a polygon from $\mathbf{a}$, as the bad block would lead to an edge of length at least $n/2$. Conversely, if $\mathbf{a} \in T_n$ has no bad blocks, then we can form a polygon from $\mathbf{a}$; for this, note that having no bad blocks forces $\mathbf{a}$ to have at least three 1's, since $n \geq 3$. Accordingly, we call an n-tuple *bad* if it has at least one bad block or *good* if it has none, and we write

$$G_n = \{\mathbf{a} \in T_n : \mathbf{a} \text{ is good}\} \quad \text{and} \quad B_n = \{\mathbf{a} \in T_n : \mathbf{a} \text{ is bad}\}.$$

So, G_n consists of the n-tuples that correspond to polygons in the manner described above.

For $\mathbf{a} = (a_1, \ldots, a_n) \in T_n$, we denote by $\Sigma\mathbf{a}$ the integer $a_1 + \cdots + a_n$, which is just the number of 1's in $\mathbf{a}$. For $0 \leq m \leq n$, we write

$$T_{m,n} = \{\mathbf{a} \in T_n : \Sigma\mathbf{a} = m\}$$

for the set of all n-tuples with exactly m 1's and we also write

$$G_{m,n} = G_n \cap T_{m,n} \quad \text{and} \quad B_{m,n} = B_n \cap T_{m,n}$$

for the sets of all good and bad such n-tuples, respectively. So $G_{m,n}$ consists of the n-tuples that correspond to m-gons in the manner described above. Note that $G_{m,n} = \varnothing$ if $m \leq 2$.

There is a natural action of the dihedral group $\mathcal{D}_n$ on T_n given by permuting the coordinates:

$$\sigma \cdot (a_1, \ldots, a_n) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}) \quad \text{for } (a_1, \ldots, a_n) \in T_n \text{ and } \sigma \in \mathcal{D}_n. \quad (2.1)$$

It is easy to see that for any $\mathbf{a} \in T_n$ and $\sigma \in \mathcal{D}_n$,

(i) $\Sigma\mathbf{a} = \Sigma(\sigma \cdot \mathbf{a})$,   (ii) $\mathbf{a} \in G_n \Leftrightarrow \sigma \cdot \mathbf{a} \in G_n$,   (iii) $\mathbf{a} \in B_n \Leftrightarrow \sigma \cdot \mathbf{a} \in B_n$.

It follows from (i) that $T_{m,n}$ is closed under the action of $\mathcal{D}_n$; by (ii) and (iii), so too are the sets G_n and B_n; by combinations of these facts, it follows that the sets $G_{m,n}$ and $B_{m,n}$ are closed as well. Thus, (2.1) defines actions of $\mathcal{D}_n$ on all of the sets T_n, G_n, B_n, $T_{m,n}$, $G_{m,n}$ and $B_{m,n}$.

Clearly, two good n-tuples are in the same orbit of the action if and only if they determine equivalent polygons. Together with Lemma 2.1, and writing p_n (respectively, $p_{m,n}$) for the number of inequivalent integer polygons (respectively, m-gons) of perimeter n,

$$p_n = |G_n/\mathcal{D}_n| = \frac{1}{2n} \sum_{\sigma \in \mathcal{D}_n} \mathrm{fix}_{G_n}(\sigma) \quad \text{and} \quad p_{m,n} = |G_{m,n}/\mathcal{D}_n| = \frac{1}{2n} \sum_{\sigma \in \mathcal{D}_n} \mathrm{fix}_{G_{m,n}}(\sigma). \quad (2.2)$$

Also, for any $\sigma \in \mathcal{D}_n$, $\mathrm{Fix}_{T_n}(\sigma) = \mathrm{Fix}_{G_n}(\sigma) \cup \mathrm{Fix}_{B_n}(\sigma)$, with a similar statement for $\mathrm{Fix}_{T_{m,n}}(\sigma)$. Since G_n and B_n are disjoint,

$$\mathrm{fix}_{G_n}(\sigma) = \mathrm{fix}_{T_n}(\sigma) - \mathrm{fix}_{B_n}(\sigma) \quad \text{and} \quad \mathrm{fix}_{G_{m,n}}(\sigma) = \mathrm{fix}_{T_{m,n}}(\sigma) - \mathrm{fix}_{B_{m,n}}(\sigma) \quad (2.3)$$

for any $\sigma \in \mathcal{D}_n$. Equations (2.2) and (2.3) form the basis of our calculation of p_n and $p_{m,n}$. We use (2.3) to calculate the values of $\mathrm{fix}_{G_n}(\sigma)$ and $\mathrm{fix}_{G_{m,n}}(\sigma)$ in Section 3, before showing in Section 4 that these, together with (2.2), yield the formulae for p_n and $p_{m,n}$ stated in Theorems 1.5 and 1.6.

3. Fix sets

The previous section reduced the enumeration of integer polygons of given perimeter to the calculation of sizes of fix sets under the action (2.1). Each result of this section gives values for $\mathrm{fix}_{G_n}(\sigma)$ and $\mathrm{fix}_{G_{m,n}}(\sigma)$ for various elements $\sigma \in \mathcal{D}_n$. In principle, it would be possible to derive the former from the latter by summing over m. However, this is easier said than done in most cases; in any event, both values may be calculated with essentially the same argument, so this is the approach we take.

In all the proofs that follow, if $\mathbf{a} \in T_r$ for some r, we generally assume that $\mathbf{a} = (a_1, \ldots, a_r)$. We also remind the reader of the convention regarding binomial coefficients being zero if the arguments fall outside of the usual ranges. We begin with the identity element.

LEMMA 3.1. *If $n \geq 3$, then:*

(i) $\quad \mathrm{fix}_{G_n}(\mathrm{id}_n) = 2^n - 1 - n \cdot 2^{\lfloor n/2 \rfloor} + \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd;} \end{cases}$

(ii) $\quad \mathrm{fix}_{G_{m,n}}(\mathrm{id}_n) = \binom{n}{m} - n\binom{\lfloor \frac{n}{2} \rfloor}{m-1}$ *for any m with $3 \leq m \leq n$.*

PROOF. We write $k = \lfloor n/2 \rfloor$ throughout the proof, so that $n = 2k$ or $2k + 1$.

