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APPROXIMATION BY SMOOTH EMBEDDED HYPERSURFACES WITH POSITIVE MEAN CURVATURE

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Here we initiate the study of the following problem. Let Ω be a compact domain in a Riemannian manifold such that $\partial\Omega$ is of minimum area for the contained volume. Can $\partial\Omega$ be approximated by smooth hypersurfaces of positive mean curvature? It reduces to the question of whether or not a stable (or minimizing) hypercone in a Euclidian space can be approximated by smooth hypersurfaces of positive mean curvature. The positive solution to the problem may be useful for studying the curvature and topology of Ω

We show in this paper that such approximation is possible provided that the given minimal cone satisfies some additional hypothesis.

1. Introduction

The aim of this note is to initiate the study of the following problem which was posed by Lawson (see [*1*, Problems section]). Let $\mathbb{C}$ be a stable (or minimizing) hypercone in $\mathbb{R}^{n+1}$. Given $\varepsilon > 0$, can one find a smooth embedded hypersurface M of positive mean curvature in $B_1(0) \cap \mathbb{C}_\varepsilon$ so that ∂M is close to $\partial(\mathbb{C} LB_1(0))$ (where $\mathbb{C}_\varepsilon$ is the ε-neighbourhood of $\mathbb{C}$ in $\mathbb{R}^{n+1}$)? This is related to the following question. Let Ω be a compact domain in a Riemannian manifold N such that $\partial\Omega$ is of a minimum area for the contained volume. Can $\partial\Omega$ be approximated by a smooth hypersurface of positive mean curvature? The

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latter is useful for studying the curvature and topoloty of Ω , see for example [6].

Under the additional hypothesis that the given cone $\mathbb{C}$ is regular, that is $\mathbb{C} = 0 \# \Sigma$ for some smooth embedded minimal hypersurface of Σ of S^n , both the local and the global approximation problems are studied. In order to make our statements more precise, we fix an orientation on the minimal cone $\mathbb{C}$ so that $\mathbb{R}^{n+1} - \mathbb{C} = E_+ \cup E_-$, where $E_\pm$ are two connected components of $\mathbb{R}^{n+1} \sim \mathbb{C}$, and ν_c points into E_+ . Suppose M is a connected, orientable hypersurface and such that $ML(B_1 \sim B_{1/2}) =$ $\{y = x + u(x)\nu_c(x): x \in (\mathbb{C})L(B_1 \sim B_{1/2})$ is a graph of a small C^2-function u over the cone. Then the orientation vector ν_M of M will be chosen so that $V_c(x)\nu_M(y) > 0$ for $y = x + u(x)\nu_c(x)$.

Now we can describe our results. In Section II, we consider the local approximation problem, and we have

THEOREM 1. *Let* $\mathbb{C}$ *be a regular minimal hypercone in* $\mathbb{R}^{n+1}$ *with* E_+ *defined as above. Then, for any integer* $k \geq 3$ *, we have:*

(i) if $n \leq 7$ *, then for every* $\varepsilon > 0$ *there is a smooth properly embedded, connected hypersurface with boundary* M_ε *supported in* $B_1(0) \cap \mathbb{C}_\varepsilon$ *so that* ∂M_ε *is* C^{k+1} *close to* $\mathbb{C} L \mathbb{S}^n$ *, and* M_ε *is of positive mean curvature;*

(ii) if $\mathbb{C}$ *in (i) is, in addition, stable (then* $n = 7$ *is automatic), then* M_ε *can be chosen to satisfy the additional property that* $spt(M_\varepsilon) \subset B_1(0) \cap \mathbb{C}_\varepsilon \cap E_+$ *;*

(iii) if $\mathbb{C}$ *is one-sided area minimizing in* $\overline{E}_+$ *, then, for each* $\varepsilon > 0$ *, there are smooth properly embedded, connected hypersurfaces* M^+ *and* M_ε^- *with positive and negative mean curvature, respectively, in* $B_1(0) \cap \mathbb{C}_\varepsilon \cap E_+$ *, and such that* $M_\varepsilon^\pm$ *are* C^{k+1} *close to* $\mathbb{C} L \mathbb{S}^n$.

