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TRANSCENDENTAL RUBAN p-ADIC CONTINUED FRACTIONS

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Abstract

We establish explicit constructions of Mahler's p-adic U_m-numbers by using Ruban p-adic continued fraction expansions of algebraic irrational p-adic numbers of degree m.

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1. Mahler's and Koksma's classifications of p-adic numbers

Let p be a prime number and let $|\cdot|_p$ denote the p-adic absolute value on the field $\mathbb{Q}$ of rational numbers, normalised such that $|p|_p = p^{-1}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is the field $\mathbb{Q}_p$ of p-adic numbers, and the unique extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ is denoted by the same notation $|\cdot|_p$. Mahler [16] gave a classification of p-adic numbers in analogy with his classification [15] of real numbers, as follows. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a nonzero polynomial in x over the ring $\mathbb{Z}$ of rational integers. We denote by $\deg(P)$ the degree of $P(x)$ with respect to x. The height $H(P)$ of $P(x)$ is defined by $H(P) = \max\{|a_n|, \ldots, |a_1|, |a_0|\}$, where $|\cdot|$ denotes the usual absolute value on the field $\mathbb{R}$ of real numbers. Let ξ be any p-adic number and let n, H be any positive rational integers. Following Bugeaud [3], set

$$w_n(H, \xi) = \min\{|P(\xi)|_p : P(x) \in \mathbb{Z}[x],\ \deg(P) \leq n, H(P) \leq H \text{ and } P(\xi) \neq 0\},$$

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log(H w_n(H, \xi))}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

Then ξ is called:

- a p-adic A-number if $w(\xi) = 0$;
- a p-adic S-number if $0 < w(\xi) < \infty$;
- a p-adic T-number if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for $n = 1, 2, 3, \ldots$; and
- a p-adic U-number if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ from some n onward.

The set of p-adic A-numbers coincides with the set of algebraic p-adic numbers. Therefore, the transcendental p-adic numbers are separated into the three disjoint classes S, T and U. If ξ is a p-adic U-number and m is the minimum of the positive integers n satisfying $w_n(\xi) = \infty$, then ξ is called a p-adic U_m-number. Alnıaçık [1, Ch. III, Theorem I] gave the first explicit constructions of p-adic U_m-numbers for each positive integer m. For further constructions of p-adic S-, T- and U-numbers, see [4, 5, 9, 10].

Assume that α is an algebraic p-adic number. Let $P(x)$ be the minimal polynomial of α over $\mathbb{Z}$. Then the degree $\deg(\alpha)$ of α and the height $H(\alpha)$ of α are defined by $\deg(\alpha) = \deg(P)$ and $H(\alpha) = H(P)$. Given a p-adic number ξ and positive rational integers n, H, in analogy with Koksma's classification [12] of real numbers and as in Bugeaud [3] and Schlickewei [21]), set

$$w_n^*(H,\xi) = \min\left\{|\xi-\alpha|_p : \begin{array}{l}\alpha \text{ is an algebraic } p\text{-adic number,}\\ \deg(\alpha) \le n, H(\alpha) \le H \text{ and } \alpha \ne \xi\end{array}\right\},$$

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(Hw_n^*(H,\xi))}{\log H} \quad \text{and} \quad w^*(\xi) = \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

Then ξ is called:

- a p-adic A^*-number if $w^*(\xi) = 0$;
- a p-adic S^*-number if $0 < w^*(\xi) < \infty$;
- a p-adic T^*-number if $w^*(\xi) = \infty$ and $w_n^*(\xi) < \infty$ for $n = 1, 2, 3, \ldots$; and
- a p-adic U^*-number if $w^*(\xi) = \infty$ and $w_n^*(\xi) = \infty$ from some n onward.

The set of p-adic A^*-numbers is equal to the set of algebraic p-adic numbers. Therefore, the transcendental p-adic numbers are separated into the three disjoint classes S^*, T^* and U^*. Let ξ be a p-adic U^*-number and let m be the minimum of the positive integers n satisfying $w_n^*(\xi) = \infty$. Then ξ is called a p-adic U_m^*-number. Mahler's classification of p-adic numbers is equivalent to Koksma's classification of p-adic numbers, that is, the classes A, S, T and U are the same as the classes A^*, S^*, T^* and U^*, respectively. Furthermore, a p-adic U_m^*-number is a p-adic U_m-number and *vice versa*. (See Bugeaud [3] for further information on Mahler's and Koksma's classifications of p-adic numbers.)