(i) Clearly, $\mathrm{fix}_{G_n}(\mathrm{id}_n) = |G_n| = |T_n| - |B_n|$. Since $|T_n| = 2^n$, it suffices to show that

$$|B_n| = 1 + n \cdot 2^k + \begin{cases} -k & \text{if } n = 2k \text{ is even,} \\ 0 & \text{if } n = 2k + 1 \text{ is odd.} \end{cases}$$

Certainly, $(0,\ldots,0) \in B_n$, so it remains to count the nonzero n-tuples from B_n. By definition, every n-tuple from B_n has at least one bad block of 0's. In fact, it is only possible for an n-tuple to have two bad blocks if n is even, in which case there are exactly $n/2$ such n-tuples (these have two 1's equally spaced around the circle). For $i \in \{1,\ldots,n\}$, write π_i for the number of nonzero n-tuples $\mathbf{a}$ from T_n that have a bad block beginning at position i: that is, reading subscripts modulo n,

$$a_{i-1} = 1 \quad \text{and} \quad a_i = a_{i+1} = \cdots = a_{i+l-1} = 0, \quad \text{where } l = k - 1 \text{ (n even) or } l = k \text{ (n odd)}.$$

Since the remaining $n - (l+1) = k$ entries $a_{i+l},\ldots,a_{i-2}$ of such an n-tuple may be chosen arbitrarily from T, it follows that $\pi_i = 2^k$ for all i. Then $\pi_1 + \cdots + \pi_n = n \cdot 2^k$ counts the nonzero n-tuples with one bad block once, but double-counts the nonzero n-tuples with two bad blocks. We have already noted that there are none of the latter if n is odd or $n/2 = k$ of them if n is even. The result quickly follows.

(ii) This time, $\text{fix}_{G_{m,n}}(\text{id}_n) = |G_{m,n}| = |T_{m,n}| - |B_{m,n}| = \binom{n}{m} - |B_{m,n}|$. Every element of $B_{m,n}$ has exactly one bad block of 0's (since $m \geq 3$). As in the proof of (i), one may show that there are $\binom{k}{m-1} = \binom{\lfloor n/2 \rfloor}{m-1}$ elements of $B_{m,n}$ with a bad block beginning at position i and the result quickly follows.   □

Next we consider the nontrivial rotations. Recall that the *order* of an element $\sigma \in \mathcal{D}_n$ is the least positive integer d such that $\sigma^d = \text{id}_n$; the order of a rotation is a divisor of n and, conversely, any such divisor can occur as the order of a rotation.

Lemma 3.2. *If $n \geq 3$ and if $\sigma \in \mathcal{D}_n$ is a rotation of order d, where $1 \neq d \mid n$, then:*

(i) $\quad \text{fix}_{G_n}(\sigma) = 2^{n/d} - 1 + \begin{cases} -n/2 & \text{if } d = 2, \\ 0 & \text{if } d \geq 3; \end{cases}$

(ii) $\quad \text{fix}_{G_{m,n}}(\sigma) = \binom{n/d}{m/d} \quad \text{for any m with } 3 \leq m \leq n.$

Proof. Let $e = n/d$. Since all rotations of order d are powers of each other, each such rotation fixes the same n-tuples from G_n and $G_{m,n}$. Thus, we may assume that $\sigma(x) \equiv x + e \pmod{n}$ for all x (see Figure 3). For use in both parts of the proof, we define a map $f : T_e \to T_n$ by $f(a_1,\ldots,a_e) = (a_1,\ldots,a_e,a_1,\ldots,a_e,\ldots,a_1,\ldots,a_e)$.

(i) It is easy to see that the set $\text{Fix}_{T_n}(\sigma)$ is precisely the image of f (see Figure 3). Since f is clearly injective, it immediately follows that $\text{fix}_{T_n}(\sigma) = |T_e| = 2^e = 2^{n/d}$. By (2.3), it remains to show that

$$\text{fix}_{B_n}(\sigma) = 1 + \begin{cases} n/2 & \text{if } d = 2, \\ 0 & \text{if } d \geq 3. \end{cases}$$

Clearly, $(0,\ldots,0) \in \text{Fix}_{B_n}(\sigma)$. If $\mathbf{a} \in T_e$ is nonzero, then the longest block of 0's in $f(\mathbf{a})$ has length at most $e - 1 = n/d - 1$ (see Figure 3). If $d > 2$, then

$$\frac{n}{d} - 1 < \frac{n}{2} - 1 = \begin{cases} k - 1 & \text{if } n = 2k \text{ is even,} \\ k - \frac{1}{2} & \text{if } n = 2k + 1 \text{ is odd} \end{cases}$$
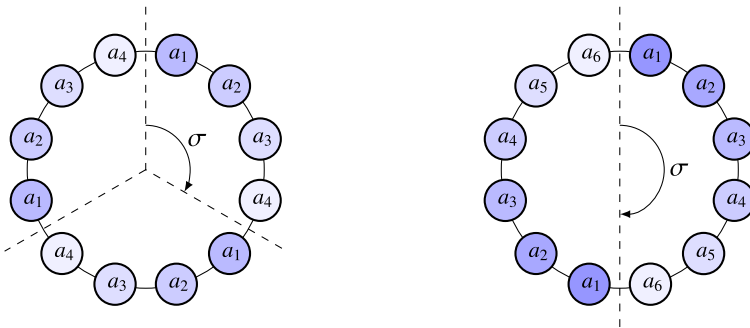
FIGURE 3. The rotation $\sigma \in \mathcal{D}_n$ and the n-tuple $f(\mathbf{a}) = (a_1, \ldots, a_e, a_1, \ldots, a_e, \ldots, a_1, \ldots, a_e) \in \text{Fix}_{T_n}(\sigma)$, from the proof of Lemma 3.2, in the case $n = 12$ and $d = 3$ (left) and $d = 2$ (right).

and so $f(\mathbf{a})$ belongs to G_n for any nonzero $\mathbf{a} \in T_e$. It follows that $\text{fix}_{B_n}(\sigma) = 1$ if $d > 2$. If $d = 2$, then $n = 2k$ must be even; it is possible for $\mathbf{a} \in T_e = T_k$ to be nonzero, but to have $f(\mathbf{a}) \in B_n$; this occurs when $\mathbf{a}$ has exactly one nonzero entry (see Figure 3, right). It follows that $\text{fix}_{B_n}(\sigma) = 1 + k = 1 + n/2$ if $d = 2$.

