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Local Solvability of Laplacian Difference Operators Arising from the Discrete Heisenberg Group

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Abstract. Differential operators D_x, D_y, and D_z are formed using the action of the 3-dimensional discrete Heisenberg group G on a set S, and the operators will act on functions on S. The Laplacian operator $L = D_x^2 + D_y^2 + D_z^2$ is a difference operator with variable differences which can be associated to a unitary representation of G on the Hilbert space $L^2(S)$. Using techniques from harmonic analysis and representation theory, we show that the Laplacian operator is locally solvable.

1 Introduction

The Heisenberg group has been instrumental in the development of a wide range of mathematical topics (see [6], [4]). Techniques involving its representation theory often yield interesting results or straightforward proofs of existing theorems, as seen in [1] and [13]. Over the past 25 years, the Heisenberg group has been a standard setting for the analysis of increasingly general operators on locally compact nilpotent Lie groups. A few examples of this work are [3], [12], and [8]. This work uses representation theory to examine loosely analogous differential operators arising from the non-type-I discrete Heisenberg group.

Let G be the 3-dimensional discrete Heisenberg group G, considered as the 3×3 upper triangular integral matrices with diagonal entries equal to one:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

For compressed notation, consider the elements as triples (x, y, z). The group operation is given by matrix multiplication:

$$(x', y', z')(x, y, z) = (x' + x, y' + y, z' + z + x'y).$$

The inverse element is then seen to be $(x, y, z)^{-1} = (-x, -y, -z + xy)$. Let H be the subgroup $\{(x, 0, 0) : x \in \mathbb{Z}\}$ and let S be the normal subgroup $\{(0, y, z) : y, z \in \mathbb{Z}\}$, so $S \cong \mathbb{Z}^2$ and $G = S \ltimes H$. If we take $S = \mathbb{Z}^2$, and let $s = (y, z)$ and $g = (x', y', z')$, then there is a left action of G on S given by multiplication and projection:

$$g \cdot s = (y + y', z + z' + x'y) \tag{1}$$

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There is a natural notion of difference operators in terms of the generators $x = (1,0,0)$, $y = (0,1,0)$, and $z = (0,0,1)$ acting on a function space of S. Let $f \in L^2(S)$ and define operators D_x, D_y, and D_z using the action of G on S:

$$D_x f(s) = f[(1,0,0)\cdot s] - f[(-1,0,0)\cdot s]$$
$$D_y f(s) = f[(0,1,0)\cdot s] - f[(0,-1,0)\cdot s]$$
$$D_z f(s) = f[(0,0,1)\cdot s] - f[(0,0,-1)\cdot s].$$

We then define the Laplacian operator $L = {D_x}^2 + {D_y}^2 + {D_z}^2$ for $f \in L^2(S)$ and $s = (y, z)$:

$$\begin{aligned} Lf(y,z) = f(y, z-2y) + f(y, z+2y) + f(y-2, z) + f(y+2, z) \\ + f(y, z-2) + f(y, z+2) - 6f(y,z). \end{aligned} \tag{2}$$

L is a difference operator with variable differences. Given $f \in L^2(S)$, consider the difference equation $Lu = f$. One goal of this paper is to describe conditions under which this equation has a solution in $L^2(S)$. These conditions lead to the primary result of this paper (Theorem 11 in the text).

Theorem *The Laplacian operator L given in equation* (2) *is locally solvable.*

Local solvability on a discrete space requires definition. In [11], the property described was *local solvability on A* for $A \subset S$. We have modified the usage slightly to avoid this restriction to subsets. An operator L is considered *locally solvable* if, for every compact subset A of S and every function f in $L^2(S)$, there exists a function u in $L^2(S)$ for which $Lu = f$ on A.

We prove this theorem by associating our particular operator L to a unitary representation of G on the Hilbert space $\mathcal{H} = L^2(S, \mu)$, where μ is counting measure, then finding an equivalent representation of G which can be written as a direct integral of representations. The Laplacian can be similarly decomposed, yielding a collection of factor equations which are shown in Section 2 to have solutions for almost every β. Section 3 presents a careful verification that the solutions of the factor equations can be recombined in a measurable way. In Section 4, we find sufficient conditions for functions $f \in L^2(S)$ such that the equation $Lu = f$ has a solution $u \in L^2(S)$ and from these arrive at the local solvability of L. After showing that L is in fact locally solvable, we demonstrate in the last section how this result leads to the determination of local solvability in sublaplacian operators.

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2 Factor Equations $L^\beta \hat{u}_\beta = \hat{f}_\beta$

Using the same action of G on S given in equation (1), define operators $U_g \in U(\mathcal{H})$ by

$$U_g f(s) = f(g^{-1}\cdot s) = f\big(y - y', z - z' - x'(y - y')\big). \tag{3}$$

Since μ is invariant for the action, the map $U\colon G \to U(\mathcal{H})$ given by $g \mapsto U_g$ is a unitary representation of G. For a proof of this, see [5]. We can now express the Laplacian in terms of the representation elements.

$$L = U_{(2,0,0)} + U_{(-2,0,0)} + U_{(0,2,0)} + U_{(0,-2,0)} + U_{(0,0,2)} + U_{(0,0,-2)} - 6I$$

The Fourier transform and its inverse are unitary maps between $l^2(\mathbb{Z}^2)$ and $L^2(\mathbb{T}^2)$, where $\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}$. The versions used in this paper will be:

$$\mathcal{F}^{-1} f(\alpha, \beta) = \sum_{y,z\in\mathbb{Z}} f(y,z)\alpha^y \beta^z$$

$$\mathcal{F}h(y,z) = \int_{\mathbb{T}^2} h(\alpha,\beta)\alpha^{-y}\beta^{-z}\, d\alpha\, d\beta.$$

Conjugation of U by $\mathcal{F}^{-1}$ yields an equivalent representation of G on $L^2(\mathbb{T}^2, \nu)$, where ν is Lebesgue measure on $\mathbb{T}^2$. Let $f \in L^2(\mathbb{T}^2)$:

$$\begin{aligned} \text{(4)}\quad \mathcal{F}^{-1}U_g\mathcal{F} f(\alpha,\beta) &= \sum_{y,z\in\mathbb{Z}} U_g\mathcal{F} f(y,z)\alpha^y\beta^z \\ &= \sum_{y,z\in\mathbb{Z}} \int_{\mathbb{T}^2} f(\alpha',\beta')\alpha^{-(y-y')}\beta^{-(z-z'-x'(y-y'))}\, d\alpha'\, d\beta' \alpha^y\beta^z \\ &= f(\alpha\beta^{x'},\beta)\alpha^{y'}\beta^{z'}. \end{aligned}$$

The new representation elements act as multiplication operators in the β variable, so $\mathcal{F}^{-1}U\mathcal{F}$ can be expressed as a direct integral of representations of G, each on the space $L^2(\mathbb{T})$. Specifically, for each $\beta \in \mathbb{T}$ and $g = (x', y', z')$ in G, define a unitary operator U_g^β on $L^2(\mathbb{T})$ by

$$U_g^\beta f(\alpha) = f(\alpha\beta^{x'})\alpha^{y'}\beta^{z'}.$$

The map $g \mapsto U_g^\beta$ is a unitary representation U^β of G on $L^2(\mathbb{T})$, and $\mathcal{F}^{-1}U\mathcal{F} = \int_{\mathbb{T}}^{\oplus} U^\beta\, d\beta$.