2. Ruban p-adic continued fractions

Ruban [20] introduced a continued fraction algorithm in $\mathbb{Q}_p$. In this section, we recall the Ruban p-adic continued fraction algorithm and its basic properties following the approach of Perron [19, Sections 29 and 30, pages 101–108] (see also [14, 17, 22, 23]). Let ξ be a nonzero p-adic number with the canonical expansion

$$\xi = \sum_{j=k}^{\infty} a_j p^j,$$

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where $a_j \in \{0, 1, \ldots, p-1\}$ for $j = k, k+1, \ldots, a_k \neq 0$ and k is the rational integer such that $|\xi|_p = p^{-k}$. If $k \leq 0$, then we write $\xi = \{\xi\} + \lfloor \xi \rfloor$, where

$$\{\xi\} = \sum_{j=k}^{0} a_j p^j \quad \text{and} \quad \lfloor \xi \rfloor = \sum_{j=1}^{\infty} a_j p^j.$$

If $k > 0$, then we write $\xi = \{\xi\} + \lfloor \xi \rfloor$, where

$$\{\xi\} = 0 \quad \text{and} \quad \lfloor \xi \rfloor = \sum_{j=k}^{\infty} a_j p^j.$$

Further, we write $0 = \{0\} + \lfloor 0 \rfloor$, where $\{0\} = \lfloor 0 \rfloor = 0$. Then, for each p-adic number ξ, $\{\xi\}$ and $\lfloor \xi \rfloor$ are uniquely determined. Let $b_0, b_1, b_2, \ldots$ be nonnegative rational numbers with

$$b_0 \in \{\{\xi\} : \xi \in \mathbb{Q}_p\} \quad \text{and} \quad b_\nu \in \{\{\xi\} : \xi \in \mathbb{Q}_p,\ |\xi|_p \geq p\} \quad (\nu = 1, 2, 3, \ldots).$$

A finite Ruban p-adic continued fraction $[b_0, b_1, \ldots, b_n]_p$ is defined by

$$[b_0, b_1, \ldots, b_n]_p = b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}.$$

Then we have the following properties.

$$[b_0]_p = b_0, \quad [b_0, b_1]_p = b_0 + \frac{1}{b_1},$$

$$[b_0, b_1, \ldots, b_n]_p = \left[b_0, b_1, \ldots, b_{n-2}, b_{n-1} + \frac{1}{b_n}\right]_p = [b_0, b_1, \ldots, b_{m-1}, [b_m, \ldots, b_n]_p]_p,$$

$$[b_0, b_1, \ldots, b_n]_p = b_0 + \frac{1}{[b_1, \ldots, b_n]_p}.$$

Hence, $[b_0, b_1, \ldots, b_n]_p$ is a nonnegative rational number, and the numbers b_ν $(\nu = 0, 1, \ldots, n)$ are called the partial quotients of the Ruban p-adic continued fraction $[b_0, b_1, \ldots, b_n]_p$. Define the nonnegative rational numbers p_ν and q_ν by

$$\begin{cases} p_{-2} = 0, & p_{-1} = 1, \quad p_\nu = b_\nu p_{\nu-1} + p_{\nu-2} \quad (\nu = 0, 1, 2, \ldots), \\ q_{-2} = 1, & q_{-1} = 0, \quad q_\nu = b_\nu q_{\nu-1} + q_{\nu-2} \quad (\nu = 0, 1, 2, \ldots). \end{cases} \tag{2.1}$$

By induction,

$$[b_0, b_1, \ldots, b_n]_p = \frac{p_n}{q_n} \quad (n = 0, 1, 2, \ldots).$$

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The nonnegative rational numbers $p_0/q_0, p_1/q_1, \ldots, p_n/q_n$ are called the convergents of the Ruban p-adic continued fraction $[b_0, b_1, \ldots, b_n]_p$; p_ν/q_ν $(\nu = 0, 1, \ldots, n)$ is called the νth convergent of $[b_0, b_1, \ldots, b_n]_p$. By induction,

$$p_\nu q_{\nu-1} - p_{\nu-1} q_\nu = (-1)^{\nu-1} \quad (\nu = -1, 0, 1, \ldots). \tag{2.2}$$

From (2.1),

$$|q_n|_p = |b_1|_p \cdot |b_2|_p \cdots |b_n|_p \quad \text{and} \quad |p_n|_p = |b_0|_p \cdot |b_1|_p \cdots |b_n|_p = |b_0|_p \cdot |q_n|_p \text{ (if } b_0 \neq 0)$$

for $n = 1, 2, 3, \ldots$. As $|b_\nu|_p \geq p$ $(\nu = 1, 2, 3, \ldots)$, we have $|q_{n+1}|_p > |q_n|_p$ and $|p_{n+1}|_p > |p_n|_p$ for $n = 1, 2, 3, \ldots$. Therefore,

$$\lim_{n\to\infty} |q_n|_p = \infty \quad \text{and} \quad \lim_{n\to\infty} |p_n|_p = \infty.$$