(ii) With the above notation, $\text{Fix}_{T_{m,n}} = \{f(\mathbf{a}) : \mathbf{a} \in T_{m/d,n/d}\}$ and so $\text{fix}_{T_{m,n}}(\sigma) = |T_{m/d,n/d}| = \binom{n/d}{m/d}$. In the proof of (i), we showed that any nonzero elements of $\text{Fix}_{B_n}(\sigma)$ have exactly two 1's and hence do not belong to $B_{m,n}$ (since $m \geq 3$). It follows that $\text{fix}_{B_{m,n}}(\sigma) = 0$. Because of (2.3), this completes the proof.   □

For the reflections, we need to consider separate cases according to the parity of n and according to the number of fixed points in the case of even n. Although the details of the proofs vary, the strategy is essentially the same in each case; we show that an n-tuple fixed by a reflection is uniquely determined by (roughly) half of its entries and identify the properties of these entries that separate the good n-tuples from the bad. We begin with the case of odd n. Recall that here there are n reflections, each of which fixes a single point of $\{1, \ldots, n\}$.

LEMMA 3.3. *If $n \geq 3$ is odd and $\sigma \in \mathcal{D}_n$ is a reflection, then:*

(i) $\text{fix}_{G_n}(\sigma) = \begin{cases} 2^{(n+1)/2} - 3 \cdot 2^{(n-1)/4} + 1 & \text{if } n \equiv 1 \ (\text{mod } 4), \\ 2^{(n+1)/2} - 2^{(n+5)/4} + 1 & \text{if } n \equiv 3 \ (\text{mod } 4); \end{cases}$

(ii) $\text{fix}_{G_{m,n}}(\sigma) = \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}}$ *for any m with $3 \leq m \leq n$.*

PROOF. Write $n = 2k + 1$ and let $l = \lfloor k/2 \rfloor$, so that $k = 2l$ or $2l + 1$. All reflections fix the same number of n-tuples from G_n and $G_{m,n}$ (and from T_n, B_n etc.), so we may assume that the fixed point of σ is n, in which case $\sigma(x) \equiv n - x \ (\text{mod } n)$ for all x (see Figure 4). This time, we define a map $f : T_{k+1} \to T_n$ by $f(a_1, \ldots, a_{k+1}) = (a_1, \ldots, a_k, a_k, \ldots, a_1, a_{k+1})$.
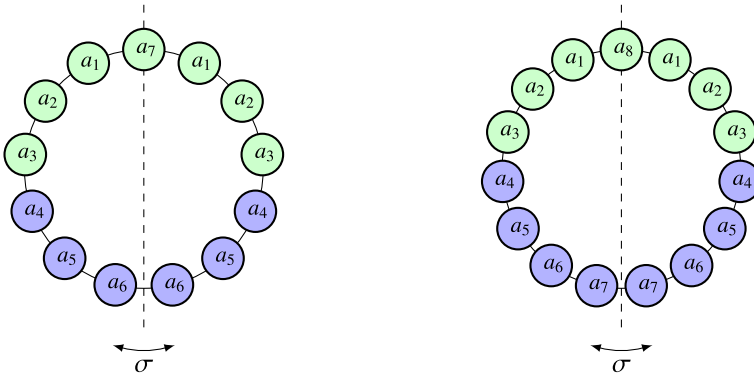
FIGURE 4. The reflection $\sigma \in \mathcal{D}_n$ and the n-tuple $f(\mathbf{a}) = (a_1, \ldots, a_k, a_k, \ldots, a_1, a_{k+1}) \in \text{Fix}_{T_n}(\sigma)$, from the proof of Lemma 3.3, in the cases $n = 13$ (left) and $n = 15$ (right). If condition (a) or (b) from the proof holds, then the green (light) or blue (dark) vertices, respectively, yield a bad block of 0's in $f(\mathbf{a})$. If neither (a) nor (b) holds, then $f(\mathbf{a})$ has no bad blocks.

(i) As in the previous proof, f is injective and its image is $\text{Fix}_{T_n}(\sigma)$. Thus, $\text{fix}_{T_n}(\sigma) = 2^{k+1} = 2^{(n+1)/2}$. By (2.3), it remains to show that

$$\text{fix}_{B_n}(\sigma) = \begin{cases} 3 \cdot 2^{(n-1)/4} - 1 & \text{if } k = 2l, \\ 2^{(n+5)/4} - 1 & \text{if } k = 2l + 1. \end{cases} \tag{3.1}$$

To do so, consider some n-tuple $f(\mathbf{a}) \in \text{Fix}_{T_n}(\sigma)$, where $\mathbf{a} \in T_{k+1}$. Then $f(\mathbf{a}) \in B_n$ if and only if at least one of the following holds (see Figure 4):

(a) $(a_1, \ldots, a_l, a_{k+1}) = (0, \ldots, 0)$ or   (b) $(a_{l+1}, \ldots, a_k) = (0, \ldots, 0)$.

There are 2^{k-l} $(k+1)$-tuples $\mathbf{a}$ satisfying (a), 2^{l+1} satisfying (b) and one satisfying both (a) and (b). It follows that $\text{fix}_{B_n}(\sigma) = 2^{k-l} + 2^{l+1} - 1$; this reduces to the expressions stated in (3.1), by checking separate cases for $k = 2l$ or $2l + 1$.

