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Simplices in the Euclidean Ball

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Abstract. We establish some inequalities for the second moment

$$\frac{1}{|K|}\int_K |x|_2^2\,dx$$

of a convex body K under various assumptions on the position of K.

1 Introduction

The starting point of this paper is the article [2], where it was shown that if all the extreme points of a convex body K in $\mathbb{R}^n$ have Euclidean norm greater than $r > 0$, then

$$\frac{1}{|K|}\int_K |x|_2^2\,dx \geqslant \frac{r^2}{9n} \tag{1.1}$$

where $|x|_2$ stands for the Euclidean norm of x and $|K|$ for the volume of K.

We improve this inequality showing that the optimal constant is $\frac{r^2}{n+2}$, with equality for the regular simplex, with vertices on the Euclidean sphere of radius r. We also prove the same inequality under the different condition that K is in Löwner position. More generally, we investigate upper and lower bounds on the quantity

$$C_2(K) := \frac{1}{|K|}\int_K |x|_2^2\,dx, \tag{1.2}$$

under various assumptions on the position of K. Some hypotheses on K are necessary because $C_2(K)$ is not homogeneous, one has $C_2(\lambda K) = \lambda^2 C_2(K)$.

Let $n \geqslant 2$. We denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$, *i.e.*, the set of compact convex sets with nonempty interior and by Δ^n the regular simplex in $\mathbb{R}^n$ with vertices in S^{n-1}, the Euclidean unit sphere. For $K \in \mathcal{K}^n$, we denote by g_K its centroid,

$$g_K = \frac{1}{|K|}\int_K x\,dx.$$

With these notations we prove the following theorem.

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Theorem 1.1 *Let $r > 0$ and $K \in \mathcal{K}^n$ such that all its extreme points have Euclidean norm greater than r. Then*

$$C_2(K) = \frac{1}{|K|}\int_K |x|_2^2\,dx \geqslant C_2(r\Delta^n) + \Big(\frac{n+1}{n+2}\Big)|g_K|_2^2 = \frac{r^2 + (n+1)|g_K|_2^2}{n+2}.$$

Moreover, if K is a polytope there is equality if and only if K is a simplex with its vertices on the Euclidean sphere of radius r.

In Theorem 1.1, for a general K, we don't have a characterization of the equality case because we deduce it by approximation from the case of polytopes. We conjecture that the equality case is still the same.

Notice that the condition imposed on K that all its extreme points have Euclidean norm greater than r is unusual. For example, if K has positive curvature, it is equivalent to either $K \supset rB_2^n$ or $K \cap rB_2^n = \varnothing$. Moreover, this hypothesis is not continuous with respect to the Hausdorff distance. Indeed, if we define $P = \operatorname{conv}(\Delta^n, x)$, where $x \notin \Delta^n$ is a point very close to the centroid of a facet of Δ^n, then the distance of Δ^n and P is very small but the point x will be an extreme point of P of Euclidean norm close to $1/n$, *i.e.*, much smaller than 1, the Euclidean norm of the vertices of Δ^n.

Other conditions on the position of K may be imposed. To state them, let us first recall the classical definitions of John and Löwner positions. Let $K \in \mathcal{K}_n$. We say that K is in *John position* if the ellipsoid of maximal volume contained in K is B_2^n. We say that K is in *Löwner position* if the ellipsoid of minimal volume that contains K is B_2^n.

It was proved by Guédon in [5] (see also [6]) that if $K \in \mathcal{K}^n$ satisfies $g_K = 0$ and if $K \cap (-K)$ is in Löwner position (which is equivalent to saying that B_2^n is the ellipsoid of minimal volume containing K and centered at the origin), then $C_2(K) \geq C_2(\Delta^n)$. Using the same ideas, we prove the following theorem.