The notion of one-sided area minimality is introduced also in Section II. By a maximum principle (see Lemma 1 in Section II) and the regularity of solutions to a parametric obstacle problem [8], we show the equivalence of the existence of M_ε^- (for suitable small positive ε's) and the one-sided area minimality of $\mathbb{C}$ in $\overline{E}_+$.

In Section III, we study the global approximation problem. There we obtain the following:

THEOREM 2. *Let* $\mathbb{C}$ *and* E_+ *be as in Theorem 1. We have:*

(i) if $\mathbb{C}$ *is not area-minimizing in* $\overline{E}_+$ *, and if* $n \le 7$ *, then for every* $\varepsilon > 0$ *there is a smooth properly embedded, complete (non-compact) hypersurface* M_ε *in* $\mathbb{C}_\varepsilon$ *with positive mean curvature;*

(ii) if $\mathbb{C}$ *is strictly one-sided area minimizing in* $\overline{E}_+$ *, then for each* $\varepsilon > 0$ *there are smooth properly embedded, complete (non-compact) hypersurfaces* M_ε^+ *and* M_ε^- *in* $\mathbb{C}_\varepsilon \cap E_+$ *with positive and negative mean-curvature, respectively.*

The strictly one-sided area minimality of a minimal cone and the proof of the existence of M_ε^- in the above Theorem are modified from the recent work [5]. See also [7] for the related discussions. It still remains as an open problem whether or not every one-sided area minimizing hypercone (not $\mathbb{R}^2$) is strictly one-sided area minimizing.

In the final section, we show how the results in Section II can be generalized. In particular, a similar approximation result is valid for 7-dimensional area minimizing oriented boundaries or minimizing oriented boundaries with a volume constraint. For $n > 7$, some additional hypotheses on singular set are needed.

II. Local approximation

DEFINITION. *Let* E *be an open set in* $\mathbb{R}^{n+1}$ *with* $\phi_E \in BV_{Loc}(\mathbb{R}^{n+1})$*, where* ϕ_E *is the characteristic function of the set* E *. Let* $T = \partial E$ *and we say* T *is area minimizing in* $\overline{E}$ *(the closure of* E*) if the following condition is satisfied: for any compact subset* K *of* $\mathbb{R}^{n+1}$ *, and any* $F \subset E$ *with* $\phi_F \in BV_{Loc}(\mathbb{R}^{n+1})$ *and* $\overline{E-F}$ *a compact subset of* $\overline{E}$ *, one has* $\int_K |D\phi_E| \le \int_K |D\phi_F|$.

It is clear that $\mathbb{C}$ is area minimizing in $\overline{E}_+$ if and only if $\mathbb{C}_1 = \mathbb{C} \llcorner B_1$ is an area minimizing integral current whose support lies in $\overline{E}_+ \cap B_1$ and whose boundary is $\mathbb{C} \llcorner S^n$. Later we will also say $\mathbb{C}$ is one-sided area minimizing in $\overline{E}_+$. By the regularity theorem for the one-sided area minimizing currents, see [8], we conclude that $\mathbb{C}_1$ is the unique such area minimizing integral current.

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Suppose $\mathbb{C}$ is area minimizing in $\overline{E}_+$, then there is a unique area minimizing, smooth properly embedded hypersurface S_+ in E_+ with $\operatorname{dist}(0,S_+) = 1$, see [5] and [7]. Moreover, S_+ is a polar graph, that is, $x \cdot v_{S_+}(x) > 0$, for all $x \in \operatorname{spt}(S_+)$; and for some constant $R_o = R_o(\mathbb{C}) > 1$,

$$S_+ \sim B_{R_o}(0) = \{x + u_+(x)v_c(x) : x \in \mathbb{C}\} L \left(\mathbb{R}^{n+1} \sim B_{R_o}(0)\right)$$

for some $C^{2,\alpha}$ function u_+ on $\mathbb{C}$ with

either $\quad u_+ = (C_1 + C_2 \log \gamma)\gamma^{-\gamma_-}\phi_1 + O(\gamma^{-\gamma_- - \alpha})$ as $\gamma \to \infty$