(ii) Now consider an element $f(\mathbf{a}) \in \text{Fix}_{T_{m,n}}(\sigma)$, where $\mathbf{a} \in T_{k+1}$. By the form of $f(\mathbf{a})$ and since n is the unique point of $\{1, \ldots, n\}$ fixed by σ, we must have

$$a_{k+1} = \begin{cases} 0 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

The remaining entries $a_1, \ldots, a_k$ of $\mathbf{a}$ may be chosen arbitrarily from T, as long as $\lfloor m/2 \rfloor$ of them are 1's. Thus, $\text{fix}_{T_{m,n}}(\sigma) = \binom{k}{\lfloor m/2 \rfloor} = \binom{\lfloor n/2 \rfloor}{\lfloor m/2 \rfloor}$. By (2.3), it remains to show that $\text{fix}_{B_{m,n}}(\sigma) = \binom{\lfloor n/4 \rfloor}{\lfloor m/2 \rfloor} + \binom{\lfloor (n+2)/4 \rfloor}{m/2}$. Now, $f(\mathbf{a})$ belongs to B_n if and only if one of (a) or (b) holds, as enumerated in the proof of (i). Condition (b) holds if and only if $\lfloor m/2 \rfloor$ of the entries $a_1, \ldots, a_l$ are 1's (recall that a_{k+1} is already fixed), so there are $\binom{l}{\lfloor m/2 \rfloor} = \binom{\lfloor n/4 \rfloor}{\lfloor m/2 \rfloor}$ $(k+1)$-tuples $\mathbf{a}$, with $f(\mathbf{a}) \in B_{m,n}$, satisfying (b). For $\mathbf{a}$ to satisfy (a), m must be even (since $a_{k+1} = 1$ if m is odd). In this case, $m/2$ of $a_{l+1}, \ldots, a_k$ must be 1's;
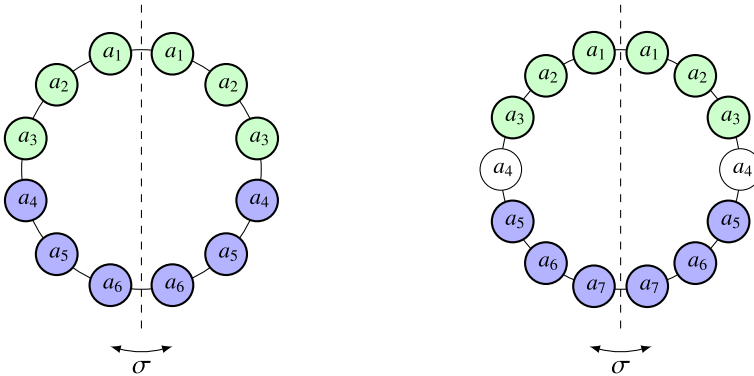
FIGURE 5. The reflection $\sigma \in \mathcal{D}_n$ and the n-tuple $f(\mathbf{a}) = (a_1, \ldots, a_k, a_k, \ldots, a_1) \in \mathrm{Fix}_{T_n}(\sigma)$, from the proof of Lemma 3.4, in the cases $n = 12$ (left) and $n = 14$ (right). If condition (a) or (b) from the proof is satisfied, then the green (light) or blue (dark) vertices, respectively, yield a bad block of 0's in $f(\mathbf{a})$. If neither (a) nor (b) holds, then $f(\mathbf{a})$ has no bad blocks.

it follows that there are $\binom{k-l}{m/2} = \binom{\lfloor (m+2)/4 \rfloor}{m/2}$ $(k+1)$-tuples $\mathbf{a}$ satisfying (a) when m is even; this is also true when m is odd, since then $m/2$ is not an integer. Since no $\mathbf{a}$ can satisfy both (a) and (b), as $m \geq 3$, it follows that $\mathrm{fix}_{B_{m,n}}(\sigma) = \binom{\lfloor n/4 \rfloor}{\lfloor m/2 \rfloor} + \binom{\lfloor (n+2)/4 \rfloor}{m/2}$. □

Recall that there are two different kinds of reflections when n is even: $n/2$ fixing no points of $\{1, \ldots, n\}$ and $n/2$ fixing two. We consider these separately.

LEMMA 3.4. *If $n \geq 4$ is even, and if $\sigma \in \mathcal{D}_n$ is a reflection with no fixed points, then:*

(i) $\mathrm{fix}_{G_n}(\sigma) = \begin{cases} 2^{n/2} - 2^{(n+4)/4} + 1 & \text{if } n \equiv 0 \ (\mathrm{mod} \ 4), \\ 2^{n/2} - 2^{(n+6)/4} + 2 & \text{if } n \equiv 2 \ (\mathrm{mod} \ 4); \end{cases}$

(ii) $\mathrm{fix}_{G_{m,n}}(\sigma) = \binom{\frac{n}{2}}{\frac{m}{2}} - 2\binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}}$ *for any m with $3 \leq m \leq n$.*

PROOF. Write $n = 2k$ and let $l = \lfloor k/2 \rfloor$. We may assume that $\sigma(x) \equiv n + 1 - x \ (\mathrm{mod} \ n)$ for all x (see Figure 5). Let $f : T_k \to T_n$ by $f(a_1, \ldots, a_k) = (a_1, \ldots, a_k, a_k, \ldots, a_1) \in T_n$.

(i) Again f is injective and has image $\mathrm{Fix}_{T_n}(\sigma)$, so that $\mathrm{fix}_{T_n}(\sigma) = 2^k = 2^{n/2}$. Thus, again by (2.3), it remains to show that

$$\mathrm{fix}_{B_n}(\sigma) = \begin{cases} 2^{(n+4)/4} - 1 & \text{if } k = 2l, \\ 2^{(n+6)/4} - 2 & \text{if } k = 2l + 1. \end{cases} \tag{3.2}$$

To do so, consider some n-tuple $f(\mathbf{a}) \in \mathrm{Fix}_{T_n}(\sigma)$, where $\mathbf{a} \in T_k$. Then $f(\mathbf{a}) \in B_n$ if and only if at least one of the following holds (see Figure 5):

(a) $(a_1, \ldots, a_l) = (0, \ldots, 0)$ or   (b) $(a_{k-l+1}, \ldots, a_k) = (0, \ldots, 0)$.