By (2.2),

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right|_p = \frac{1}{|q_n|_p \cdot |q_{n-1}|_p} \quad (n = 1, 2, 3, \ldots).$$

Then

$$\lim_{n\to\infty} \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right|_p = 0.$$

Thus, $\{p_n/q_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}_p$ and has a limit in $\mathbb{Q}_p$. An infinite Ruban p-adic continued fraction $[b_0, b_1, b_2, \ldots]_p$ is defined as the limit of the sequence $\{p_n/q_n\}_{n=0}^{\infty}$, that is,

$$[b_0, b_1, b_2, \ldots]_p := \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [b_0, b_1, \ldots, b_n]_p.$$

Further, for $\xi \in \mathbb{Q}_p \setminus \{0\}$,

$$[b_0, \ldots, b_n, \xi]_p = \frac{p_n \cdot \xi + p_{n-1}}{q_n \cdot \xi + q_{n-1}} \quad (n = 0, 1, 2, \ldots). \tag{2.3}$$

Let ξ_0 be a p-adic number. If $\xi_0 \neq \{\xi_0\}$, then we write

$$\xi_0 = b_0 + \frac{1}{\xi_1},$$

where $b_0 = \{\xi_0\}$, $\xi_1 = 1/\lfloor \xi_0 \rfloor$, $|\xi_1|_p \geq p$ and $\{\xi_1\} \neq 0$. If $\xi_1 \neq \{\xi_1\}$, then we write

$$\xi_1 = b_1 + \frac{1}{\xi_2},$$

where $b_1 = \{\xi_1\}$, $\xi_2 = 1/\lfloor \xi_1 \rfloor$, $|\xi_2|_p \geq p$ and $\{\xi_2\} \neq 0$. If the process continues, then

$$\xi_\nu = b_\nu + \frac{1}{\xi_{\nu+1}} \quad (\nu \geq 0), \tag{2.4}$$

where $b_\nu = \{\xi_\nu\}$ $(\nu \geq 0)$ and $\xi_{\nu+1} = 1/\lfloor \xi_\nu \rfloor$ $(\nu \geq 0)$, and

$$|\xi_\nu|_p = |b_\nu|_p \geq p \quad (\nu \geq 1).$$

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The p-adic numbers $\xi_1, \xi_2, \ldots$ are called complete quotients, and the nonnegative rational numbers $b_0, b_1, b_2, \ldots$ are called partial quotients. It follows from (2.4) that

$$\xi_0 = [b_0, \xi_1]_p = [b_0, b_1, \xi_2]_p = [b_0, b_1, \ldots, b_n, \xi_{n+1}]_p \tag{2.5}$$

and

$$\xi_\nu = [b_\nu, b_{\nu+1}, \ldots, b_n, \xi_{n+1}]_p \quad (\nu = 0, 1, \ldots, n).$$

By (2.5), (2.3) and (2.2),

$$\xi_0 - \frac{p_n}{q_n} = \frac{p_n\xi_{n+1} + p_{n-1}}{q_n\xi_{n+1} + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(q_n\xi_{n+1} + q_{n-1})}.$$

Then

$$\left|\xi_0 - \frac{p_n}{q_n}\right|_p = \frac{1}{|\xi_{n+1}|_p \cdot |q_n|_p^2} = \frac{1}{|b_{n+1}|_p \cdot |q_n|_p^2} = \frac{1}{|q_{n+1}|_p \cdot |q_n|_p} < \frac{1}{|q_n|_p^2}. \tag{2.6}$$

We now have two cases to consider.

Case (i). Some ξ_{n+1} appears with $\xi_{n+1} = \{\xi_{n+1}\} = b_{n+1}$ and the process stops with $\xi_{n+1} = b_{n+1}$. Then it follows from (2.5) that

$$\xi_0 = [b_0, b_1, \ldots, b_n, b_{n+1}]_p.$$

Case (ii). $\xi_{n+1} \neq \{\xi_{n+1}\}$ for every $n \geq -1$ and the process never stops. Then it follows from (2.6) that

$$\xi_0 = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [b_0, b_1, \ldots, b_n]_p = [b_0, b_1, b_2, \ldots]_p.$$

The Ruban continued fraction expansion of a p-adic number is unique because the canonical expansion of a p-adic number is unique. Laohakosol [14] and Wang [22] proved that a p-adic number is rational if and only if its Ruban continued fraction expansion is finite or ultimately periodic with the period $p - p^{-1}$. Ooto [17] recently proved that an analogue of Lagrange's theorem does not hold for the Ruban p-adic continued fraction: that is, there are quadratic irrational p-adic numbers whose Ruban continued fraction expansions are not ultimately periodic.