or $\quad u_+ = C\gamma^{-\gamma_+}\phi_1 + O(\gamma^{-\gamma_+ - \alpha})$ as $\gamma \to \infty$

where $C_1 + C_2 \log \gamma > 0$ in the first identity, and $C_2 = 0$ unless $\gamma_+ = \gamma_- = \frac{n-2}{2}$, and $C > 0$ in the second identity. Moreover $\gamma_+ \geq \gamma_- > 0$ in our case, see [5].

The following Lemma will be needed in the proof of Theorem 1.

LEMMA 3. *Let $T = \partial E$ be an oriented boundary with E an open subset of $\mathbb{R}^{n+1}$. Let $\operatorname{spt} T \sim \{0\}$ be a C^2-hypersurface in B_2 and $\operatorname{spt}(T) \sim \{0\}$ have non-negative mean curvature with respect to the unit normal vector field pointing into E. Suppose $\operatorname{spt} T$ intersects $\mathbb{S}^n$ transversely, and let T^* be an area minimizing integral current whose support lies in $\overline{E}$ and whose boundary is $T L \mathbb{S}^n$. Then either $T^* = TLB_1$ or $\operatorname{spt}(T^*) \cap \operatorname{spt} T = \operatorname{spt}(T L \mathbb{S}^n)$.*

The proof of the above Lemma is contained in the following more general maximum-principle:

MAXIMUM PRINCIPLE. *Let $\partial A, \partial B$ be oriented boundaries of open sets A, B and let $x_o \in \partial A \cap \partial B$. Suppose that $\partial A, \partial B$ are area-minimizing in $U^{n+1}(x_o, 2\rho)$, where $0 < 2\rho < \min\{\operatorname{dist}(x_o, \partial(\partial A)), \operatorname{dist}(x_o, \partial(\partial B))\}$. In addition, we assume $A \subset B$. Then $\operatorname{dist}(\partial A L \partial B_\rho(x_o), \partial B L \partial B_\rho(x_o))$ can not be positive.*

Proof. Suppose $\operatorname{dist}(\partial A \llcorner \partial B_\rho(x_0), \partial B \llcorner \partial B_\rho(x_0)) > 0$. By [11] we have regularity of an oriented area minimizing boundary, so we see $\partial A \llcorner \partial B_\rho(x_0)$ and $\partial B \llcorner \partial B_\rho(x_0)$ are $(n-1)$-dimensional integral cycles, and the subregion of $S^n_\rho(x_0)$ which is bounded by $\partial A \llcorner \partial B_\rho(x_0)$ and $\partial B \llcorner \partial B_\rho(x_0)$ is well-defined. Now we can choose $(n-1)$-dimensional integral cycles which are supported strictly in the above subregion and which are arbitrary close to $\partial A \llcorner \partial B_\rho(x_0)$ in flat norm. Let Γ be such an integral cycle, we solve the oriented plateau problem with boundary Γ. Let T be a solution. By [8] we see $T = \partial E$ for some open set $E \subset \mathbb{R}^{n+1}$. The regularity theorem, see [11], for least area boundary implies that $A \llcorner B_\rho(x_0) \subset E \subset B \llcorner B_\rho(x_0)$. Let $\tau_a : x \to x + a$ be a translation with $|a|$ small, then $\tau_{a\#} T$ is a solution of the oriented plateau problem with boundary $\tau_{a\#}\Gamma$. By our choice of Γ, and a similar argument to that above, we have

$$A \llcorner B_\rho(x_0) \subset \tau_{a\#} E \llcorner B_\rho(x_0) \subset B \llcorner B_\rho(x_0) , \qquad \text{for all a } \mathbb{R}^{n+1}$$

with $|a|$ small.

In particular, $x_0 \in \operatorname{spt}(\tau_{a\#} T)$, that is, $\tau_a^{-1}(x_0) \in \operatorname{spt}(T)$, for all $a \in \mathbb{R}^{n+1}$ with $|a|$ small. This is impossible. □

Now we would like to show that one-sided area minimality and one-sided approximability by a non-positive mean curvature, smooth hypersurface are equivalent for minimal hypercones.