The Laplacian can be similarly conjugated by $\mathcal{F}^{-1}$ and decomposed into operators on $L^2(\mathbb{T})$. For each β, we form a factor operator L^β such that $\mathcal{F}^{-1}L\mathcal{F} = \int_{\mathbb{T}}^{\oplus} L^\beta\, d\beta$:

$$\text{(5)}\quad L^\beta f(\alpha) = f(\alpha\beta^2) + f(\alpha\beta^{-2}) + \alpha^2 f(\alpha) + \alpha^{-2} f(\alpha) + \beta^2 f(\alpha) + \beta^{-2} f(\alpha) - 6f(\alpha).$$

The difference equation $Lu = f$ can now be analyzed through the collection of factor equations:

$$L^\beta v_\beta = \hat{f}_\beta \tag{6}$$

where $\hat{f} = \mathcal{F}^{-1}f \in L^2(\mathbb{T}^2)$ and we define $\hat{f}_\beta(\alpha) = \hat{f}(\alpha,\beta)$, which are elements of $L^2(\mathbb{T})$ for almost every β. The solutions v_β will be denoted more suggestively as $\hat{u}_\beta$, as they will comprise the solution u to $Lu = f$.

The following lemma is a well-known result which we use several times to prove invertibility of operators.

Lemma 1 *If T is an operator on a Hilbert space $\mathcal{H}$, and if there exists a $\delta > 0$ for which $|\langle Tx, x\rangle| > \delta$ for all $x \in \mathcal{H}$ with $\|x\| = 1$, then T has a bounded inverse and $\|T^{-1}\| \leq \frac{1}{\delta}$.*

Proof It is clear that, given such a δ, we also have $|\langle T^* x, x\rangle| > \delta$ for all unit vectors $x \in \mathcal{H}$. Therefore, both T and T^* have trivial null space. Since the orthogonal complement of the null space of T^* is the closure of the range of T, we know that the range of T is dense in $\mathcal{H}$. Then, since $\delta < |\langle Tx, x\rangle| \leq \|Tx\|$, we find that the range of T is closed, and is therefore all of $\mathcal{H}$.

T has now been shown to have a well-defined bounded inverse, and the bound on the norm of T^{-1} is straightforward to verify:

$$\|T^{-1}\| = \sup_x \frac{\|T^{-1}x\|}{\|x\|} = \sup_y \frac{\|y\|}{\|Ty\|} \leq \sup_y \frac{\|y\|}{\delta\|y\|} = \frac{1}{\delta}. \qquad \blacksquare$$

Proposition 2 *For almost every $\beta \in \mathbb{T}$, L^β has a bounded inverse.*

Proof Let $\|f\|_2 = 1$. Note that L^β is self-adjoint, so $\langle L^\beta f, f\rangle \in \mathbb{R}$.

$$\begin{aligned}\langle L^\beta f, f\rangle &= \int_{\mathbb{T}} [f(\alpha\beta^2) + f(\alpha\beta^{-2}) + \alpha^2 f(\alpha) + \alpha^{-2} f(\alpha) \\ &\qquad + \beta^2 f(\alpha) + \beta^{-2} f(\alpha) - 6 f(\alpha)]\overline{f(\alpha)}\, d\alpha \\ &= 2\,\mathrm{Re}\,\beta^2 - 6 + \int_{\mathbb{T}} [f(\alpha\beta^2) + f(\alpha\beta^{-2}) + \alpha^2 f(\alpha) + \alpha^{-2} f(\alpha)]\bar{f}(\alpha)\, d\alpha \\ &\leq 2\,\mathrm{Re}\,\beta^2 - 2.\end{aligned}$$

The Cauchy–Schwartz inequality gives the last integral ≤ 4. For $\beta \neq \pm 1$, there is a $\delta_\beta > 0$ such that $\mathrm{Re}\,\beta^2 < 1 - \frac{\delta_\beta}{2}$, which will give $\langle L^\beta f, f\rangle < -\delta_\beta$. So, by Lemma 1, for $\beta \neq \pm 1$, the operator L^β has a bounded inverse and $\|(L^\beta)^{-1}\| \leq \frac{1}{\delta_\beta}$. $\blacksquare$

Proposition 2 gives, for almost every $\beta \in \mathbb{T}$, a solution to equation (6), $\hat{u}_\beta = (L^\beta)^{-1}\hat{f}_\beta$. The next step is to combine these solutions to form the function $\hat{u}$ on $\mathbb{T}^2$:

$$\hat{u}(\alpha, \beta) = \hat{u}_\beta(\alpha).$$

Before proceeding, we must ensure that $\hat{u}$ can always be formed in a measurable way. The following section addresses this detail, and the question of a solution to $Lu = f$ is continued in Section 4.

3 Measurability of $\hat{u}$

It is not obvious that the function $\hat{u}$, as defined in the previous section, is measurable in both variables. This is a detail requiring careful attention. The proof will use four technical results, which we state here as lemmas. These are not new results, but involve some subtleties from functional analysis. In order to preserve the flow of the paper, the proofs of the lemmas have been included in an appendix.

Lemma 3 *The Borel sets of a separable Hilbert space $\mathcal{H}$ with the norm topology coincide with the weak Borel sets.*

The map from an operator T to its adjoint T^* is not, in general, continuous in the strong operator topology. We establish here, however, that the adjoint map is Borel as a map on a norm-bounded set $S_M \subset \mathcal{B}(\mathcal{H})$, where S_M is given the strong operator topology.

Lemma 4 *Given $\mathcal{H}$, a separable Hilbert space, and S_M, the set of operators in $\mathcal{B}(\mathcal{H})$ bounded in norm by M, the map $T \mapsto T^*$ is Borel in the strong operator topology on S_M.*

The next lemma considers the map from an operator T to its inverse, where T comes from a norm-bounded set of invertible operators. If the inverses were also uniformly bounded, the inverse map would be continuous in the strong operator topology. For our purposes, there is no such bound on the norms of the inverses, but we can show that the inverse map is Borel as a map from a set of uniformly bounded invertible operators with the strong operator topology to $\mathcal{B}(\mathcal{H})$.

Lemma 5 *Given $\mathcal{H}$ a separable Hilbert space and S_M the set of operators in $\mathcal{B}(\mathcal{H})$ bounded uniformly in norm by M, the map $T \mapsto T^{-1}$ is a Borel map in the strong operator topology from the invertible operators in S_M to $\mathcal{B}(\mathcal{H})$.*

Note that the composition of the Borel maps from Lemmas 4 and 5 gives an additional result that $T \mapsto (T^{-1})^*$ is also a Borel map in the strong operator topology from S_M to $\mathcal{B}(\mathcal{H})$.