3. Our main results

Alnıaçık [2, Theorem] gave a construction of real U_m-numbers by using continued fraction expansions of algebraic irrational real numbers of degree m. In the present paper, we establish the following p-adic analogue.

THEOREM 3.1. *Let α be an algebraic irrational p-adic number with $|\alpha|_p \geq 1$ and the Ruban p-adic continued fraction expansion*

$$\alpha = [a_0, a_1, a_2, \ldots]_p. \tag{3.1}$$

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Let $(r_n)_{n=0}^{\infty}$ *and* $(s_n)_{n=0}^{\infty}$ *be two infinite sequences of nonnegative rational integers such that*

$$0 = r_0 < s_0 < r_1 < s_1 < r_2 < s_2 < r_3 < s_3 < \cdots \quad \text{and} \quad r_{n+1} - s_n \geq 2.$$

Denote by p_n/q_n $(n = 0, 1, 2, \ldots)$ *the nth convergent of the Ruban p-adic continued fraction (3.1). Assume that*

$$\lim_{n\to\infty} \frac{\log |q_{s_n}|_p}{\log |q_{r_n}|_p} = \infty \tag{3.2}$$

and

$$\limsup_{n\to\infty} \frac{\log |q_{r_{n+1}}|_p}{\log |q_{s_n}|_p} < \infty. \tag{3.3}$$

Define the rational numbers b_j $(j = 0, 1, 2, \ldots)$ *by*

$$b_j = \begin{cases} a_j & \text{if } r_n \leq j \leq s_n \quad (n = 0, 1, 2, \ldots), \\ \upsilon_j & \text{if } s_n < j < r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases} \tag{3.4}$$

where υ_j *is a rational number of the form*

$$\upsilon_j = c_{-d}p^{-d} + c_{-d+1}p^{-d+1} + \cdots + c_{-1}p^{-1} + c_0.$$

Here, $d \in \mathbb{Z}$, $d > 0$, $c_{-d} \neq 0$ *and* $c_i \in \{0, 1, \ldots, p-1\}$ *for* $i = -d, -d+1, \ldots, -1, 0$. *Note that* $|\upsilon_j|_p \geq p$. *Suppose that* $|\upsilon_j|_p \leq \kappa_1 |a_j|_p^{\kappa_2}$ *and* $\sum_{j=s_n+1}^{r_{n+1}-1} |a_j - \upsilon_j|_p \neq 0$, *where* κ_1 *and* κ_2 *are fixed positive rational integers. Then the irrational p-adic number* $\xi = [b_0, b_1, b_2, \ldots]_p$ *is a p-adic* U_m*-number, where m denotes the degree of the algebraic irrational p-adic number* α.

REMARK 3.2. Let $\mathbb{F}_q$ be the finite field with q elements and let $\mathbb{F}_q((x^{-1}))$ be the field of formal power series over $\mathbb{F}_q$. In $\mathbb{F}_q((x^{-1}))$, Can and Kekeç [6, Theorem 1.1] recently established the formal power series analogue of Alnıaçık [2, Theorem].

Recently, Kekeç [11, Theorem 1.5] modified the hypotheses in Alnıaçık [2, Theorem] and gave a construction of transcendental real numbers that are not U-numbers by using continued fraction expansions of irrational algebraic real numbers. Our second main result in the present paper is the following partial p-adic analogue of Kekeç [11, Theorem 1.5].

THEOREM 3.3. *Let* α *be an algebraic p-adic number of degree* $m \geq 2$ *with* $|\alpha|_p \geq 1$ *and the Ruban p-adic continued fraction expansion*

$$\alpha = [a_0, a_1, a_2, \ldots]_p.$$

Let $(r_n)_{n=0}^{\infty}$ *and* $(s_n)_{n=0}^{\infty}$ *be two infinite sequences of nonnegative rational integers such that*

$$0 = r_0 < s_0 < r_1 < s_1 < r_2 < s_2 < r_3 < s_3 < \cdots \quad \text{and} \quad r_{n+1} - s_n \geq 2.$$