PROPOSITION 4. *Let* $\mathbb{C}$ *and* E_+ *be as in Theorem 1. If* $\mathbb{C}$ *is not area minimizing in* $\overline{E}_+$, *then there is an* $\varepsilon_0 = \varepsilon_0(\mathbb{C}) > 0$ *such that any smooth embedded hypersurface* M *with non-positive mean curvature in* $\overline{E}_+$, *with boundary* ∂M *outside* B_1, *satisfies* $\operatorname{spt}(M) \cap B_{\varepsilon_0} = \phi$.

Proof. Let R be an oriented boundary which solves the obstacle problem:

$$(*) \ \operatorname{Min}\{\operatorname{Mass}(Q) : Q \in I_n(\mathbb{R}^{n+1}) \text{ with } \operatorname{spt}(Q) \subseteq \overline{E}_+ \text{ and } \partial Q = \mathbb{C} \llcorner S^n\}$$

The existence of such an R and its boundary regularity were shown in [8].

We claim spt $(M) \cap$ spt $(R) = \phi$. This can be verified as follows. We let $\lambda_o > 0$ be such that $\lambda_o = \sup\{\lambda \in (0,1], \mathrm{spt}(C_t^*) \cap \mathrm{spt}\,(M) = \phi$ for all $t \le \lambda\}$, where $C_t^* = \mu_t \# R + (\mathbb{C}_1 \sim C_t)$. The existence of such λ_o is obvious. If $\lambda_o < 1$, then we see spt $(\mu_{\lambda_o} \# R) \cap$ spt $(M) \neq \phi$ and spt $(\mu_{\lambda_o} \# R)$ lies in one side of spt (M) , this contradicts Lemma 3.

Now take $\varepsilon_o = \sup\,\{\varepsilon > 0, B_\varepsilon(0) \cap \mathrm{spt}\,(R) = \phi\}$, and $\varepsilon_o > 0$ by the Maximum-Principle. □

Remark. (1) In Proposition 4, $\mathbb{C}$ can be replaced by any stationary oriented boundary.

(2) Proposition 4 also shows that locally area minimality is essential in proofs of the classical "Bridge-Principle," as in [9], [6]. See also [2].

Proof of Theorem 1. Part (i) is contained in Theorem 2 which will be proved in the next section.

We first consider the case where $\mathbb{C}$ is area minimizing in $\overline{E}_+$. Then we have $S_\lambda = \mu_\lambda \# S_+$ is an area minimizing hypersurface in $E_+ \cap \mathbb{C}_{\varepsilon/2}$ for a suitable small $\lambda > 0$. For λ small, one can assume $\partial\left(S_\lambda L B_2(0)\right)$ is C^{k+1} close to $\mathbb{C} L S_2^n$, and since $S_\lambda L B_2(0)$ is strictly stable, the implicit function theorem applies, see [9] , to yield the existence of smooth embedded hypersurfaces M_ε^+, M_ε^- lying in $\mathbb{C}_\varepsilon \cap E_+ \cap B_2(0)$ with constant (small) positive and negative mean curvature respectively. Moreover $\partial M_\varepsilon^+ = \partial M_\varepsilon^- = \partial\left(S_\lambda L B_2(0)\right)$. This proves (iii).

To show (ii), we consider, for small $\delta > 0$, the obstacle problems $\min\{M(Q) : Q \in I_n(\mathbb{R}^{n+1})$ with spt $(Q) \subseteq \overline{E}_+$ and $\partial Q = \Gamma_\delta\}$, where $\Gamma_\delta = \{x + \delta v_c(x) : x \in \partial\mathbb{C}_1\}$. By [5, Section 5] and [8] we obtain solutions T_δ to the above problems with sing $(T_\delta) = \phi$, for all $\delta > 0$ sufficiently small. It is at this step we need the assumption $n = 7$.