Lemma 6 *Let A, B be σ-finite measure spaces. For each $\beta \in B$, let f_β be an element of $L^2(A)$.*

(1) *If $f_\beta(\alpha)$ as a map on $A \times B$ is in $L^2(A \times B)$, then the map $\beta \mapsto f_\beta$ is a Borel map into $L^2(A)$.*
(2) *If the map $\beta \mapsto f_\beta$ is Borel, then $f_\beta(\alpha)$ can be selected to be a Borel map on the product space $A \times B$.*

These four results are now combined to prove the measurability of $\hat{u}$.

Proposition 7 *Define the function $\hat{u}$ by*

$$\hat{u}(\alpha, \beta) = \hat{u}_\beta(\alpha) = (L^\beta)^{-1} f_\beta(\alpha)$$

where $f_\beta(\alpha) = f(\alpha, \beta)$ for a function $f \in L^2(\mathbb{T}^2)$. Then $\hat{u}$ is a Borel measurable function on $\mathbb{T}^2$.

Proof Consider the inner product:

$$\langle f_\beta, g \rangle = \int_{\mathbb{T}} f_\beta(\alpha) \bar{g}(\alpha)\, d\alpha.$$

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The integrand is Borel on $\mathbb{T}^2$, so the integral is Borel on $\mathbb{T}$ by Fubini's Theorem, and Lemma 3 gives f_β a Borel map from $\mathbb{T}$ to $L^2(\mathbb{T})$.

Using the definition of L^β, we have:

$$L^\beta f(\alpha) = f(\alpha\beta^2) + f(\alpha\beta^{-2}) + \alpha^2 f(\alpha) + \alpha^{-2} f(\alpha) + \beta^2 f(\alpha) + \beta^{-2} f(\alpha) - 6f(\alpha).$$

This function is Borel (and in fact continuous) in β, so L^β is a Borel function of β in the strong operator topology. Next, since almost every L^β is invertible, we can write the following inner product for almost every β.

$$\begin{aligned} \langle (L^\beta)^{-1} f_\beta, g\rangle &= \left\langle f_\beta, ((L^\beta)^{-1})^* g\right\rangle \\ &= \int_{\mathbb{T}} f_\beta(\alpha)\overline{((L^\beta)^{-1})^* g(\alpha)}\, d\alpha \end{aligned} \tag{7}$$

By Lemmas 4 and 5, we have $((L^\beta)^{-1})^*$ Borel in the strong operator topology, which gives $((L^\beta)^{-1})^* g \in L^2(\mathbb{T})$ a Borel function of β. Lemma 6 then gives $((L^\beta)^{-1})^* g(\alpha)$ jointly Borel on $\mathbb{T}^2$, and thus the integrand in equation (7) is also Borel. By Fubini's theorem, then, the inner product is a Borel function of β and Lemma 3 gives $\hat{u}(\alpha,\beta) = (L^\beta)^{-1} f_\beta(\alpha)$ a Borel function on $\mathbb{T}^2$. ■

4 Local Solvability of L

Now that we have established the measurability of the function $\hat{u}$, we determine sufficient conditions on a function $f \in l^2(\mathbb{Z}^2)$ under which the equation $Lu = f$ has a solution.

Theorem 8 *Let L be the Laplacian operator and let $f \in l^2(\mathbb{Z}^2)$ have compact support and satisfy the following conditions for every $y \in \mathbb{Z}$:*

$$\sum_z f(y,z) = 0, \qquad \sum_z f(y,z)z = 0,$$
$$\sum_z (-1)^z f(y,z) = 0, \quad \sum_z (-1)^z f(y,z)z = 0.$$

Then there exists a function u for which $Lu = f$.

Proof We have found a measurable function satisfying $\mathcal{F}^{-1}L\mathcal{F}\hat{u} = \hat{f}$. If the solution $\hat{u}$ is an element of $L^2(\mathbb{T}^2)$, then we can take its Fourier transform to get a function $u \in l^2(\mathbb{Z}^2)$ which satisfies $Lu = f$.

$$\begin{aligned} \|\hat{u}\|_2^2 = \int_{\mathbb{T}^2} |\hat{u}|^2 &= \int_{\mathbb{T}^2} |(L^\beta)^{-1}\hat{f}_\beta(\alpha)|^2\, d\alpha\, d\beta \\ &= \int_{\mathbb{T}} \|(L^\beta)^{-1}\hat{f}_\beta\|_2^2\, d\beta \\ &\le \int_{\mathbb{T}} \|(L^\beta)^{-1}\|^2 \|\hat{f}_\beta\|_2^2\, d\beta \end{aligned}$$

therefore,

$$\|\hat{u}\|_2^2 \le \int_{\mathbb{T}} \frac{\|\hat{f}_\beta\|_2^2}{\delta_\beta^2}\, d\beta \qquad \text{by Lemma 1.} \tag{8}$$

The existence of a δ_β for which $2 - 2\,\mathrm{Re}\,\beta^2 < -\delta_\beta$ is clearly seen for $\beta \neq \pm 1$, but we must now state the bound explicitly. Let $\beta = e^{i\theta}$ for $-\frac{\pi}{2} \le \theta < \frac{3\pi}{2}$, and define:

$$\delta_\beta = \min\bigl(\theta^2, (\pi-\theta)^2\bigr) = \begin{cases} \theta^2 & -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ (\pi-\theta)^2 & \frac{\pi}{2} \le \theta < \frac{3\pi}{2}. \end{cases}$$

We wish to show that the function $\hat{u}$ is an element of $L^2(\mathbb{T}^2)$. We split the integral over β into two parts, depending on the definition of δ_β. If $\delta_\beta = \theta^2$, we expand $\hat{f}$ in a Taylor series around $\theta = 0$,

$$\hat{f}(\alpha,\theta) = \hat{f}(\alpha,0) + \frac{\partial \hat{f}}{\partial\theta}(\alpha,0)\theta + \frac{\partial^2 \hat{f}}{\partial\theta^2}(\alpha,0)\frac{\theta^2}{2} + \cdots + R_n(\alpha,\theta)\theta^n.$$

Since f has compact support, $\hat{f}$ is a finite sum of terms with convergent Taylor series, so this series will converge. By the assumptions of this theorem, however, the first two terms in the expansion are zero:

$$\hat{f}(\alpha,0) = \sum_{y,z} f(y,z)\alpha^y = \sum_y \Bigl(\sum_z f(y,z)\Bigr)\alpha^y = 0,$$

$$\frac{\partial \hat{f}}{\partial\theta}(\alpha,0)\theta = \sum_{y,z} f(y,z)\alpha^y(iz) = \sum_y \Bigl(\sum_z f(y,z)z\Bigr) i\alpha^y = 0.$$

This allows us to write the integral over the set $A = [-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\begin{aligned}\int_A \Bigl\| \frac{\hat{f}_\beta}{\theta^2}\Bigr\|_2^2 d\beta &= \int_A \Bigl\| \frac{\frac{\partial^2 \hat{f}}{\partial\theta^2}(\alpha,0)\frac{\theta^2}{2} + \cdots + R_n(\alpha,\theta)\theta^n}{\theta^2}\Bigr\|_2^2 d\beta \\ &\le \frac{1}{4}\int_A \Bigl\| \frac{\partial^2 \hat{f}}{\partial\theta^2}(\alpha,0) + \cdots + R_n(\alpha,\theta)\theta^{n-2}\Bigr\|_2^2 d\beta \\ &< \infty.\end{aligned}$$