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Denote by p_n/q_n $(n = 0, 1, 2, \ldots)$ the nth convergent of the Ruban p-adic continued fraction α. Define the rational numbers b_j $(j = 0, 1, 2, \ldots)$ by

$$b_j = \begin{cases} a_j & \text{if } r_n \le j \le s_n \quad (n = 0, 1, 2, \ldots), \\ \upsilon_j & \text{if } s_n < j < r_{n+1} \quad (n = 0, 1, 2, \ldots), \end{cases} \tag{3.5}$$

where υ_j is a rational number of the form

$$\upsilon_j = c_{-d}p^{-d} + c_{-d+1}p^{-d+1} + \cdots + c_{-1}p^{-1} + c_0.$$

Here $d \in \mathbb{Z}$, $d > 0$, $c_{-d} \neq 0$ and $c_i \in \{0, 1, \ldots, p-1\}$ for $i = -d, -d+1, \ldots, -1, 0$. Note that $|\upsilon_j|_p \ge p$. Suppose that $|\upsilon_j|_p \le \kappa_1 |a_j|_p^{\kappa_2}$ and $\sum_{j=s_n+1}^{r_{n+1}-1} |a_j - \upsilon_j|_p \neq 0$, where κ_1 and κ_2 are fixed positive rational integers. Assume that

$$\liminf_{n\to\infty} \frac{\log |q_{s_n}|_p}{\log |q_{r_n}|_p} > 2 + 4m\Big(m + \kappa_2 + \frac{\log \kappa_1}{\log 2}\Big). \tag{3.6}$$

Then the irrational p-adic number $\xi = [b_0, b_1, b_2, \ldots]_p$ is transcendental.

In the next section, we cite some auxiliary results that we need to prove our results. In Section 5, we prove Theorems 3.1 and 3.3.

4. Auxiliary results

The following lemma is a p-adic analogue of Alnıaçık [2, Lemma IV].

LEMMA 4.1. *Let p/q and u/v be two rational numbers with Ruban p-adic continued fraction expansions*

$$\frac{p}{q} = [a_0, a_1, \ldots, a_n]_p \quad \text{and} \quad \frac{u}{v} = [b_0, b_1, \ldots, b_n]_p \quad (|a_0|_p \ge 1, |b_0|_p \ge 1).$$

Assume that

$$|b_j|_p \le \kappa_1 |a_j|_p^{\kappa_2} \quad (j = 0, 1, \ldots, n), \tag{4.1}$$

where κ_1 and κ_2 are fixed positive rational integers. Then

$$|u|_p \le |a_0|_p^{\kappa_2} \kappa_1 |q|_p^{\kappa_2 + \log \kappa_1/\log 2}.$$

PROOF. It follows from (4.1) that

$$|u|_p = |b_0|_p \cdot |b_1|_p \cdots |b_n|_p \le \kappa_1^{n+1} \cdot (|a_0|_p \cdot |a_1|_p \cdots |a_n|_p)^{\kappa_2}.$$

As $|q|_p = |a_1|_p \cdots |a_n|_p \ge p^n \ge 2^n$,

$$|u|_p \le (2^{n+1})^{\log \kappa_1/\log 2} |a_0|_p^{\kappa_2} |q|_p^{\kappa_2} \le |a_0|_p^{\kappa_2} \kappa_1 |q|_p^{\kappa_2 + \log \kappa_1/\log 2}. \qquad \square$$

THEOREM 4.2 (Içen [8, page 25] and [7, Lemma 1, page 71]). *Let L be a p-adic algebraic number field of degree m and let $\alpha_1, \ldots, \alpha_k$ be algebraic p-adic numbers in L. Let η be any algebraic p-adic number. Suppose that $F(\eta, \alpha_1, \ldots, \alpha_k) = 0$, where*

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$F(x, x_1, \ldots, x_k)$ is a polynomial in $x, x_1, \ldots, x_k$ over $\mathbb{Z}$ with degree at least one in x. Then

$$H(\eta) \leq 3^{2dm+(l_1+\cdots+l_k)m} H^m H(\alpha_1)^{l_1 m} \cdots H(\alpha_k)^{l_k m},$$

where d is the degree of $F(x, x_1, \ldots, x_k)$ in x, l_i is the degree of $F(x, x_1, \ldots, x_k)$ in x_i $(i = 1, \ldots, k)$ and H is the maximum of the usual absolute values of the coefficients of $F(x, x_1, \ldots, x_k)$.

LEMMA 4.3 (Pejkovic [18, Lemma 2.5]). *Let α_1 and α_2 be two distinct algebraic p-adic numbers. Then*

$$|\alpha_1 - \alpha_2|_p \geq (\deg(\alpha_1) + 1)^{-\deg(\alpha_2)} (\deg(\alpha_2) + 1)^{-\deg(\alpha_1)} H(\alpha_1)^{-\deg(\alpha_2)} H(\alpha_2)^{-\deg(\alpha_1)}.$$

LEMMA 4.4 (Ooto [17, Lemma 7 and page 1058]). *Let α be a p-adic number with $|\alpha|_p \geq 1$ and let p_n/q_n be the nth convergent of its Ruban p-adic continued fraction expansion. Then $p_n \leq |p_n|_p$, $q_n \leq |q_n|_p$ and*

$$p_n \cdot |p_n|_p \in \mathbb{Z} \quad q_n \cdot |q_n|_p \in \mathbb{Z}.$$

THEOREM 4.5 (Lang [13, page 32]). *Let K be a p-adic algebraic number field and let α be any algebraic p-adic number. Then, for each $\varepsilon > 0$, the inequality*

$$|\alpha - \beta|_p < \frac{1}{H(\beta)^{2+\varepsilon}}$$

has only finitely many solutions β in K.