Next we apply perturbation results of [12] to obtain M_δ which are smooth embedded hypersurfaces in E_+ with positive mean curvature and $\partial M_\delta = \partial T_\delta = \Gamma_\delta$. We also observe that $\mathbb{C}_{1,2/\varepsilon} = \mathbb{C}(B_{2/\varepsilon} \sim B_1)$ is strictly stable, hence for all sufficiently small $\delta > 0$, one can find a smooth embedded hypersurface $\tilde{M}_\delta$ in E_+ with positive mean-curvature and (which is close to $\mathbb{C}_{1,2/\varepsilon}$) such that $\partial\tilde{M}_\delta = -\Gamma_\delta + \tilde{\Gamma}_\delta$, $\tilde{\Gamma}_\delta = \{x + \delta v_c(x) ;$ $x \in \partial\mathbb{C}_{2/\varepsilon}\}$.

Now we claim the angle made by M_δ and $\tilde{M}_\delta$ along Γ_δ in the positive mean curvature direction is strictly less than π for all small $\delta > 0$. This can be verified by a contradiction argument. Since as $\delta \to 0^+$, $\tilde{M}_\delta \to \mathbb{C}_{1,2/\varepsilon}$ and $M_\delta \to R$ a one-sided area minimizing integral current with boundary $\Gamma_0 = \partial\mathbb{C}_1$ (by taking a sub-sequence if necessary). By the boundary regularity of R, see [8], and the Hopf-boundary point lemma, we see spt (R) and spt $(\mathbb{C})$ intersect transversely along Γ_0. Therefore we can smooth the corner made by M_δ and $\tilde{M}_\delta$ to obtain M which is a smooth embedded hypersurface with positive mean curvature in E_+. Finally we let $M_\varepsilon = \mu_\varepsilon \# M$ to yield (ii).

III. Global approximation

Let $\mathbb{C}$ be a regular minimal hypercone, by [5], the main result of [3] can be generalized to the case of a faster decay solution at infinity, that is, for a C^α-function f on $\mathbb{C}$ which decays at ∞ at a sufficiently faster rate, and with $\|f\|_{C^\alpha}$ small, then one can find a solution of $M_c u = f$ on $\mathbb{C}_{1,\infty}$, with $\|u\|_{C^{2,\alpha}}$ small, and u decays to $\mathbb{C}$ at infinity at a sufficiently faster rate.

A regular minimal hypercone $\mathbb{C}$ is called one-sided strictly area minimizing in $\overline{E}_+$ if there is $\theta > 0$ such that

$$M(\mathbb{C}_1) \le M(S) - \theta\varepsilon^n$$

whenever $1 > \varepsilon > 0$ and S is an integral current with spt $(S) \subseteq \overline{E}_+ \sim B_\varepsilon$

and $\partial S = \partial \mathbb{C}_1$, see [7], [5] .

As for strictly area minimizing hypercones, we have the following consequence of [5, Theorem 3.2] which gives various characterizations of one-sided strictly area minimizing hypercones.

LEMMA 5. *Let $\mathbb{C}$, E_+ , S_+ , U_+ be as in Section II, then the following are equivalent:*

(i) $\mathbb{C}$ *is strictly area minimizing in* $\overline{E}_+$.

(ii) U_+ *has slower decay at infinity. That is*

$$\lim_{|x|\to\infty} U_+(x)/|x|^{-\gamma} \equiv c > 0 , \qquad \text{in the case } \gamma_+ > \gamma_-$$

$$\lim_{|x|\to\infty} U_+(x)/(\log |x|)\cdot|x|^{-(n-2)/2} = c > 0 , \text{ in the case } \gamma_+ = \gamma_- = \frac{n-2}{2} .$$

(iii) For any non-negative $g_+ \in C^1(S_+)$ *which decays at infinity faster than* $|x|^{-\gamma_+-3}$ *, there exists a positive solution* W_+ *of* $L_{S_+} W_+ = -g_+$.