An identical argument shows that the integral is also finite over the set $B = [\frac{\pi}{2}, \frac{3\pi}{2})$, for which $\delta_\beta = (\pi-\theta)^2$, by writing the Taylor series about $\theta = \pi$ and using the other two assumed conditions. Since each integral is finite, we have shown that $\hat{u} \in L^2(\mathbb{T}^2)$, hence its Fourier transform $u = \mathcal{F}\hat{u} \in l^2(\mathbb{Z}^2)$, and u satisfies our equation $Lu = f$. ∎

Let f be an element in $L^2(\mathbb{Z}^2)$ and A any compact subset of $\mathbb{Z}^2$. Let $\tilde{f}$ be the restriction of f to A. We will show L is locally solvable by proving the existence of an extension g of $\tilde{f}$ in $L^2(\mathbb{Z}^2)$ which satisfies the conditions in Theorem 8. Given a fixed $y \in \operatorname{supp}(\tilde{f})$, these conditions can be written in complex-valued matrix form. Define the matrix A_n with dimensions $4 \times (2n+1)$ to have entries

$$A_n[s,t] = (-1)^j t^k \tag{9}$$

where $t \in \{-n, \dots, n\}$ and $k, j \in \{0, 1\}$, so that the index s, which will be given by $s = 2j + k + 1$, is in $\{1, 2, 3, 4\}$. For example, when $n = 4$ we have the 4×9 matrix:

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -4 & 3 & -2 & 1 & 0 & -1 & 2 & -3 & 4 \end{bmatrix}.$$

For each fixed y, there is an m such that $\tilde{f}(y,z) = 0$ for all $|z| > m$. Define a column vector v of length $2m+1$ by $v[t] = \tilde{f}(y,t)$ for $t = -m, \dots, m$. The conditions of Theorem 8 are now concisely written as

$$A_m v = 0.$$

Given $n \geq 2$, we can construct a square 4×4 matrix S_n from the 2 outermost columns on each end of A_n (columns $-n, -n+1, n-1, n$).

Lemma 9 *S_n is nonsingular for all but finitely many positive integers $n \geq 2$.*

Proof

$$S_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -n & -n+1 & n-1 & n \\ \pm 1 & \mp 1 & \mp 1 & \pm 1 \\ \pm(-n) & \mp(-n+1) & \mp(n-1) & \pm n \end{bmatrix}.$$

The entries in S_n are polynomials in n, and therefore so is the determinant. When $n = 2$, the determinant is nonzero, so the polynomial has nonzero values. It can therefore have only finitely many (positive integral) roots. ∎

Lemma 10 *For any complex vector v of length $2m+1$, there exists $n \geq m+2$ such that there is an extension w of v to length $2n+1$ with $A_n w = 0$.*

Proof Let $A_m v = b$, where b will be a column vector of length 4. If $b = 0$, no extension of v is necessary and we are done. If not, by Lemma 9, there exists an $n \geq m+2$ for which the square matrix S_n is nonsingular. Let w_1 and w_2 be vectors of length 2 which comprise the solution to the equation

$$S_n \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -b.$$

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If we denote by 0_k the zero column vector of length $k = n - m - 2$, then construct the column vector w of length $2n + 1$:

$$w = \begin{bmatrix} w_1 \\ 0_k \\ v \\ 0_k \\ w_2 \end{bmatrix}.$$

Clearly w is an extension of v and we have $A_n w = 0$. ■

Theorem 11 *The Laplacian operator L is locally solvable.*

Proof For each fixed y in the support of $\tilde{f}$, as above we form the vector v of length $2m + 1$, where m is a positive integer for which $\tilde{f}(y, z) = 0$ when $|z| < m$. Lemma 10 gives an extension w of v with length $2n + 1$ for which $A_n w = 0$, so we define an extension g of $\tilde{f}$ by $g(y, z) = w[z]$ for $-n \le z \le n$. We repeat this process on the finitely many $y \in \operatorname{supp}(\tilde{f})$, which gives an extension g satisfying the conditions of Theorem 8. Therefore, the operator L is locally solvable. ■

5 Sublaplacian Operators

It is possible to determine the local solvability of the sublaplacian operators on $l^2(\mathbb{Z}^2)$ which arise from the same representation of the discrete Heisenberg group.

D_z^2, $D_x^2 + D_z^2$, and $D_y^2 + D_z^2$. The local solvability of the operator D_z^2 becomes clear with the observation that the definition of our bound δ_β in Section 4 for the full Laplacian involved only the term from the z component. The exact same argument used previously will therefore prove that the sublaplacian operators D_z^2, $D_x^2 + D_z^2$, and $D_y^2 + D_z^2$ are locally solvable.

D_x^2 and D_y^2. First, let $L = D_x^2$ and $f \in l^2(\mathbb{Z}^2)$,

$$\begin{aligned} L &= U_{(2,0,0)} + U_{(-2,0,0)} + 2I, \\ Lf(y,z) &= f(y, z - 2y) + f(y, z + 2y) - 2f(y,z). \end{aligned} \tag{10}$$

The result can be determined directly by counterexample.

Corollary 12 *The operator $L = D_x^2$ is not locally solvable.*

Proof Consider any function $f \in l^2(\mathbb{Z}^2)$ for which $f(0, 0) = k \neq 0$. Given any function u on $\mathbb{Z}^2$, equation (10) gives:

$$\begin{aligned} Lu(0,0) &= u(0,0) + u(0,0) - 2u(0,0) \\ &= 0 \\ &\neq k. \end{aligned}$$

Thus, there is no u for which $Lu = f$ at the point $(0,0)$. ■

Next, let $L = D_y^2$. Conjugation by the Fourier transform will provide insight, although decomposition into factor operators will not prove helpful since we cannot bound the inner products $\langle L^\beta f, f\rangle$ away from zero.

$$
\begin{aligned}
L = D_y^2 &= U_{(0,2,0)} + U_{(0,-2,0)} - 2I,\\
Lf(y,z) &= f(y-2,z) + f(y+2,z) - 2f(y,z), \qquad (11)\\
\mathcal{F}^{-1}L\mathcal{F}f(\alpha,\beta) &= \alpha^2 f(\alpha,\beta) + \alpha^{-2} f(\alpha,\beta) - 2f(\alpha,\beta).
\end{aligned}
$$

Corollary 13 *The operator $L = D_y^2$ is locally solvable.*

Proof The transformed operator

$$\mathcal{F}^{-1}L\mathcal{F}f(\alpha,\beta) = \bigl(2\operatorname{Re}(\alpha^2) - 2\bigr) f(\alpha,\beta)$$

is a multiplication operator with a clear inverse for $\alpha \neq 1$. The transformed equation $\mathcal{F}^{-1}L\mathcal{F}\hat{u} = \hat{f}$ then has a solution $\hat{u}$ which can be defined to be jointly measurable.

$$\hat{u}(\alpha,\beta) = \frac{\hat{f}(\alpha,\beta)}{\bigl(2\operatorname{Re}(\alpha^2) - 2\bigr)} \quad \text{a.e.}$$

It remains to be shown, however, whether such solutions are square integrable, so that we have a solution $u = \mathcal{F}\hat{u}$ to the equation $Lu = f$.