There are 2^{k-l} k-tuples $\mathbf{a}$ satisfying (a) and also 2^{k-l} satisfying (b). Only one $\mathbf{a}$ satisfies both (a) and (b) if $k = 2l$, but there are two such $\mathbf{a}$ if $k = 2l + 1$ (compare the left and right diagrams in Figure 5). It follows that $\mathrm{fix}_{B_n}(\sigma) = 2 \cdot 2^{k-l} - (1 \text{ or } 2)$, as appropriate, which reduces to (3.2).

(ii) Now, $\mathrm{Fix}_{T_{m,n}}(\sigma)$ is empty if m is odd (see Figure 5), in which case the stated formula for $\mathrm{fix}_{G_{m,n}}(\sigma)$ holds, as $m/2$ is not an integer. From now on, we assume that $m = 2h$ is even. Clearly, $\mathrm{fix}_{T_{m,n}}(\sigma) = \binom{k}{h}$. If $\mathbf{a} \in T_k$ is such that $f(\mathbf{a})$ belongs to $B_{m,n}$, then exactly one of (a) or (b) holds, as above. There are $\binom{k-l}{h}$ k-tuples $\mathbf{a}$ satisfying (a) and the same number satisfying (b). Thus, by (2.3),

$$\mathrm{fix}_{G_{m,n}}(\sigma) = \mathrm{fix}_{T_{m,n}}(\sigma) - \mathrm{fix}_{B_{m,n}}(\sigma) = \binom{k}{h} - 2\binom{k-l}{h} = \binom{\frac{n}{2}}{\frac{m}{2}} - 2\binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}}. \qquad \square$$

For the final lemma, we must consider separate cases for $\mathrm{fix}_{G_{m,n}}(\sigma)$ according to the parity of m.

LEMMA 3.5. *If $n \geq 4$ is even, and if $\sigma \in \mathcal{D}_n$ is a reflection with two fixed points, then:*

(i) $\quad \mathrm{fix}_{G_n}(\sigma) = \begin{cases} 2^{(n+2)/2} - 2^{(n+8)/4} + 1 & \text{if } n \equiv 0 \pmod 4, \\ 2^{(n+2)/2} - 2^{(n+6)/4} & \text{if } n \equiv 2 \pmod 4; \end{cases}$

(ii) $\quad \mathrm{fix}_{G_{m,n}}(\sigma) = \begin{cases} \binom{\frac{n}{2}}{\frac{m}{2}} - 2\binom{\lfloor \frac{n}{4} \rfloor}{\frac{m}{2}} & \text{if } m \geq 3 \text{ is even,} \\ 2\binom{\frac{n}{2}-1}{\lfloor \frac{m}{2} \rfloor} - 2\binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} & \text{if } m \geq 3 \text{ is odd.} \end{cases}$

PROOF. Write $n = 2k$ and let $l = \lfloor k/2 \rfloor$. This time we may assume that the fixed points of σ are 1 and $k + 1$, in which case $\sigma(x) \equiv n + 2 - x \pmod n$ for all x (see Figure 6). Define a map $f : T_{k+1} \to T_n$ by $f(a_1, \ldots, a_{k+1}) = (a_1, a_2, \ldots, a_k, a_{k+1}, a_k, \ldots, a_2)$.

(i) As usual, f is injective and has image $\mathrm{Fix}_{T_n}(\sigma)$, so that $\mathrm{fix}_{T_n}(\sigma) = 2^{k+1} = 2^{(n+2)/2}$. Thus, by (2.3), it remains to show that

$$\mathrm{fix}_{B_n}(\sigma) = \begin{cases} 2^{(n+8)/4} - 1 & \text{if } k = 2l, \\ 2^{(n+6)/4} & \text{if } k = 2l + 1. \end{cases} \tag{3.3}$$

To do so, consider some n-tuple $f(\mathbf{a}) \in \mathrm{Fix}_{T_n}(\sigma)$, where $\mathbf{a} \in T_{k+1}$. Then $f(\mathbf{a}) \in B_n$ if and only if at least one of the following holds (see Figure 6):

(a) $(a_1, \ldots, a_{k-l}) = (0, \ldots, 0)$,   (b) $(a_{l+2}, \ldots, a_{k+1}) = (0, \ldots, 0)$,
(c) $\mathbf{a} = (1, 0, \ldots, 0, 1)$.

For $k = 2l$ or $2l + 1$, respectively, there are $2 \cdot 2^{l+1} - (2 \text{ or } 1)$ elements $\mathbf{a} \in T_{k+1}$ satisfying (a) or (b). Since $\mathbf{a} = (1, 0, \ldots, 0, 1)$ satisfies neither (a) nor (b), it follows that $\mathrm{fix}_{B_n}(\sigma) = 2^{l+1} - (2 \text{ or } 1) + 1$, which reduces to (3.3).
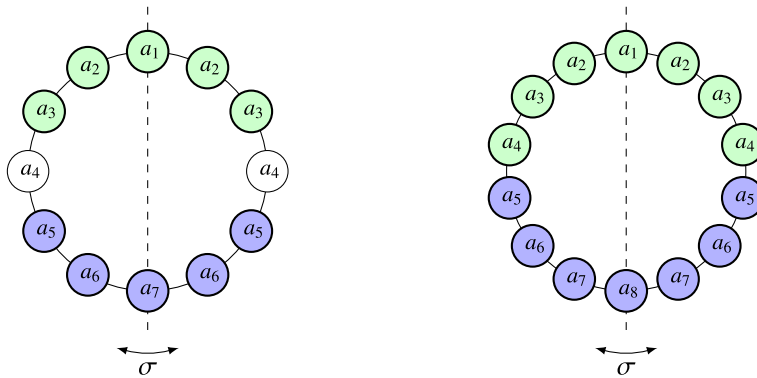
FIGURE 6. The reflection $\sigma \in \mathcal{D}_n$ and the n-tuple $f(\mathbf{a}) = (a_1, a_2, \ldots, a_k, a_{k+1}, a_k, \ldots, a_2) \in \mathrm{Fix}_{T_n}(\sigma)$, from the proof of Lemma 3.5, in the cases $n = 12$ (left) and $n = 14$ (right). If condition (a) or (b) from the proof is satisfied, then the green (light) or blue (dark) vertices, respectively, yield a bad block of 0's in $f(\mathbf{a})$. If neither (a) nor (b) holds, then $f(\mathbf{a})$ has no bad blocks.