5. Proofs of Theorems 3.1 and 3.3

PROOF OF THEOREM 3.1. We prove Theorem 3.1 by adapting the method of the proof of Alnıaçık [2, Theorem] to the non-Archimedean p-adic case. Define the algebraic p-adic numbers

$$\alpha_{r_n} := [b_0, b_1, \ldots, b_{r_n}, a_{r_n+1}, a_{r_n+2}, \ldots]_p \in \mathbb{Q}(\alpha) \quad (n = 0, 1, 2, \ldots)$$

and

$$\beta_{r_n} := [a_{r_n+1}, a_{r_n+2}, \ldots]_p \in \mathbb{Q}(\alpha) \quad (n = 0, 1, 2, \ldots).$$

Then $\deg(\alpha_{r_n}) = \deg(\beta_{r_n}) = m$ $(n = 0, 1, 2, \ldots)$. By (2.3),

$$\alpha = [a_0, a_1, \ldots, a_{r_n}, \beta_{r_n}]_p = \frac{p_{r_n}\beta_{r_n} + p_{r_n-1}}{q_{r_n}\beta_{r_n} + q_{r_n-1}} \quad (n = 0, 1, 2, \ldots)$$

and thus

$$\alpha q_{r_n}\beta_{r_n} + \alpha q_{r_n-1} - p_{r_n}\beta_{r_n} - p_{r_n-1} = 0 \quad (n = 0, 1, 2, \ldots).$$

Therefore, $F(\beta_{r_n}, \alpha) = 0$, where, by Lemma 4.4,

$$F(x, x_1) = |p_{r_n}|_p q_{r_n} x_1 x + |p_{r_n}|_p q_{r_n-1} x_1 - |p_{r_n}|_p p_{r_n} x - |p_{r_n}|_p p_{r_n-1}$$

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is a polynomial in x, x_1 over $\mathbb{Z}$. It follows from Theorem 4.2 and Lemma 4.4 that

$$H(\beta_{r_n}) \le c_1 |q_{r_n}|_p^{2m}, \tag{5.1}$$

where $c_1 = 3^{3m}|a_0|_p^{2m} H(\alpha)^m$. Set

$$\frac{p'_n}{q'_n} := [b_0, b_1, \ldots, b_n]_p \quad (n = 0, 1, 2, \ldots).$$

Then

$$\alpha_{r_n} = [b_0, b_1, \ldots, b_{r_n}, \beta_{r_n}]_p = \frac{p'_{r_n}\beta_{r_n} + p'_{r_n-1}}{q'_{r_n}\beta_{r_n} + q'_{r_n-1}} \quad (n = 0, 1, 2, \ldots)$$

and

$$\alpha_{r_n} q'_{r_n}\beta_{r_n} + \alpha_{r_n} q'_{r_n-1} - p'_{r_n}\beta_{r_n} - p'_{r_n-1} = 0 \quad (n = 0, 1, 2, \ldots).$$

Thus, $F(\alpha_{r_n}, \beta_{r_n}) = 0$, where, by Lemma 4.4,

$$F(x, x_1) = |p'_{r_n}|_p q'_{r_n} x_1 x + |p'_{r_n}|_p q'_{r_n-1} x - |p'_{r_n}|_p p'_{r_n} x_1 - |p'_{r_n}|_p p'_{r_n-1}$$

is a polynomial in x, x_1 over $\mathbb{Z}$. It follows from Theorem 4.2, Lemma 4.4 and (5.1) that

$$H(\alpha_{r_n}) \le 3^{3m} |p'_{r_n}|_p^{2m} c_1^m |q_{r_n}|_p^{2m^2}. \tag{5.2}$$

From (3.4),

$$|b_j|_p \le \kappa_1 |a_j|_p^{\kappa_2} \quad (j = 0, 1, 2, \ldots).$$

By Lemma 4.1,

$$|p'_{r_n}|_p \le |a_0|_p^{\kappa_2} \kappa_1 |q_{r_n}|_p^{\kappa_2 + \log \kappa_1 / \log 2} \quad (n = 0, 1, 2, \ldots). \tag{5.3}$$

Using (5.2), (5.3) and $\lim_{n\to\infty} |q_{r_n}|_p = \infty$, we obtain, for sufficiently large n,

$$H(\alpha_{r_n}) \le |q_{r_n}|_p^{c_2}, \tag{5.4}$$

where $c_2 = 1 + (m + \kappa_2 + \log \kappa_1 / \log 2)2m$.