Theorem 1.2 *Let K be a convex body in Löwner position. Then*

$$\frac{n}{n+2} = C_2(B_2^n) \geq C_2(K) \geq C_2(\Delta^n) + \frac{(n+1)^2}{n(n+2)}|g_K|_2^2 = \frac{n + (n+1)^2|g_K|_2^2}{n(n+2)}.$$

Moreover, if K is symmetric, then

$$\frac{n}{n+2} = C_2(B_2^n) \geq C_2(K) \geq C_2(B_1^n) = \frac{2n}{(n+1)(n+2)}.$$

Let K be a convex body in John position. Then

$$\frac{n}{n+2} = C_2(B_2^n) \leq C_2(K) \leq C_2(n\Delta^n) + 2\Big(\frac{n+1}{n+2}\Big)|g_K|_2^2 = \frac{n^2 + 2(n+1)|g_K|_2^2}{n+2}.$$

Moreover, if K is symmetric, then

$$\frac{n}{n+2} = C_2(B_2^n) \leq C_2(K) \leq C_2(B_\infty^n) = \frac{n}{3}.$$

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The inequalities involving the Euclidean ball in Theorem 1.2 are deduced from the following proposition.

Proposition 1.3 *Let K be a convex body.*

(i) *If $K \subset B_2^n$ and $0 \in K$ then $C_2(K) \le C_2(B_2^n) = \frac{n}{n+2}$ with equality if and only if $K = \{tx : 0 \le t \le 1, x \in S\}$, where $S \subset S^{n-1}$.*

(ii) *If $K \supset B_2^n$ then $C_2(K) \ge C_2(B_2^n) = \frac{n}{n+2}$, with equality if and only $K = B_2^n$.*

In view of Proposition 1.3, it could be conjectured that for every centrally symmetric convex bodies K, L such that $K \subset L$ one has $C_2(K) \le C_2(L)$. But this is not the case. It can be seen already in dimension 2, by taking

$$L = \operatorname{conv}\big((a,0),(-a,0),(0,1),(0,-1)\big)$$
$$K = \{(x,y) \in L : |y| \le 1/2\}$$

with a large enough. Indeed, $C_2(L) = \frac{1+a^2}{6}$ and $C_2(K) = \frac{5+15a^2}{72}$.

The paper is organized as follows. In Section 2, we gather some background material needed in the rest of the paper. We prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4 and Proposition 1.3 in Section 5.

2 Preliminaries

As mentioned before the quantity $C_2(K)$ is not affine invariant. Let us investigate the behaviour of $C_2(K)$ under affine transform. We start with translations. For $a \in \mathbb{R}^n$, and $K \in \mathcal{K}^n$, one has

$$C_2(K-a) = \frac{1}{|K|}\int_K |x-a|_2^2\,dx = C_2(K) - 2\langle g_K, a\rangle + |a|_2^2.$$

Hence

$$C_2(K-g_K) = C_2(K) - |g_K|_2^2 \tag{2.1}$$

minimizes $C_2(K-a)$ among translation $a \in \mathbb{R}^n$. Let T be a nonsingular linear transform, then

$$C_2(TK) = \frac{1}{|TK|}\int_{TK} |x|_2^2\,dx = \frac{1}{|K|}\int_K |Tx|_2^2\,dx.$$

The preceding quantity may be computed in terms of $C_2(K)$ if K is in isotropic position (see below).

2.1 Decomposition of Identity

Let $u_1, \dots, u_N$ be N points in the unit sphere S^{n-1}. We say that they form a representation of the identity if there exist $c_1, \dots, c_N$ positive integers such that

$$I = \sum_{i=1}^N c_i u_i \otimes u_i \quad \text{and} \quad \sum_{i=1}^N c_i u_i = 0.$$

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Notice that in this case, one has, for $x \in \mathbb{R}^n$

$$x = \sum_{i=1}^{N} c_i \langle x, u_i \rangle u_i, \quad |x|_2^2 = \sum_{i=1}^{N} c_i \langle x, u_i \rangle^2 \quad \text{and} \quad \sum_{i=1}^{N} c_i = n. \tag{2.2}$$

Moreover, for any linear map T on $\mathbb{R}^n$, its Hilbert–Schmidt norm is given by

$$\|T\|_{HS}^2 := \operatorname{tr}(T^\star T) = \sum_{i=1}^{N} c_i \langle u_i, T^\star T u_i \rangle = \sum_{i=1}^{N} c_i |Tu_i|_2^2.$$