(iv) For any non-negative $g_+ \in C^1(S_+)$ *which decays at* ∞ *faster than* $|x|^{-\gamma_+-3}$ *, there exists on* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0,\varepsilon_0)$, *there is a positive* $C^2(S_+)$ *solution of* $M_{S_+} W_+^{\varepsilon} = -\varepsilon g_+$.

Remark. (1) The proof of the above Lemma can be found in [5, Section 3], except that (iii) and (iv) should be slightly modified. The reason why we can allow the condition "g_+ decays sufficiently faster at ∞ " instead of "compact support" is that $\phi = (\gamma^{-\gamma_+} - \gamma^{-\gamma_+})^{-\alpha}\phi_1$ satisfies $L_{S_+}\phi \le -C_0\gamma^{-\gamma_+-2-\alpha}$ on $S_+ \sim B_{R_0}$ where $\alpha \in (0,1)$, R_0, C_0 are two positive constants depending only on α and the cone $\mathbb{C}$. For the details the reader should see [5, Section 3].

(2) By the same proof as for (iv), one can show there is a negative $C^2(S_+)$ solution of $M_{S_+} W_-^{\varepsilon} = \varepsilon g_+$, for $\varepsilon \in (0,\varepsilon_0)$.

Now we can prove Theorem 2.

Proof. of Theorem 2. Part (i): For given $\varepsilon > 0$, we let u be a $C^{2,\alpha}$ solution of $M_{\mathbb{C}}u = f > 0$ in $\mathbb{C}_{1,\infty}$, and $u|_{\partial\mathbb{C}_1} = 0$, such that $\|u\|_{C^{2,\alpha}} \le \varepsilon$, where f is a smooth positive function defined on the cone $\mathbb{C}$ which decays faster at ∞. Hence $\text{graph}_{\mathbb{C}}U = \{x + u(x)v_c(x) : x \in \mathbb{C}_{1,\infty}\}$ is a smooth embedded hypersurface with boundary and positive mean-curvature.

Now let R be as in the Proof of Proposition 4, so that $\partial R = \partial\mathbb{C}_1$ and R is area-minimizing in E_+. Suppose spt (R) is a smooth embedded hypersurface with boundary (this will be the case if $n \le 6$), then we apply the perturbation of the theorem of [12] to obtain a smooth embedded hypersurface M of positive mean curvature in E_+ such that $\partial M = \partial\mathbb{C}_1$ and M is sufficiently close to R. As in the Proof of Theorem 1, Part (ii), we see that the corner angle made by M and $\text{graph}_{\mathbb{C}}U$ is strictly less than π if ε is sufficiently small. Therefore we can smooth the corner to obtain a new surface $\tilde{M}$ with positive mean curvature. Let $M_\varepsilon = U_\varepsilon \# \tilde{M}$, then M_ε satisfies the conclusion (i).

Now suppose sing $(R) \ne \phi$, then we consider the obstacle problem:

$$\min\left\{M(Q) : Q \in I_n(\mathbb{R}^{n+1}) \text{ with spt } (Q) \subset \mathbb{C}_\varepsilon \cup \overline{E}_+ \text{ and } \partial Q = \{x - \delta v_0(x) : x \in \partial\mathbb{C}_2\}\right\}$$

where $\delta \in (0,\varepsilon)$. Let T_δ be a solution of above problem, then $T_\delta \to R^*$ as $\delta \to 0^+$, for some R^* a solution to the corresponding problem with $\delta = 0$. Then, by [8], we see the uniformly boundary regularity of T_δ's for δ positive and small, and by [5, Section 5] we can assume sing $(T_{\delta_m}) = \phi$ for some $\delta_m \to 0^+$. Since we can choose $\text{graph}_{\mathbb{C}}U$ as close to $\mathbb{C}_{1,\infty}$ as we want, and spt (R^*) is smooth near $\partial\mathbb{C}_2$ and intersects $\mathbb{C}$ transversely slong $\partial\mathbb{C}_2$, we can choose a proper $\delta_m > 0$ and $\text{graph}_{\mathbb{C}}U$, so that spt (T_{δ_m}) intersects $\text{graph}_{\mathbb{C}}U$ transversely along some submanifold which is close to $\partial\mathbb{C}_2$. Moreover we can smooth the corner to obtain a new surface of positive mean-survature $\tilde{M}$ as before, and finally we set $M_\varepsilon = \mu_\varepsilon \# \tilde{M}$. This finishes the Proof of Part (i).