Given $f \in l^2(\mathbb{Z}^2)$ and a compact subset $A \subset \mathbb{Z}^2$, let $\tilde{f}$ be the restriction of f to A. Then the Fourier transform of $\tilde{f}$ is a polynomial in α and β with a finite number of terms. Let every power of α be between M and $-M$, and let $N > 2M$. Define

$$\hat{g}(\alpha,\beta) = (1-\alpha^{2N})^2 \hat{f}(\alpha,\beta).$$

This function $\hat{g}$ maintains all the nonzero coefficients of $\hat{\tilde{f}}$, which means that $g = \mathcal{F}\hat{g}$ is compactly supported and equal to f on A. We claim that the solution $\hat{u}$ to the transformed equation $\mathcal{F}^{-1}L\mathcal{F}\hat{u} = \hat{f}$ is an element of $L^2(\mathbb{T}^2)$.

$$
\begin{aligned}
\hat{u}(\alpha,\beta) &= \frac{\hat{g}(\alpha,\beta)}{\bigl(2\operatorname{Re}(\alpha^2) - 2\bigr)} \quad \text{a.e.}\\
&= \frac{(1-\alpha^{2N})^2 \hat{f}(\alpha,\beta)}{\bigl(2\operatorname{Re}(\alpha^2) - 2\bigr)}\\
&= \frac{(1-\alpha^2)^2(1+\alpha^2+\alpha^4+\cdots+\alpha^{2N-2})^2 \hat{f}(\alpha,\beta)}{\bigl(2\operatorname{Re}(\alpha^2) - 2\bigr)}\\
&= \frac{1}{2}\frac{(1-\alpha^2)^2}{\bigl(\operatorname{Re}(\alpha^2) - 1\bigr)}(1+\alpha^2+\alpha^4+\cdots+\alpha^{2N-2})^2 \hat{f}(\alpha,\beta)
\end{aligned}
$$

The factor $\frac{(1-\alpha^2)^2}{(\mathrm{Re}(\alpha^2)-1)}$ must be shown to be bounded near $\alpha^2 = 1$, and thus for all α. Let $\alpha^2 = e^{i\theta}$.

$$\begin{aligned}\left|\frac{(1-\alpha^2)^2}{\mathrm{Re}(\alpha^2)-1}\right| &= \frac{|1-e^{\theta}|^2}{1-\cos\theta} = \frac{|1-\cos\theta-\sin\theta|^2}{1-\cos\theta}\\ &= \frac{(1-\cos\theta)^2+\sin^2\theta}{1-\cos\theta} = \frac{2-2\cos\theta}{1-\cos\theta}\\ &= 2 \quad \theta \neq 0.\end{aligned}$$

The expression is bounded as $\theta \to 0$, so we have $\hat{u} \in L^2(\mathbb{T}^2)$. The Fourier transform $u = \mathcal{F}\hat{u}$ has finite support and is a solution to $Lu = g$. Therefore, the operator $L = D_y^2$ is locally solvable. ■

$D_x^2 + D_y^2$. If $L = D_x^2 + D_y^2$, we can take the appropriate terms from equation (5) to find the factor operators L^β on $\mathbb{T}$:

$$L^\beta f(\alpha) = f(\alpha\beta^2) + f(\alpha\beta^{-2}) - 2f(\alpha) + [2\,\mathrm{Re}(\alpha^2) - 2]f(\alpha).$$

Fix $\beta \in \mathbb{T}$ and let f be a function in $L^2(\mathbb{T}^2)$ with $\|f\|_2 = 1$. Recall that, by Lemma 1, if there is a bound δ_β for which $\langle L^\beta f, f\rangle < -\delta_\beta$, then L^β has a bounded inverse. Although we cannot bound almost every inner product away from zero when considering D_x^2 and D_y^2 individually, there does exist such a bound for almost every β when our operator is the sum $L = D_x^2 + D_y^2$. To find the bound, use the expansion of f in its Fourier series: $f(\alpha) = \sum_{n\in\mathbb{Z}} c_n\alpha^n$.

$$\begin{aligned}\langle L^\beta f, f\rangle &= \int_{\mathbb{T}} [f(\alpha\beta^2) + f(\alpha\beta^{-2}) + (\alpha^2+\alpha^{-2})f(\alpha) - 4f(\alpha)]\overline{f(\alpha)}\,d\alpha\\ &= -4 + \int_{\mathbb{T}} f(\alpha\beta^2)\overline{f(\alpha)} + f(\alpha\beta^{-2})\overline{f(\alpha)} + (\alpha^2+\alpha^{-2})|f(\alpha)|^2\,d\alpha\\ &= -4 + \int_{\mathbb{T}} \sum_{n,k\in\mathbb{Z}} [c_n\bar{c}_k(\alpha\beta^2)^n\alpha^{-k} + c_n\bar{c}_k(\alpha\beta^{-2})^n\alpha^{-k}\\ &\qquad + c_n\bar{c}_k\alpha^{n+2}\alpha^{-k} + c_n\bar{c}_k\alpha^{n-2}\alpha^{-k}]\,d\alpha\\ &= -4 + \sum_{n,k}\Big[\int_{\mathbb{T}} c_n\bar{c}_k\alpha^{n-k}\beta^{2n}\,d\alpha + \int_{\mathbb{T}} c_n\bar{c}_k\alpha^{n-k}\beta^{-2n}\,d\alpha\\ &\qquad + \int_{\mathbb{T}} c_n\bar{c}_k\alpha^{n+2-k}\,d\alpha + \int_{\mathbb{T}} c_n\bar{c}_k\alpha^{n-2-k}\,d\alpha\Big]\\ &= -4 + \sum_n [(\beta^{2n}+\beta^{-2n})|c_n|^2 + c_n\bar{c}_{n+2} + c_n\bar{c}_{n-2}],\end{aligned}$$

therefore,

$$\langle L^\beta f, f\rangle = -4 + \sum_n [2\,\mathrm{Re}(\beta^{2n})|c_n|^2 + c_n\bar{c}_{n+2} + c_n\bar{c}_{n-2}]. \tag{12}$$

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Lemma 14 *Let $\beta = e^{i\theta}$ and $f \in L^2(\mathbb{T}^2)$ with $\|f\|_2 = 1$. There exists a neighborhood N of $\theta = 0$ such that for all $\theta \in N \setminus \{0\}$, the inner product $\langle L^\beta f, f\rangle$ for the corresponding β is bounded by some δ_β away from zero.*

Proof If, for each $\beta \in N \setminus \{0\}$, there exists a δ_β for which at least one of

$$\sum_n \operatorname{Re} \beta^{2n} |c_n|^2 \quad \text{or} \quad \sum_n c_n \bar{c}_{n+2}$$

is less than $1 - \frac{\delta_\beta}{2}$, then by equation (12) we have $\langle L^\beta f, f\rangle < -\delta_\beta$.