(ii) Write $h = \lfloor m/2 \rfloor$ and consider some n-tuple $f(\mathbf{a}) \in \mathrm{Fix}_{T_{m,n}}(\sigma)$, where $\mathbf{a} \in T_{k+1}$. If $m = 2h + 1$ is odd, then exactly one of a_1, a_{k+1} must be 1 and h of the remaining entries $a_2, \ldots, a_k$ must be 1's. If $m = 2h$ is even, then either $a_1 = a_{k+1} = 1$ and $h - 1$ of $a_2, \ldots, a_k$ are 1's, or else $a_1 = a_{k+1} = 0$ and h of $a_2, \ldots, a_k$ are 1's. Thus,

$$\mathrm{fix}_{T_{m,n}}(\sigma) = \begin{cases} \binom{k-1}{h-1} + \binom{k-1}{h} = \binom{k}{h} & \text{if } m = 2h \text{ is even,} \\ 2\binom{k-1}{h} & \text{if } m = 2h + 1 \text{ is odd.} \end{cases}$$

By (2.3), and since $k = n/2$ and $h = \lfloor m/2 \rfloor$, it remains to show that $\mathrm{fix}_{B_{m,n}}(\sigma) = 2\binom{\lfloor n/4 \rfloor}{\lfloor m/2 \rfloor}$. To do so, consider an n-tuple $f(\mathbf{a}) \in \mathrm{Fix}_{B_{m,n}}(\sigma)$, where $\mathbf{a} \in T_{k+1}$. So, $\mathbf{a}$ satisfies either (a) or (b), as above, and these are mutually exclusive (since $m \geq 3$); note that (c) cannot hold (also since $m \geq 3$). Note that $\mathbf{a}$ satisfies (a) if and only if $a_{k+1} = 0$ or 1 (for $m = 2h$ or $2h + 1$, respectively), and h of $a_{k-l+1}, \ldots, a_k$ are 1's. There are thus $\binom{l}{h} = \binom{\lfloor n/4 \rfloor}{\lfloor m/2 \rfloor}$ such $(k + 1)$-tuples $\mathbf{a}$ satisfying (a) and there are the same number satisfying (b). □

4. Proofs of the main results

We can now complete the proofs of our main results, as stated in Section 1.

PROOF OF THEOREM 1.5. As discussed in Section 2, the proof uses Burnside's lemma. Specifically, Equation (2.2) says that $p_{m,n} = 1/(2n) \sum_{\sigma \in \mathcal{D}_n} \mathrm{fix}_{G_{m,n}}(\sigma)$. From Lemmas 3.1 and 3.2 and the fact that the cyclic group C_n has $\varphi(d)$ elements of order d if $d \mid n$, the contribution to (2.2) made by the rotations from $\mathcal{D}_n$ is

$$\frac{1}{2n} \sum_{\sigma \in C_n} \mathrm{fix}_{G_{m,n}}(\sigma) = \frac{1}{2n} \left(\binom{n}{m} - n\binom{\lfloor \frac{n}{2} \rfloor}{m-1} + \sum_{1 \neq d \mid n} \varphi(d) \binom{\frac{n}{d}}{\frac{m}{d}} \right) = \sum_{d \mid n} \frac{\varphi(d)}{2n} \binom{\frac{n}{d}}{\frac{m}{d}} - \frac{1}{2} \binom{\lfloor \frac{n}{2} \rfloor}{m-1}.$$

Since $\binom{n/d}{m/d} = 0$ if $d \nmid m$, '$\sum_{d|n}$' may be replaced by '$\sum_{d|\gcd(m,n)}$'. It therefore remains to show that the contribution to (2.2) made by the reflections from $\mathcal{D}_n$ is

$$\frac{1}{2n} \sum_{\sigma \in \mathcal{D}_n \setminus C_n} \text{fix}_{G_{m,n}}(\sigma) = \frac{1}{2} \left(\binom{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n-m}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} \right) - \binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}} \right).$$

If n is odd, then $\mathcal{D}_n$ has n reflections; if n is even, then $\mathcal{D}_n$ has $n/2$ reflections with no fixed points and $n/2$ with two fixed points. Combining Lemmas 3.3, 3.4 and 3.5, and keeping in mind that $\binom{\lfloor (n+2)/4 \rfloor}{m/2} = 0$ if m is odd, we calculate case by case that

$$\frac{1}{2n} \sum_{\sigma \in \mathcal{D}_n \setminus C_n} \text{fix}_{G_{m,n}}(\sigma) = \begin{cases} \frac{1}{2} \left(\binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}} \right) & \text{if } n \text{ is odd,} \\[2mm] \frac{1}{2} \left(\binom{\frac{n}{2}}{\frac{m}{2}} - \binom{\lfloor \frac{n}{4} \rfloor}{\frac{m}{2}} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}} \right) & \text{if } n \text{ and } m \text{ are both even,} \\[2mm] \frac{1}{2} \left(\binom{\frac{n}{2} - 1}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n}{4} \rfloor}{\lfloor \frac{m}{2} \rfloor} - \binom{\lfloor \frac{n+2}{4} \rfloor}{\frac{m}{2}} \right) & \text{if } n \text{ is even and } m \text{ is odd.} \end{cases}$$