Part (ii): Follows directly from Lemma 5 and Remark 2 following Lemma 5. □

IV. General cases

In this final section, we would like to make a few remarks about one-side local approximation of an area minimizing oriented boundary or area minimizing boundary with contained volume by smooth properly embedded hypersurfaces of positive mean curvature.

Let N be a complete $(n+1)$-dimensional smooth Riemannian manifold, and let E,U,V be open subsets of N with $E \subset U$, $V \subset\subset U$, and let $T = \partial E \llcorner U$ be an oriented boundary of least area in U with spt $(T) = \partial E \llcorner U$. We also assume that $\xi \in$ sing (T) implies that there is a regular tangent cone $C(\xi)$ at ξ for T. Also we fix a C^2-comain $W \subset V$ such that spt $T \cap \partial W \subset \mathrm{reg}(T)$ and the intersection is transverse, with $\Gamma_o = \partial(T \llcorner W)$.

Then we have the following.

PROPOSITION 6. *For any* $\varepsilon > 0$ *there is a smooth, properly embedded hypersurface* M_ε^+ *(respectively* M_ε^- *), in an* ε*-neighbourhood of* spt (T)*, of positive (respectively negative) mean curvature, and such that* $\partial M_\varepsilon^{\pm} = \Gamma = \phi \# \Gamma_o$ *for some* $\phi \in C^2(\Gamma_o)$ *with* $|\phi - i_{\Gamma_o}|_{C^2} < \varepsilon$. *Moreover* $M_\varepsilon^{\pm} \subset E$.

Proof. This is an easy consequence of a result of [5, Section 5] and the implicit function theorem. □

PROPOSITION 7. *For* $n \le 7$, *let* $\Omega \subset N$ *be such that* $\partial\Omega$ *has least area for the contained volume. Then* $\partial\Omega$ *can be approximated by a sequence of smooth embedded hypersurfaces* $\{M_m\}_{m=1}^{\infty}$, *lie inside* $\bar{\Omega}$, *and each* M_m *has positive mean curvature.*

Proof. By [4] and [11], the conclusing is trivial when $n \le 6$. For the case $n = 7$, $\partial\Omega$ has at most isolated singularities (hence

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finite since $\partial\Omega$ is compact). Let $\xi \in$ sing $(\partial\Omega)$, $C(\xi)$ be the regular minimizing tangent cone of $\partial\Omega$ at ξ , see [10]. After a suitable scaling, we can assume that $\partial\Omega L B_1(\xi)$ sufficiently close to $C(\xi) L B_1(\xi)$, and that $\Gamma = \partial\Omega L S_1(\xi)$ is a smooth 6-dimensional submanifold of $S_1(\xi) = \partial B_1(\xi)$. One solves the oriented plateau problem with boundary Γ in $\bar{\Omega} \cap B_1(\xi)$, the resulting solutions M will be close to $C(\xi)$ since Γ is close to $\partial C_1(\xi)$. By the maximum-principle (see Lemma 1 also), spt $(M) \cap \partial\Omega = \Gamma$ and the intersection is transverse by the Hopf-boundary point lemma. We can assume sing $(M) = \phi$ otherwise replace M by a solution of the oriented plateau problem with boundary $\Gamma_\varepsilon =$ $\partial\Omega L S_{1+\varepsilon}(\xi)$ for $\varepsilon > 0$ small, this follows from [5, Section 5].

Since M is smooth, one can apply the implicit function theorems as in [12] to obtain a new hypersurface $\bar{M}\ \bar{\Omega}$ with positive mean curvature and $\partial\bar{M} = \Gamma$, $\bar{M} \cap \partial\Omega = \Gamma$, and the intersection is transverse. By the same argument as before we can smooth corners to find resulting hypersurfaces properly embedded in $\bar{\Omega}$, of positive mean curvature. □

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