Let $\epsilon = \theta^2$. Define $F = \{n \in \mathbb{Z} \mid 1 - \operatorname{Re} \beta^{2n} < \epsilon\}$ and let $\tilde{F}$ be the complement of F.

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \operatorname{Re} \beta^{2n}|c_n|^2 &= \sum_F \operatorname{Re} \beta^{2n}|c_n|^2 + \sum_{\tilde{F}} \operatorname{Re} \beta^{2n}|c_n|^2 - |c_n|^2 + |c_n|^2 \\
&\le \sum_F |c_n|^2 + \sum_{\tilde{F}} |c_n|^2 + \sum_{\tilde{F}} (\operatorname{Re} \beta^{2n} - 1)|c_n|^2 \\
&\le 1 - \epsilon \sum_{\tilde{F}} |c_n|^2.
\end{aligned}$$

Define δ_β to be ϵ^2. For functions such that $\sum_{\tilde{F}} |c_n|^2 \ge \frac{\epsilon}{2}$, we will have

$$\sum_n \operatorname{Re} \beta^{2n}|c_n|^2 \le 1 - \frac{\delta_\beta}{2}.$$

Otherwise, we are considering the functions such that $\sum_{\tilde{F}} |c_n|^2 < \frac{\epsilon}{2}$. We want to show that for all of these functions, $\sum_n c_n\bar{c}_{n+2} < 1 - \frac{\delta_\beta}{2}$.

$$\sum_n c_n\bar{c}_{n+2} = \sum_A c_n\bar{c}_{n+2} + \sum_B c_n\bar{c}_{n+2} + \sum_C c_n\bar{c}_{n+2} + \sum_D c_n\bar{c}_{n+2},$$

where

$$\begin{aligned}
A &= \{n : n, n+2 \in F\},\\
B &= \{n : n \in F, n+2 \in \tilde{F}\},\\
C &= \{n : n \in \tilde{F}, n+2 \in F\},\\
D &= \{n : n, n+2 \in \tilde{F}\}.
\end{aligned}$$

The neighborhood N can be defined such that, when $\theta \in N \setminus \{0\}$, the set A must be empty. Define the set $G = \{\phi : 1 - \operatorname{Re} \phi < \theta^2\}$, so that $n \in A$ implies that both β^{2n} and β^{2n+4} are in G. For angles sufficiently near (but not equal) zero, however, $\theta^2 < 1 - \cos 2\theta$, so the set $G \subset (e^{-2i\theta}, e^{2\theta})$. Since β^{2n+4} is a 4θ rotation from β^{2n}, they cannot both be in G.

Next, the Cauchy–Schwartz Inequality gives:

$$\begin{aligned}
\sum_n c_n \bar{c}_{n+2} &\le \Big(\sum_B |c_n|^2 |\bar{c}_{n+2}|^2\Big)^{\frac{1}{2}} + \Big(\sum_C |c_n|^2 |\bar{c}_{n+2}|^2\Big)^{\frac{1}{2}} + \Big(\sum_D |c_n|^2 |\bar{c}_{n+2}|^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_F |c_n|^2 \sum_{\tilde{F}} |c_{n+2}|^2\Big)^{\frac{1}{2}} + \Big(\sum_{\tilde{F}} |c_n|^2 \sum_F |c_{n+2}|^2\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_{\tilde{F}} |c_n|^2 \sum_{\tilde{F}} |c_{n+2}|^2\Big)^{\frac{1}{2}} \\
&\le \sqrt{\frac{\epsilon}{2}} + \sqrt{\frac{\epsilon}{2}} + \frac{\epsilon}{2},
\end{aligned}$$

therefore,

$$\begin{aligned}
\sum_n c_n \bar{c}_{n+2} &\le 1 - \frac{\epsilon^2}{2} \\
&= 1 - \frac{\delta_\beta}{2}.
\end{aligned}$$

This last inequality holds for ϵ sufficiently small. We define the neighborhood N to be $-\frac{1}{2} \le \theta \le \frac{1}{2}$, so $\epsilon \in (0, \frac{1}{4})$ and $\delta_\beta \in (0, \frac{1}{16})$. For $\theta \in N \setminus \{0\}$ and $\beta = e^{i\theta}$, we have $\langle L^\beta f, f\rangle < -\delta_\beta$. ■

Lemma 15 *For almost every β, L^β has a bounded inverse.*

Proof Lemmas 1 and 14 combine to give the result for β in a neighborhood of 1. An identical argument can be made for β near i, -1, and $-i$, using $\epsilon = (\theta - \frac{\pi}{2})^2$, $(\theta - \pi)^2$, $(\theta - \frac{3\pi}{2})^2$ respectively, and for all other $\beta \in \mathbb{T}$, we just restrict ϵ to a constant (we use $\epsilon = \frac{1}{4}$). Letting $\delta_\beta = \epsilon^2$ gives $\langle L^\beta f, f\rangle < -\delta_\beta$ for all $\beta \neq \pm 1, \pm i$. ■

Proposition 2 gives, for almost every $\beta \in \mathbb{T}$, a solution to equation (6), $\hat{u}_\beta = (L^\beta)^{-1} \hat{f}_\beta$. Combine these solutions as we did in the full Laplacian case to form the measurable function $\hat{u}$ on $\mathbb{T}^2$ given by $\hat{u}(\alpha, \beta) = \hat{u}_\beta(\alpha)$.

Theorem 16 *Let $L = D_x^2 + D_y^2$ and let $f \in l^2(\mathbb{Z}^2)$ have compact support. Let f satisfy the following* 16 *conditions for each y in the support of f:*

$$\sum_z [f(y,z) z^j (i^k)^z] = 0 \quad j = 0,1,2,3, \; k = 0,1,2,3.$$

Then there exists a function u for which $Lu = f$.