It finally remains to observe that $\lfloor m/2 \rfloor + \lfloor (n - m)/2 \rfloor$ is equal to $n/2 - 1$ if n is even and m is odd, or to $\lfloor n/2 \rfloor$ otherwise; this may be checked on a case-by-case basis. □

PROOF OF THEOREM 1.6. With some careful algebra, (2.2) and Lemmas 3.1–3.5 yield

$$p_n = \sum_{d|n} \frac{\varphi(d)}{2n} (2^{n/d} - 1) - 2^{\lfloor n/2 \rfloor - 1} + \begin{cases} 3 \cdot 2^{(n-4)/2} - 3 \cdot 2^{(n-4)/4} + \frac{1}{2} & \text{if } n \equiv 0 \pmod 4, \\ 2^{(n-1)/2} - 3 \cdot 2^{(n-5)/4} + \frac{1}{2} & \text{if } n \equiv 1 \pmod 4, \\ 3 \cdot 2^{(n-4)/2} - 2^{(n+2)/4} + \frac{1}{2} & \text{if } n \equiv 2 \pmod 4, \\ 2^{(n-1)/2} - 2^{(n+1)/4} + \frac{1}{2} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

(4.1)

(Note that the '$n/2$' from the even case of Lemma 3.1(i) cancels out with the '$n/2$' from the $d = 2$ case of Lemma 3.2(i).) Using the identity $\sum_{d|n} \varphi(d) = n$,

$$\sum_{d|n} \frac{\varphi(d)}{2n} (2^{n/d} - 1) = \sum_{d|n} \frac{\varphi(d) \cdot 2^{n/d}}{2n} - \frac{1}{2n} \sum_{d|n} \varphi(d) = \sum_{d|n} \frac{\varphi(d) \cdot 2^{n/d-1}}{n} - \frac{1}{2}.$$

The '$1/2$' here cancels those in (4.1). If n is even, then $-2^{\lfloor n/2 \rfloor - 1} + 3 \cdot 2^{(n-4)/2} = -2 \cdot 2^{(n-4)/2} + 3 \cdot 2^{(n-4)/2} = 2^{(n-4)/2} = 2^{\lfloor (n-3)/2 \rfloor}$. A similar calculation shows that if n is odd, then $-2^{\lfloor n/2 \rfloor - 1} + 2^{(n-1)/2} = 2^{(n-3)/2} = 2^{\lfloor (n-3)/2 \rfloor}$. □

PROOF OF THEOREM 1.7. The dominant term in Theorem 1.6 is the $d = 1$ term of the sum, that is, $2^{n-1}/n$. All other terms (of which there are at most n) are at most a constant multiple of $2^{n/2}$. □

The formula $p_n \sim 2^{n-1}/n = 2^n/(2n)$ given in Theorem 1.7 can be interpreted as saying that: (i) practically all of the 2^n subsets of $\{1, \ldots, n\}$ correspond to polygons (see Lemma 3.1(i)); (ii) practically all such polygons are completely asymmetric, so that each equivalence class of polygons is counted approximately $2n$ times.

PROOF OF THEOREM 1.8. Note that for integers $0 \leq k \leq x$, the binomial coefficient $\binom{x}{k}$ is a polynomial of degree k in x; thus, it is asymptotic to its leading term $x^k/k!$. Examining the expression for $p_{m,n}$ in Theorem 1.5, there are two dominant terms: the $d = 1$ term of the sum, and the second bracketed binomial coefficient, both of which are polynomials in n of degree $m - 1$. Thus, for fixed m and as $n \to \infty$,

$$p_{m,n} \sim \frac{1}{2n}\binom{n}{m} - \frac{1}{2}\binom{\lfloor \frac{n}{2} \rfloor}{m-1} \sim \frac{1}{2n} \cdot \frac{n^m}{m!} - \frac{1}{2} \cdot \frac{n^{m-1}/2^{m-1}}{(m-1)!}$$

$$= \frac{n^{m-1}}{2m!} - \frac{n^{m-1}}{2^m(m-1)!} = \frac{2^{m-1} - m}{2^m m!}n^{m-1}. \qquad \square$$

5. Triangles and quadrilaterals

Specialising Theorem 1.5 to the cases $m = 3$ and $m = 4$ allows us to recover Honsberger's theorem on triangles (stated in Theorem 1.2 above) and also the new result (Theorem 1.9) on quadrilaterals.

PROOF OF THEOREM 1.2. By Theorem 1.5, the number of inequivalent (that is, incongruent) integer triangles of perimeter n is

$$p_{3,n} = \frac{1}{2n}\binom{n}{3} + \frac{1}{n}\binom{\frac{n}{3}}{1} + \frac{1}{2}\left(\binom{1 + \lfloor \frac{n-3}{2} \rfloor}{1} - \binom{\lfloor \frac{n}{2} \rfloor}{2}\right) - \binom{\lfloor \frac{n}{4} \rfloor}{1} - 0\right)$$

$$= \frac{n^2 - 3n + 2}{12} + \frac{(0 \text{ or } 1)}{3} + \frac{1 + \lfloor \frac{n-3}{2} \rfloor}{2} - \frac{\lfloor \frac{n}{2} \rfloor(\lfloor \frac{n}{2} \rfloor - 1)}{4} - \frac{\lfloor \frac{n}{4} \rfloor}{2}$$

$$= \begin{cases} \dfrac{n^2}{48} + \dfrac{-4 + (0 \text{ or } 4) + (0 \text{ or } 3)}{12} & \text{if } n \text{ is even,} \\[3mm] \dfrac{(n + 3)^2}{48} + \dfrac{-4 + (0 \text{ or } 4) + (0 \text{ or } 3)}{12} & \text{if } n \text{ is odd.} \end{cases}$$

Since $p_{3,n}$ is an integer, and since $-\frac{1}{2} < -\frac{4}{12} \leq (-4 + (0 \text{ or } 4) + (0 \text{ or } 3))/12 \leq \frac{3}{12} < \frac{1}{2}$, it follows that $p_{3,n}$ is the nearest integer to $n^2/48$ or $(n + 3)^2/48$, as appropriate. $\square$