If A is an affine transformation and T its linear part, *i.e.*, $A(x) = T(x) + A(0)$, then

$$\sum_{i=1}^{N} c_i |Au_i|_2^2 = \|T\|_{HS}^2 + n|A(0)|_2^2. \tag{2.3}$$

Indeed,

$$\begin{aligned}\sum_{i=1}^{N} c_i |Au_i|_2^2 = \sum_{i=1}^{N} c_i |Tu_i + A(0)|_2^2 &= \|T\|_{HS}^2 + 2\sum_{i=1}^{N} c_i \langle Tu_i, A(0)\rangle + \sum_{i=1}^{N} c_i |A(0)|_2^2 \\ &= \|T\|_{HS}^2 + n|A(0)|_2^2.\end{aligned}$$

2.2 John, Löwner and Isotropic Positions

Let $K \in \mathcal{K}_n$. Recall that K is in John position if the ellipsoid of maximal volume contained in K is B_2^n and that K is in Löwner position if the ellipsoid of minimal volume that contains K is B_2^n. The following theorem ([7], see also [1]) characterizes these positions.

Theorem 2.1 *Let $K \in \mathcal{K}_n$. Then K is in John position if and only if $B_2^n \subseteq K$ and there exist $u_1, \dots, u_N \in \partial K \cap S^{n-1}$ that form a representation of identity.*

Also K is in Löwner position if and only if $B_2^n \supseteq K$ and there exist $u_1, \dots, u_N \in \partial K \cap S^{n-1}$ that form a representation of identity.

Let $K \in \mathcal{K}_n$. We say that K is in isotropic position if $|K| = 1$, $g_K = 0$ and

$$\int_K \langle x, \theta \rangle^2 \, dx = L_K^2, \forall \theta \in S^{n-1}. \tag{2.4}$$

If K is in isotropic position, then $C_2(K) = nL_K^2$. Note that the isotropic position is unique up to orthogonal transformations and that for any convex body $K \in \mathcal{K}_n$ there exist an affine transformation A such that AK is in isotropic position (see [9] or [4]). The quantity L_K is called the isotropic constant of K.

For any nonsingular linear transform T on $\mathbb{R}^n$,

$$\int_K \langle x, Tx \rangle \, dx = L_K^2 \operatorname{tr} T.$$

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In particular if K is isotropic and $T \in \mathrm{GL}_n$, then

$$C_2(TK) = \frac{1}{|K|}\int_K |Tx|_2^2\,dx = L_K^2 \mathrm{tr} T^*T = \frac{\|T\|_{HS}^2}{n} C_2(K). \tag{2.5}$$

From the arithmetic geometric inequality, it implies that $C_2(TK) \geq |\det(T)|^{\frac{2}{n}} C_2(K)$. With (2.1), it gives, as it is well known, that the isotropic position minimizes $C_2(AK)$ among affine transforms A that preserve volume.

3 Proof of Theorem 1.1

We start with the following lemma.

Lemma 3.1 *For every $n \geqslant 1$*

$$C_2(\Delta^n) = \frac{1}{n+2}.$$

Proof The volume of the regular simplex Δ^n with vertices $u_1, \dots, u_{n+1}$ in S^{n-1} is

$$|\Delta^n| = \frac{\sqrt{n+1}}{n!}\left(\frac{n+1}{n}\right)^{\frac{n}{2}}.$$

Let $f(t) = |\{x \in \Delta^n : \langle x, u_1\rangle = t\}|$. One has

$$\begin{aligned} f(t) &= \left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}} |\Delta^{n-1}|(1-t)^{n-1}\mathbf{1}_{[-\frac{1}{n},1]}(t) \\ &= |\Delta^n|\left(\frac{n}{n+1}\right)^n n(1-t)^{n-1}\mathbf{1}_{[-\frac{1}{n},1]}(t). \end{aligned}$$

Hence, by Fubini

$$\frac{1}{|\Delta^n|}\int_{\Delta^n} \langle x, u_1\rangle^2\,dx = \frac{1}{|\Delta^n|}\int_{-\frac{1}{n}}^1 t^2 f(t)\,dt = \frac{1}{n(n+2)}.$$