We approximate ξ by the algebraic p-adic numbers α_{r_n}. We infer from (2.6) and (3.4) that

$$|\xi - \alpha_{r_n}|_p \le \max\left\{\left|\xi - \frac{p'_{s_n}}{q'_{s_n}}\right|_p, \left|\alpha_{r_n} - \frac{p'_{s_n}}{q'_{s_n}}\right|_p\right\} < \frac{1}{|q'_{s_n}|_p^2} \quad (n = 0, 1, 2, \ldots). \tag{5.5}$$

Put

$$\frac{d_{r_n}}{e_{r_n}} := [a_{r_n+1}, a_{r_n+2}, \ldots, a_{s_n}]_p = [b_{r_n+1}, b_{r_n+2}, \ldots, b_{s_n}]_p.$$

We have

$$\frac{p_{s_n}}{q_{s_n}} = [a_0, a_1, \ldots, a_{r_n}, a_{r_n+1}, a_{r_n+2}, \ldots, a_{s_n}]_p$$

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and

$$\frac{p'_{s_n}}{q'_{s_n}} = [b_0, b_1, \ldots, b_{r_n}, b_{r_n+1}, b_{r_n+2}, \ldots, b_{s_n}]_p.$$

Then

$$|q_{s_n}|_p = |a_1 \cdots a_{r_n}|_p |a_{r_n+1} \cdots a_{s_n}|_p = |q_{r_n}|_p |a_{r_n+1}|_p |e_{r_n}|_p$$

and

$$|q'_{s_n}|_p = |b_1 \cdots b_{r_n+1}|_p |b_{r_n+2} \cdots b_{s_n}|_p > |e_{r_n}|_p.$$

Therefore,

$$|q_{s_n}|_p < |a_{r_n+1}|_p |q_{r_n}|_p |q'_{s_n}|_p \quad (n = 0, 1, 2, \ldots). \tag{5.6}$$

It follows from Lemmas 4.3 and 4.4 that

$$\left|\alpha - \frac{p_{r_n}}{q_{r_n}}\right|_p \geq \frac{1}{c_3 |q_{r_n}|_p^{2m}}, \tag{5.7}$$

where $c_3 = (m+1)2^m H(\alpha)|a_0|_p^{2m}$. On the other hand, by (2.6),

$$\left|\alpha - \frac{p_{r_n}}{q_{r_n}}\right|_p = \frac{1}{|a_{r_n+1}|_p |q_{r_n}|_p^2} \quad (n = 0, 1, 2, \ldots). \tag{5.8}$$

Combining (5.6), (5.7) and (5.8), we get

$$|q_{s_n}|_p < c_3 |q_{r_n}|_p^{2m-1} |q'_{s_n}|_p. \tag{5.9}$$

By (3.2) and (5.9),

$$c_3 |q_{r_n}|_p^{2m-1} \leq |q'_{s_n}|_p$$

for sufficiently large n. So, for sufficiently large n,

$$|q_{s_n}|_p < |q'_{s_n}|_p^2. \tag{5.10}$$

We see from (3.2), (5.4), (5.5) and (5.10) that

$$0 < |\xi - \alpha_{r_n}|_p < \frac{1}{|q_{s_n}|_p} \leq \frac{1}{H(\alpha_{r_n})^{\phi_n}}$$

for sufficiently large n, where

$$\phi_n = \frac{\log |q_{s_n}|_p}{c_2 \log |q_{r_n}|_p} \quad \text{and} \quad \lim_{n \to \infty} \phi_n = \infty.$$

As $\deg(\alpha_{r_n}) = m$ $(n = 0, 1, 2, \ldots)$, this shows that ξ is a p-adic U^*-number with

$$w_m^*(\xi) = \infty. \tag{5.11}$$

We wish to show that ξ is a p-adic U_m^*-number. We must prove that $w_t^*(\xi) < \infty$ for $t = 1, \ldots, m-1$. Let β be any algebraic p-adic number with $1 \leq \deg(\beta) \leq m-1$ and with sufficiently large height $H(\beta)$. We deduce from Lemma 4.3 and (5.4) that

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$$|\alpha_{r_n} - \beta|_p \geq \frac{1}{c_4 |q_{r_n}|_p^{c_5} H(\beta)^m} \tag{5.12}$$

for sufficiently large n, where $c_4 = (m+1)^{m-1} m^m$ and $c_5 = c_2(m-1)$. By (3.3), there exists a real number $T > 1$ such that

$$|q_{s_n}|_p^T \geq |q_{r_{n+1}}|_p \tag{5.13}$$

for sufficiently large n. We have

$$|\xi - \beta|_p = |(\xi - \alpha_{r_n}) + (\alpha_{r_n} - \beta)|_p. \tag{5.14}$$