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Proof We have satisfied $\mathcal{F}^{-1}L\mathcal{F}\hat{u} = \hat{f}$. If the solution $\hat{u}$ is an element of $L^2(\mathbb{T}^2)$, then we can take its Fourier transform to get a function $u \in l^2(\mathbb{Z}^2)$. As in equation (8), we have:

$$\|\hat{u}\|_2^2 \le \int_{\mathbb{T}} \frac{\|\hat{f}_\beta\|_2^2}{\delta_\beta^2}\, d\beta. \tag{13}$$

From the definition of ϵ in Lemmas 14 and 15, we can express

$$\delta_\beta = \epsilon^2 = \min\Big\{ \frac{1}{16}, \Big|\theta - \frac{k\pi}{2}\Big|^4 ; k = 0, 1, 2, 3 \Big\},$$

so the integral in equation (13) must be shown finite over the intervals on $\mathbb{T}$ near $\frac{k\pi}{2}$, $k = 0, 1, 2, 3$. For each k, expand $\hat{f}$ in a Taylor series about $\theta = \frac{k\pi}{2}$:

$$\hat{f}(\alpha,\theta) = \hat{f}\Big(\alpha, \frac{k\pi}{2}\Big) + \frac{\partial \hat{f}}{\partial\theta}\Big(\alpha, \frac{k\pi}{2}\Big)\Big(\theta - \frac{k\pi}{2}\Big) + \frac{\partial^2 \hat{f}}{\partial\theta^2}\Big(\alpha, \frac{k\pi}{2}\Big)\frac{(\theta - \frac{k\pi}{2})^2}{2} + \cdots$$
$$\cdots + R_n(\alpha,\theta)\Big(\theta - \frac{k\pi}{2}\Big)^n.$$

Since f has compact support, $\hat{f}$ is a finite sum of terms with convergent Taylor series and this series will converge. Note that the partial derivatives of these expansions have the form:

$$\frac{\partial^j \hat{f}}{\partial\theta^j}\Big(\alpha, \frac{k\pi}{2}\Big) = \sum_{y,z} f(y,z)\alpha^y (i^k)^z (iz)^j = \sum_y \Big(\sum_z f(y,z)(i^k)^z z^j\Big) i^j \alpha^y.$$

The hypotheses on f force the first four terms of each Taylor expansion to be zero. The integral over region $B_k = \{\beta : \delta_\beta = |\theta - \frac{k\pi}{2}|^4\}$ becomes

$$\begin{aligned}\int_{B_k} \Big\| \frac{\hat{f}_\beta}{(\theta - \frac{k\pi}{2})^4} \Big\|_2^2 d\beta &= \int_{B_k} \Big\| \frac{\frac{\partial^4 \hat{f}}{\partial\theta^4}(\alpha, \frac{k\pi}{2})\frac{(\theta-\frac{k\pi}{2})^4}{4!} + \cdots + R_n(\alpha,\theta)(\theta - \frac{k\pi}{2})^n}{(\theta - \frac{k\pi}{2})^4} \Big\|_2^2 d\beta \\ &= \int_{B_k} \Big\| \frac{1}{4!}\frac{\partial^4 \hat{f}}{\partial\theta^4}\Big(\alpha, \frac{k\pi}{2}\Big) + \cdots + R_n(\alpha,\theta)\Big(\theta - \frac{k\pi}{2}\Big)^{n-4} \Big\|_2^2 d\beta \\ &< \infty.\end{aligned}$$

This gives $\hat{u} \in L^2(\mathbb{T}^2)$, and therefore its Fourier transform $u = \mathcal{F}\hat{u}$ is in $l^2(\mathbb{Z}^2)$, and u satisfies our equation $Lu = f$. ■

Given a function $f \in L^2(\mathbb{Z}^2)$ and a compact set $A \subset \mathbb{Z}^2$, let $\tilde{f}$ be the restriction of f to A. An identical argument to that used for the Laplacian will prove that there exists an extension g of $\tilde{f}$ which satisfies the conditions in Theorem 16.

Theorem 17 *The operator $L = D_x^2 + D_y^2$ is locally solvable.*

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Proof Given a function $f \in L^2(\mathbb{Z}^2)$ and a compact subset $A \subset \mathbb{Z}^2$, let $\tilde{f}$ be the restriction of f to A and, as we did for the Laplacian, we write the conditions of Theorem 16 in matrix form. Define the matrix A_n of size $16 \times (2n+1)$:

$$A_n[s,t] = (i^k)^t t^j,$$

where $t \in \{-n, \dots, n\}$ and $j, k \in \{0, 1, 2, 3\}$, so that the index $s = 4k + j + 1$ is in $\{1, 2, \dots, 16\}$. For each fixed y in the support of $\tilde{f}$, define the column vector v containing the values $v[t] = \tilde{f}(y,t) t = -m, \dots, m$, where $\tilde{f}(y,z) = 0$ for $|z| > m$. The conditions are now expressed for each y as the matrix equation $A_m v = 0$. The square matrix S_n will now be composed of the 8 outermost columns on each side of A_n. The argument of Lemma 9 is valid for the larger matrix as well, so S_n is nonsingular for some $n \geq m + 8$. Following the steps in Lemma 10, for each y in the support of $\tilde{f}$ we extend the vector v to w, which in this case must have length $2n+1$ for $n \geq m + 8$. We then define the function $g(y,z) = w[z]$ for $z \in \{-n, \dots, n\}$. Since $A_n g = 0$, by Theorem 16, the equation $Lu = g$ has a solution $u \in L^2(\mathbb{Z}^2)$, and for this u, we have $Lu = f$ on A. ■

Appendix

This appendix contains the proofs of results from Section 3.

Proof of Lemma 3 The weak topology on a Hilbert space is the smallest topology such that the functionals in $\mathcal{H}^*$ are all continuous. Since these are continuous in the norm topology, we have that every weak open set is also open in norm.

We next show that any norm open set in $\mathcal{H}$ is weakly Borel. Let B_ϵ be the closed ball $\{x \in \mathcal{H} : \|x\| \leq \epsilon\}$ and let I_ϵ be the closed disc of radius ϵ in $\mathbb{C}$. If $\{x_i\}$ is a countable dense subset of the set of unit vectors in $\mathcal{H}$, then we have $\|x\| = \sup_{\|y\|=1} |\langle x, y\rangle| = \sup_i |\langle x, x_i\rangle|$. This gives $x \in B_\epsilon \Leftrightarrow |\langle x, x_i\rangle| \leq \epsilon$ for all i. Let x_i^* be the functional given by $x_i^*(y) = \langle y, x_i\rangle$.

$$\begin{aligned} B_\epsilon &= \bigcap_i \{x : |\langle x, x_i\rangle| \leq \epsilon\} \\ &= \bigcap_i (x_i^*)^{-1}(I_\epsilon). \end{aligned}$$

Thus, B_ϵ is weakly closed. Let $\mathcal{O}$ be the open ball $\{x \in \mathcal{H} : \|x\| < \epsilon\}$.

$$\mathcal{O} = \bigcup_n B_{\epsilon - \frac{1}{n}}$$

The open ball $\mathcal{O}$ is Borel in the weak topology. $\mathcal{H}$ is separable in the norm topology, so every open set is a countable union of open balls and we can conclude that every norm open set is weakly Borel. ■

Proof of Lemma 4 The strong operator topology on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that all of the seminorms ρ_x given by $\rho_x(T) = \|Tx\|$ are continuous. When restricted to the bounded set S_M, the strong operator topology is second countable. To verify this, let $\{x_i\}$ be a countable dense subset of $\mathcal{H}$ and let $\{\rho_i\}$ be the corresponding seminorms given by $\rho_i(T) = \|Tx_i\|$. The weakest topology on S_M such that each ρ_i is continuous is a second countable topology, and we show that it is, in fact, the strong operator topology.