Since $\lambda\Delta^n$ is in isotropic position for some $\lambda > 0$ we conclude that

$$C_2(\Delta^n) = \frac{1}{|\Delta^n|}\int_{\Delta^n} |x|_2^2\,dx = \frac{n}{|\Delta^n|}\int_{\Delta^n} \langle x, u_1\rangle^2\,dx = \frac{1}{n+2}.$$

Lemma 3.2 *Let $S = \mathrm{conv}(x_1, \dots, x_{n+1}) \subset \mathbb{R}^n$ be a nondegenerate simplex. Then*

$$C_2(S) = \frac{1}{n+1}\Big(\sum_{i=1}^{n+1} |x_i|_2^2\Big) C_2(\Delta^n) + |g_S|_2^2\big(1 - C_2(\Delta^n)\big).$$

In particular, if $\frac{1}{n+1}\sum_{i=1}^{n+1} |x_i|_2^2 \geq r^2$, then

$$C_2(S) \geq r^2 C_2(\Delta^n) + \left(\frac{n+1}{n+2}\right)|g_S|_2^2.$$

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Remark This implies that a nondegenerate simplex $S = \operatorname{conv}(x_1, \dots, x_{n+1})$ such that $|x_i|_2 \geq r$ for all i satisfies $C_2(S) \geq C_2(r\Delta^n) = \frac{r^2}{n+2}$, with equality if and only if $|x_i|_2 = r$ for all i and $g_S = 0$. In dimension 2, these conditions for equality imply that S is regular, but in dimension $n \geq 3$, it is not the case anymore. For example in dimension 3, if one takes the regular simplex and moves two vertices symmetrically along the geodesic between them to make them closer, and if one does the same to the two other opposite vertices, then the centroid stays at 0 and the vertices stay on the sphere. In any dimension $n \geq 4$, one chooses the north pole as the first vertex of our simplex and the n other vertices as the vertices of a simplex in dimension $n-1$ that is not regular and satisfies the equality case, we put this simplex in a horizontal hyperplane in such a way that the centroid is at 0.

Proof Let A be an affine map such that $S = A\Delta^n$, and denote by T its linear part. One has $g_S = Ag_{\Delta^n} = A(0)$, hence $A = T + g_S$ and $S = g_S + T\Delta^n$. Denote by $u_i \in S^{n-1}$ the vertices of Δ^n, so that $x_i = g_S + Tu_i$, for $1 \leq i \leq n+1$. Hence, by (2.1) and (2.5)

$$C_2(S) = |g_S|_2^2 + C_2(T\Delta^n) = |g_S|_2^2 + \frac{\|T\|_{HS}^2}{n} C_2(\Delta^n).$$

Since $u_1, \dots, u_{n+1}$ form a decomposition of identity

$$I = \frac{n}{n+1} \sum_{i=1}^{n+1} u_i \otimes u_i,$$

one has

$$\|T\|_{HS}^2 = \frac{n}{n+1} \sum_{i=1}^{n+1} |Tu_i|_2^2 = \frac{n}{n+1} \sum_{i=1}^{n+1} |Au_i|_2^2 - n|g_S|_2^2.$$

Therefore, we get the equality. The inequality is obvious. ∎

Proof of Theorem 1.1 We first consider the case where K is a polytope. Let us prove by induction on the dimension that there exist nondegenerate simplices $S_1, \dots, S_m$ with the following properties: $K = \bigcup_{i=1}^m S_i$, the interiors of S_i and S_j are mutually disjoint for $i \neq j$, and the vertices of the S_i's are among the vertices of K.

In dimension two, it is enough to fix a vertex x of K, then to consider the edges $\Delta_1, \dots, \Delta_m$ which don't contain x, and choose $S_i = \operatorname{conv}(x, \Delta_i)$, for $i = 1, \dots, m$.