PROOF OF THEOREM 1.9. As in the above proof of Theorem 1.2, we use Theorem 1.5 to show, case by case, that

$$p_{4,n} = \begin{cases} (n^3 - 3n^2 + 20n)/96 & \text{if } n \equiv 0 \pmod 4, \\ (n^3 - 7n + 6)/96 & \text{if } n \equiv 1 \pmod 4, \\ (n^3 - 3n^2 + 20n - 36)/96 & \text{if } n \equiv 2 \pmod 4, \\ (n^3 - 7n - 6)/96 & \text{if } n \equiv 3 \pmod 4. \end{cases} \qquad (5.1)$$

For example, if $n \equiv 2 \pmod 4$, then

$$p_{4,n} = \frac{1}{2n}\binom{n}{4} + \frac{1}{2n}\binom{\frac{n}{2}}{2} + \frac{1}{2}\left(\binom{\frac{n}{2}}{2} - \binom{\frac{n}{2}}{3} - \binom{\frac{n-2}{4}}{2} - \binom{\frac{n+2}{4}}{2}\right),$$

TABLE 1. The number $p_{m,n}$ of inequivalent integer m-gons with perimeter n for $3 \le m \le n \le 20$.

$m \backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1	0	1	1	2	1	3	2	4	3	5	4	7	5	8	7	10	8
4		1	1	2	3	5	7	9	13	16	22	25	34	38	50	54	70	75
5			1	1	3	4	9	13	23	29	48	60	92	109	158	186	258	296
6				1	1	4	7	15	25	46	72	113	172	248	360	491	686	896
7					1	1	4	8	20	37	75	129	228	359	584	868	1324	1870
8						1	1	5	10	29	57	125	231	435	745	1261	2031	3195
9							1	1	5	12	35	79	185	374	749	1382	2489	4237
10								1	1	6	14	47	111	280	600	1281	2493	4746
11									1	1	6	16	56	147	392	912	2052	4261
12										1	1	7	19	72	196	561	1368	3260
13											1	1	7	21	84	252	756	1980
14												1	1	8	24	104	324	1032
15													1	1	8	27	120	406
16														1	1	9	30	145
17															1	1	9	33
18																1	1	10
19																	1	1
20																		1

TABLE 2. The number p_n of inequivalent integer polygons with perimeter n for $3 \le n \le 20$. These are column sums of Table 1.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1	3	5	10	16	32	54	102	180	336	607	1144	2098	3960	7397	14 022	26 452

which reduces to $(n^3 - 3n^2 + 20n - 36)/96$. Equation (5.1) clearly completes the proof in the $n \equiv 0 \pmod 4$ case. In the $n \equiv 1 \pmod 4$ case, since $(n^3 - 7n + 6)/96$ is an integer, and since $\frac{6}{96} < \frac{1}{2}$, certainly $(n^3 - 7n + 6)/96$ is the nearest integer to $(n^3 - 7n)/96$. The other cases are analogous.   □

6. Calculated values and concluding remarks

Tables 1 and 2 give calculated values of $p_{m,n}$ and p_n, respectively, for $3 \le n \le 20$. Because of the powers of 2 in Theorem 1.6, the p_n sequence is very easy to compute; for example, the millionth term can be calculated in well under 20 seconds on a standard laptop. The first four rows of Table 1 are Sequences A005044, A057886, A124285 and A124286, respectively, in the Online Encyclopedia of Integer Sequences [15], while the whole of Table 1 is Sequence A124287. At the time of writing, the sequence p_n (Table 2) did not appear in [15]; it is now Sequence A293818. Out of all these sequences, as far as we are aware, a formula had only previously been proven for Sequence A005044 (Honsberger's theorem 1.2 concerning triangles, quoted in Section 1 above).

If we were interested in polygon equivalence under cyclic re-ordering of edges but not reversals, then the inequivalent integer m-gons (respectively, polygons) of

perimeter n are in one–one correspondence with the orbits of $G_{m,n}$ (respectively, G_n) under the action of the cyclic group C_n given by (2.1). If we write $p'_{m,n}$ (respectively, p'_n) for the number of such inequivalent m-gons (respectively, polygons), then

$$p'_{m,n} = |G_{m,n}/C_n| = \frac{1}{n} \sum_{\sigma \in C_n} \mathrm{fix}_{G_{m,n}}(\sigma) = \sum_{d \mid \gcd(m,n)} \frac{\varphi(d)}{n} \binom{n/d}{m/d} - \binom{\lfloor n/2 \rfloor}{m-1},$$

$$p'_n = |G_n/C_n| = \frac{1}{n} \sum_{\sigma \in C_n} \mathrm{fix}_{G_n}(\sigma) = \sum_{d \mid n} \frac{\varphi(d)}{n}(2^{n/d} - 1) - 2^{\lfloor n/2 \rfloor}$$

$$= \sum_{d \mid n} \frac{\varphi(d) \cdot 2^{n/d}}{n} - 1 - 2^{\lfloor n/2 \rfloor}.$$

The sequence $p'_{3,n}$ (triangles up to rotation) is Sequence A008742 in [15]; curiously, however, A008742 is listed as the Molien series of a certain three-dimensional point group. More computed values may be found in Sequences A293819–A293823 in [15]. (Note that Sequence A124278 of [15] counts $|P_{m,n}/\mathcal{S}_m|$, that is, polygons up to arbitrary re-orderings of the sides, under which $(1, 1, 2, 2)$ and $(1, 2, 1, 2)$ are considered equivalent, for example; see also [2].)

Finally, we observe that Theorem 1.6 has an interesting number-theoretic consequence: namely, that $\sum_{d \mid n} \varphi(d) \cdot 2^{n/d-1}$ is a multiple of n for $n \geq 3$; compare Sequences A000031, A053634 and A053635 in [15].

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