From (5.5), (5.10) and (5.13), for sufficiently large n,

$$|\xi - \alpha_{r_n}|_p < \frac{1}{|q'_{s_n}|_p^2} < \frac{1}{|q_{s_n}|_p} \leq \frac{1}{|q_{r_{n+1}}|_p^{1/T}}. \tag{5.15}$$

Let i be the unique positive rational integer satisfying $|q_{r_i}|_p \leq H(\beta) < |q_{r_{i+1}}|_p$. Put $T_1 := T(m + c_5 + 1)$. If $|q_{r_i}|_p \leq H(\beta) < |q_{r_{i+1}}|_p^{1/T_1}$, then it follows from (5.12), (5.14) and (5.15) with $n = i$ that

$$|\xi - \beta|_p \geq \frac{1}{c_4 H(\beta)^{m+c_5}}. \tag{5.16}$$

If $|q_{r_{i+1}}|_p^{1/T_1} \leq H(\beta) < |q_{r_{i+1}}|_p$, then it follows from (3.2), (5.12), (5.14) and (5.15) with $n = i + 1$ that

$$|\xi - \beta|_p \geq \frac{1}{c_4 H(\beta)^{m+c_5 T_1}}. \tag{5.17}$$

We deduce from (5.16) and (5.17) that

$$|\xi - \beta|_p \geq \frac{1}{c_4 H(\beta)^{m+c_5 T_1}}$$

for all algebraic p-adic numbers β with $\deg(\beta) \leq m - 1$ and with sufficiently large height $H(\beta)$. This gives

$$w_t^*(\xi) < \infty \quad (t = 1, \ldots, m-1). \tag{5.18}$$

We infer from (5.11) and (5.18) that ξ is a p-adic U_m^*-number. As the set of p-adic U_m-numbers is equal to the set of p-adic U_m^*-numbers, ξ is a p-adic U_m-number. □

EXAMPLE 5.1. This example illustrates Theorem 3.1. In Theorem 3.1, take the algebraic p-adic number α as the quadratic irrational

$$\alpha = [a_0, a_1, a_2, \ldots]_p = [p^{-2}, p^{-2}, p^{-2}, \ldots]_p$$

and the sequences $(r_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ as

$$r_0 = 0, r_n = 2(n+1)!\ \ (n = 1, 2, 3, \ldots) \quad \text{and} \quad s_n = (n+2)!\ \ (n = 0, 1, 2, \ldots).$$

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Define the rational numbers b_j $(j = 0, 1, 2, \ldots)$ by

$$b_j = \begin{cases} p^{-2} & \text{if } r_n \le j \le s_n \quad (n = 0, 1, 2, \ldots), \\ p^{-4} & \text{if } s_n < j < r_{n+1} \quad (n = 0, 1, 2, \ldots). \end{cases}$$

Take $\kappa_1 = 1$ and $\kappa_2 = 2$. Then all the conditions of Theorem 3.1 are satisfied and therefore the irrational p-adic number $\xi = [b_0, b_1, b_2, \ldots]_p$ is a p-adic U_2-number.

REMARK 5.2. In Theorem 3.1, if we replace $\lim_{n\to\infty}(\log |q_{s_n}|_p / \log |q_{r_n}|_p) = \infty$ by

$$\liminf_{n\to\infty} \frac{\log |q_{s_n}|_p}{\log |q_{r_n}|_p} > T(1 + m + c_5 T_1) \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log |q_{s_n}|_p}{\log |q_{r_n}|_p} = \infty,$$

then we see from the proof that Theorem 3.1 still holds true.

PROOF OF THEOREM 3.3. We replace (3.2) by (3.6) and keep all the lines of the proof of Theorem 3.1 up to (5.10). By (3.6), there exists a positive real number ε such that

$$\frac{\log |q_{s_n}|_p}{\log |q_{r_n}|_p} > (2 + \varepsilon)c_2 \tag{5.19}$$

for sufficiently large n. We deduce from (5.4), (5.5), (5.10) and (5.19) that

$$0 < |\xi - \alpha_{r_n}|_p < \frac{1}{H(\alpha_{r_n})^{2+\varepsilon}}$$

for sufficiently large n. It follows from the definition of α_{r_n} and (3.5) that the algebraic p-adic numbers α_{r_n} in $\mathbb{Q}(\alpha)$ are all distinct. Then, by Theorem 4.5, the irrational p-adic number ξ is transcendental. □

Finally, we pose the following question.

PROBLEM 5.3. Does an exact analogue of Kekeç [11, Theorem 1.5] hold in $\mathbb{Q}_p$?

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