Assuming that the property has been proved in dimension $n-1$, let us prove it in dimension n. We proceed in the same way as in dimension two. Let x be a fixed vertex of K. Let $F_1, \dots, F_N$ be the facets of K which don't contain x. From the induction hypothesis, we can split each of them into simplices with the required properties. Let $\Delta_1, \dots, \Delta_m$ be the collection of these simplices. Then it is not difficult to check that the simplices $S_i = \operatorname{conv}(x, \Delta_i)$, for $i = 1, \dots, m$ have the required properties.

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Now we may apply Lemma 3.2 to the S_i's and we get

$$C_2(K) = \frac{1}{|K|}\sum_{i=1}^m |S_i| C_2(S_i) \geq \frac{1}{|K|}\sum_{i=1}^m |S_i|\left(r^2 C_2(\Delta^n) + \left(\frac{n+1}{n+2}\right)|g_{S_i}|_2^2\right)$$
$$= r^2 C_2(\Delta^n) + \left(\frac{n+1}{n+2}\right)\sum_{i=1}^m \frac{|S_i|}{|K|}|g_{S_i}|_2^2.$$

Then we use the convexity of the function $x \mapsto |x|_2^2$ to deduce that

$$C_2(K) \geq r^2 C_2(\Delta^n) + \left(\frac{n+1}{n+2}\right)\Big|\sum_{i=1}^m \frac{|S_i|}{|K|} g_{S_i}\Big|_2^2 = r^2 C_2(\Delta^n) + \left(\frac{n+1}{n+2}\right)|g_K|_2^2.$$

This proves the inequality. If there is equality, then from the strict convexity of the function $x \mapsto |x|_2^2$ one deduces that $g_{S_i} = g_K$, for every i. Since the S_i's have disjoint interiors, this implies that there is only one of them. Hence K is a simplex and its vertices have Euclidean norm r.

Let us prove the general case. Let K be a convex body such that all its extreme points have Euclidean norm greater than r. Then there is a sequence of polytopes $(P_n)_n$ converging to K in the Hausdorff metric in $\mathcal{K}^n$ such that for every $n \in N$, the extreme points of P_n have Euclidean norm greater than r. Since C_2 is continuous with respect to the Hausdorff distance we get the inequality for K. ■

4 Proof of Theorem 1.2

As in [5], our main tools are the following inequalities, proved by Milman and Pajor [9] in the symmetric case and by Kannan, Lovász, and Simonovits [8] in the non-symmetric case. Recall that if K is a convex body and $u \in \mathbb{R}^n$, the support function of K is defined by

$$h_K(u) = \sup_{x\in K}\langle x, u\rangle.$$

Lemma 4.1 *Let K be a convex body and $u \in S^{n-1}$.*

(1) *If K is symmetric then*

$$\frac{2h_K(u)^2}{(n+1)(n+2)} \leq \frac{1}{|K|}\int_K \langle x, u\rangle^2\, dx \leq \frac{h_K(u)^2}{3},$$

with equality on the left-hand side if and only if K is a double-cone in direction u, which means that there exists x in $\mathbb{R}^n$ and a symmetric convex body L in $u^\perp$ such that $\langle x, u\rangle \neq 0$ and $K = \operatorname{conv}(L, x, -x)$ and equality in the right-hand side if and only if K is a cylinder in direction u, which means that there exists x in $\mathbb{R}^n$ and a symmetric convex body L in $u^\perp$ such that $\langle x, u\rangle \neq 0$ and $K = L + [-x, x]$.

(2) *If $g_K = 0$, then*

$$\frac{h_K(u)^2}{n(n+2)} \leq \frac{1}{|K|}\int_K \langle x, u\rangle^2\, dx \leq \frac{n h_K(u)^2}{n+2},$$

with equality on the left-hand side if and only if K is a cone in direction u, which means that there exists x in $\mathbb{R}^n$ and a convex body L in $u^\perp$ such that $\langle x, u\rangle > 0$, $g_L = 0$, and $K = \operatorname{conv}(L, x) - \frac{x}{n}$ and equality in the right-hand side if and only if K is a cone in direction $-u$.