Given $x \in \mathcal{H}$ and $\epsilon > 0$, there is an x_i such that $\|x - x_i\| < \frac{\epsilon}{3M}$. Let $\{T_\alpha\} \subset S_M$ be a net of operators converging to T, which also gives $\rho_i(T_\alpha)$ converging to $\rho_i(T)$ since ρ_i is continuous:

$$\begin{aligned}
|\rho_x(T_\alpha) - \rho_x(T)| &= \big|\, \|T_\alpha x\| - \|Tx\| \,\big| \\
&\le \|T_\alpha x - Tx\| \\
&\le \|T_\alpha x - T_\alpha x_i\| + \|T_\alpha x_i - Tx_i\| + \|Tx_i - Tx\| \\
&\le \|T_\alpha\|\,\|x - x_i\| + \rho_i(T_\alpha - T) + \|T\|\,\|x_i - x\| \\
&\le \epsilon \quad \text{for sufficient choice of } \alpha.
\end{aligned}$$

Every seminorm ρ_x is continuous, so this second countable topology on S_M is actually the strong operator topology.

Next, we verify that open sets in the strong operator topology on S_M are Borel sets in the weak operator topology. A countable base of open sets around zero on S_M is given by:

$$U_{i,n} = \Big\{\, T : \rho_i(T) < \frac{1}{n} \Big\} = \bigcup_{m=n+1}^{\infty} \Big\{\, T : \|Tx_i\| < \frac{1}{n} - \frac{1}{m} \Big\}.$$

Let $\{y_j\}$ be a dense subset of the unit sphere in $\mathcal{H}$. Using the fact that $\|Tx_i\| = \sup_j |\langle Tx_i, y_j\rangle|$, we can write $U_{i,n}$ as a union of an intersection of open sets in the weak operator topology:

$$U_{i,n} = \bigcup_{m=n+1}^{\infty} \bigcap_{j=1}^{\infty} \Big\{\, T : |\langle Tx_i, y_j\rangle| < \frac{1}{n} - \frac{1}{m} \Big\}.$$

This is sufficient to show that every strongly open set is weakly Borel.

Given any open set $\mathcal{O}$ in the strong operator topology on S_M, $\mathcal{O}$ is also a Borel set in the weak operator topology. Let $C = \{T : T^* \in \mathcal{O}\}$. Note that the adjoint map $T \mapsto T^*$ is continuous from the strong operator topology to the weak operator topology on S_M, since $T_n \to_s T$ implies $\langle (T_n - T)x, y\rangle = \langle x, (T_n^* - T^*)y\rangle \to 0$ for every $x, y \in \mathcal{H}$. As a result of this continuity, C is Borel in the strong operator topology, which proves the adjoint map is Borel. ■

Proof of Lemma 5 Given an open set $\mathcal{O}$ in the strong operator topology on $\mathcal{B}(\mathcal{H})$, let $C = \{T \in S_M : T \text{ invertible and } T^{-1} \in \mathcal{O}\}$. The goal is to prove that C is a Borel set in the strong operator topology on S_M.

For each natural number K, define

$$S_{MK} = \{T \in S_M : T \text{ invertible and } \|T^{-1}\| \le K\}.$$

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We must show that S_{MK} is closed in the strong operator topology on S_M. Recall from the proof of Lemma 4 that S_M is second countable (hence first countable) in the strong operator topology. Let $\{T_n\} \subset S_{MK}$ be a sequence converging in the strong operator topology to an operator T. Since S_M is strongly closed, $\|T\| \leq M$. It remains to be shown that T is also invertible with $\|T^{-1}\| \leq K$.

Notice that for each $T_n \in S_{MK}$, since $\|T_n^{-1}\| \leq K$, we also have for every $x \in \mathcal{H}$:

$$\frac{\|T_n^{-1}x\|}{\|x\|} = \frac{\|y\|}{\|T_n y\|} \leq K \quad \text{where } y = T_n^{-1}x. \tag{14}$$

Thus, for all unit vectors y, we have $\|T_n y\| \geq \frac{1}{K}$, and a straightforward contradiction argument shows that $\|Ty\| \geq \frac{1}{K}$ for $\|y\| = 1$ as well. Therefore, T has trivial kernel and closed range. This also implies that T^* will have trivial kernel, and since the orthogonal complement of the kernel of T^* is the closure of the range of T, we now have T a bijection. Therefore, T is invertible with bounded inverse. Equation (14) applied to T gives $\|T^{-1}\| \leq K$, so $T \in S_{MK}$, and therefore S_{MK} is closed in the strong operator topology.

Next, observe that the sequence $\{T_n^{-1}\}$ above converges to T^{-1} in the strong operator topology. This calculation makes use of the identity

$$T_n^{-1} - T^{-1} = T_n^{-1}(T - T_n)T^{-1}.$$

For every $x \in \mathcal{H}$:

$$\begin{aligned} \|T_n^{-1}x - T^{-1}x\| &= \|T_n^{-1}(T - T_n)T^{-1}x\| \\ &\leq \|T_n^{-1}\| \, \|(T_n - T)(T^{-1}x)\| \\ &\leq K\|(T_n - T)(T^{-1}x)\| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, the inverse map $T \mapsto T^{-1}$ is continuous in the strong operator topology from S_{MK} to $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{O}$ be an open set in $\mathcal{B}(\mathcal{H})$ and $C \subset \mathcal{B}(\mathcal{H})$ defined previously. C can be written as a union of sets which are the intersection of S_{MK} with open subsets of S_M:

$$\begin{aligned} C &= \{T \in S_M : T \text{ invertible and } T^{-1} \in \mathcal{O}\} \\ &= \bigcup_K \{T \in S_{MK} : T^{-1} \in \mathcal{O}\}. \end{aligned}$$

Each set in the union is open in the strong operator topology on S_{MK} by the continuity of the inverse map, and is therefore the intersection of an open set in S_M with the closed set S_{MK}. Since each of these sets is Borel on S_M, so is their union. Therefore, the inverse map is Borel from S_M to $\mathcal{B}(\mathcal{H})$ in the strong operator topology. ■

Proof of Lemma 6 (1) An immediate consequence of Lemma 3 is that the function $\beta \mapsto f_\beta$ is s.o.t.-Borel if and only if the map $\beta \mapsto \langle f_\beta, g\rangle$ is Borel.

$$\langle f_\beta, g\rangle = \int_A f_\beta(\alpha)\bar{g}(\alpha)\, d\alpha$$

We are assuming $f_\beta(\alpha)$ is Borel on $A \times B$, so the integrand above is also a Borel function on the product space $A \times B$, so by Fubini's theorem, the integral is a Borel function on B.

(2) Let $\beta \mapsto f_\beta$ be a Borel function, and recall $f_\beta \in L^2(A)$. If ϕ_n is an orthonormal basis of $L^2(A)$, then we can write (with possible changes on sets of measure zero on each f_β):

$$f_\beta(\alpha) = \sum_n \langle f_\beta, \phi_n \rangle \phi_n(\alpha).$$

By Lemma 3, $\langle f_\beta, \phi_n \rangle$ is a Borel function on B. We also have each ϕ_n a Borel function on A. Since functions of only one variable are Borel on the product space, each term in the sum is Borel on $A \times B$, and thus so is the sum. ■

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