Notice that the proof of (1) given in [9] is beautiful and elementary but the proof of (2) given in [8] is not as simple; it used a much more elaborate tool called the localization lemma. A simple proof of (2) was given in [3], but since it is not easily available, we reproduce it partially here for completeness.

Proof of (2) With a change of variable, we may assume that $|K| = 1$ and $h_K(u) = 1$. Let $f(t) = |\{x \in K : \langle x, u\rangle = t\}|$. The support of f is $[-h_K(-u), h_K(u)]$, and from Brunn's theorem, $f^{1/(n-1)}$ is concave on its support. Moreover, from Fubini, for any continuous function ϕ on $\mathbb{R}$

$$\int_K \phi(\langle x, u\rangle)\, dx = \int_{-\infty}^{\infty} \phi(t) f(t)\, dt.$$

Since $|K| = 1$ we have $\int f = 1$ and since $g_K = 0$, $\int t f(t)\, dt = 0$. Define

$$g_1(t) = \frac{n^{n+1}}{(n+1)^n}(1-t)^{n-1}\mathbf{1}_{[-\frac{1}{n},1]}, \quad g_2(t) = \frac{n}{(n+1)^n}(t+n)^{n-1}\mathbf{1}_{[-n,1]}.$$

Then the functions $g_i^{1/(n-1)}$ are affine on their support and satisfy $\int g_i = 1$ and $\int t g_i(t)\, dt = 0$. Let $h_1 = f - g_1$ and $h_2 = g_2 - f$. Assume that $h_i \neq 0$. Since $\int h_i = \int t h_i = 0$, the function h_i changes sign at least twice at $s_i < t_i$ for $i = 1, 2$. Moreover because of the concavity of $f^{1/(n-1)}$, the function h_i is negative in (s_i, t_i) and positive outside. By looking at its variations, one sees that the function

$$W_i(t) := \int_{-\infty}^{t} \int_{-\infty}^{s} h_i(x)\, dx\, ds$$

is nonnegative on its support. Integrating by parts twice and assuming that ϕ is twice differentiable and convex, one has

$$\int \phi(t) h_i(t)\, dt = \int \phi''(t) W_i(t)\, dt \geq 0.$$

Therefore $\int \phi(t) g_1(t)\, dt \leq \int \phi(t) f(t)\, dt \leq \int \phi(t) g_2(t)\, dt$. For $\phi(t) = t^2$, we get the inequality. If there is equality for example in the left-hand side, then $W_1 = 0$, thus $h_1 = 0$, hence $f = g_1$. From the equality case in Brunn's theorem we deduce that all the sets $\{x \in K : \langle x, u\rangle = t\}$ are homothetic. ■

Using Lemma 4.1, we prove the following proposition.

Proposition 4.2 *Let K be a convex body such that there exist vectors $u_1, \dots, u_m \in S^{n-1}$, with $h_K(u_i) = 1$ which form a representation of identity $I = \sum_{i=1}^m c_i u_i \otimes u_i$, with $\sum_{i=1}^m c_i u_i = 0$.*

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(1) *If K is symmetric then*

$$\frac{2n}{(n+1)(n+2)} = C_2(B_1^n) \le C_2(K) \le C_2(B_\infty^n) = \frac{n}{3}.$$

(2) *In general one has*

$$\frac{1}{n+2} + \frac{|g_K|_2^2}{n(n+2)} \le C_2(K) - |g_K|_2^2 \le \frac{n^2}{n+2} + \frac{n}{n+2}|g_K|_2^2.$$

Proof (1) Using the decomposition of identity, we deduce that

$$C_2(K) = \frac{1}{|K|}\int_K |x|_2^2\,dx = \sum_{i=1}^m c_i \frac{1}{|K|}\int_K \langle x, u_i\rangle^2\,dx.$$

We apply the preceding lemma to the vectors u_i and use that $h_K(u_i) = 1$ to get

$$\frac{2n}{(n+1)(n+2)} \le C_2(K) \le \frac{n}{3}.$$

For $K = B_1^n$ or $K = B_\infty^n$, the calculation is trivial but we could also check that we are in the case of equality of Lemma 4.1.

(2) One has

$$C_2(K) - |g_K|_2^2 = \frac{1}{|K|}\int_{K-g_K} |x|_2^2\,dx = \sum_{i=1}^m c_i \frac{1}{|K|}\int_{K-g_K} \langle x, u_i\rangle^2\,dx.$$

We apply the preceding lemma to the vectors u_i

$$C_2(K) - |g_K|_2^2 \le \sum_{i=1}^m c_i \frac{n}{n+2} h_{K-g_K}(u_i)^2 = \frac{n}{n+2}\sum_{i=1}^m c_i \big(h_K(u_i) - \langle g_K, u_i\rangle\big)^2.$$

Since $h_K(u_i) = 1$ and $\sum c_i u_i = 0$

$$C_2(K) - |g_K|_2^2 \le \frac{n}{n+2}(n + |g_K|_2^2).$$

The lower estimate follows in the same way. ■

Theorem 1.2 follows from Proposition 1.3 and Proposition 4.2.

5 Proof of Proposition 1.3

We first state and prove a standard lemma that we shall use in the proof.

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Lemma 5.1 *Let K be a Borel set such that $0 < |K| < +\infty$, $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function and $\lambda = (|K|/|B_2^n|)^{1/n}$. Then*

$$\frac{1}{|K|}\int_K \varphi(|x|_2)\,dx \geq \frac{1}{|B_2^n|}\int_{B_2^n} \varphi(\lambda|x|_2)\,dx = n\int_0^1 \varphi(\lambda r)r^{n-1}\,dr.$$

Proof One has $|\lambda B_2^n| = |K|$, hence $|K \setminus (\lambda B_2^n)| = |(\lambda B_2^n) \setminus K|$. Since φ is nondecreasing, we deduce that

$$\int_{K\setminus(\lambda B_2^n)} \varphi(|x|_2)\,dx \geq \varphi(\lambda)|K \setminus (\lambda B_2^n)| = \varphi(\lambda)|(\lambda B_2^n) \setminus K| \geq \int_{(\lambda B_2^n)\setminus K} \varphi(|x|_2)\,dx.$$

Therefore

$$\int_K \varphi(|x|_2)\,dx = \int_{K\cap(\lambda B_2^n)} \varphi(|x|_2)\,dx + \int_{K\setminus(\lambda B_2^n)} \varphi(|x|_2)\,dx \geq \int_{\lambda B_2^n} \varphi(|x|_2)\,dx.$$

Two changes of variables finish the proof. ■

Proof of Proposition 1.3

(1) Let $\|\cdot\|_K$ be the gauge function of K, *i.e.*, $\|x\|_K = \inf\{t > 0 : x \in tK\}$, for every $x \in \mathbb{R}^n$. Since $K \subset B_2^n$ one has $|x|_2 \leq \|x\|_K$, for every $x \in \mathbb{R}^n$. Hence

$$\int_K |x|_2^2\,dx \leq \int_K \|x\|_K^2\,dx = \int_K\int_0^{\|x\|_K} 2t\,dt\,dx = \int_0^1 2t|\{x \in K; \|x\|_K \geq t\}|\,dt$$
$$= |K|\int_0^1 2t(1-t^n)\,dt = |K|\frac{n}{n+2}.$$

This gives the inequality. If there is equality, then $\|x\|_K = |x|_2$ for every $x \in K$ hence $\{x : \|x\|_K = 1\} = K \cap S^{n-1} := S$ therefore $K = \{tx : 0 \leq t \leq 1, x \in S\}$, with $S \subset S^{n-1}$.

(2) Applying Lemma 5.1 to $\varphi(t) = t^2$, we deduce that

$$C_2(K) = \frac{1}{|K|}\int_K |x|_2^2\,dx \geq \lambda^2 C_2(B_2^n) = \Big(\frac{|K|}{|B_2^n|}\Big)^{\frac{2}{n}} C_2(B_2^n).$$

If we assume that $K \supset B_2^n$ then it follows that $C_2(K) \geq C_2(B_2^n)$, with equality if and only if $K = B_2^n$, which is the content of the second part of Proposition 1.3. ■

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