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DIFFUSION ON LIE GROUPS (III)

Dedicated to the memory of Carl Herz

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ABSTRACT. For amenable Lie groups of NC-type the heat kernel satisfies $p_t \sim t^{-a}$. We find the exact value of $a \geq 0$.

RÉSUMÉ. Pour les groupes de Lie amenables de type NC le noyeau de la chaleur satisfait $p_t \sim t^{-a}$. On trouve la valeur exacte de $a \geq 0$.

0. Introduction.

0.1 *Statement of the main result.* Let G be some connected real Lie group and let $\Delta = -\sum_{j=1}^{k} X_j^2$ be some left invariant subelliptic sublaplacian, *i.e.*, the X_j are left invariant fields which, after identification with elements of the Lie algebra $\mathfrak{g}$, generate the algebra (*cf.* [10], [21]). Let $T_t = e^{-t\Delta}$ and ϕ_t be the corresponding semigroup and the heat diffusion kernel:

$$T_t f(x) = \int \phi_t(y^{-1}x) f(y)\, dy; \quad f \in C_0^\infty(G)$$

where dy [resp.: $dy^r = dy^{-1} = m(y)\, dy$] denotes the left [resp.: right] invariant Haar measure on G.

In this paper I shall assume throughout that G is amenable, *i.e.*, that the semisimple Lie algebra $\mathfrak{g}/\mathfrak{q} = \mathfrak{s}$, where $\mathfrak{q} \subset \mathfrak{g}$ is the radical of $\mathfrak{g}$, is of compact type. Observe that we can find $\mathfrak{s} \subset \mathfrak{g}$ some Lie subalgebra s.t. $\mathfrak{g} = \mathfrak{q} \rtimes \mathfrak{s}$, $\mathfrak{s} \cong \mathfrak{g}/\mathfrak{q}$ (*cf.* [15]).

In [17], [18], [16] I have classified such groups and corresponding Lie algebras into two classes: the C-algebras and groups and the Non C-algebras and groups (the latter denoted by NC) .

For the convenience of the reader, here is the definition: we consider $\mathfrak{n} \subset \mathfrak{q}$ the nil radical and $V = \mathfrak{q}/\mathfrak{n}$, $W = \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ the two corresponding abelian algebras. The adjoined action then induces

$$\mathrm{ad}\colon V \otimes \mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{C}}[W \otimes \mathbb{C}].$$

Let $\lambda_1 \cdots \lambda_r \in \mathrm{Hom}_{\mathbb{C}}[V \otimes \mathbb{C}, \mathbb{C}]$ be the corresponding roots (*i.e.*, $\exists\, 0 \neq w \in W \otimes \mathbb{C}$ s.t. $\big(\mathrm{ad}\, v - \lambda_j(v)\big) w = 0\ \forall v \in V \otimes \mathbb{C}$) and let $L_j, j = 1, \ldots, s$, be the distinct *non zero* real parts ($0 \neq L = \mathrm{Re}\,\lambda$) $\mathcal{L} = (L_1, \ldots, L_s) \subset \mathrm{Hom}_{\mathbb{R}}[V, \mathbb{R}] \setminus \{0\} = V^* \setminus \{0\}$. When $\mathcal{L} = \emptyset$

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we say (following [7]) that $\mathfrak{g}$ is an R-algebra. When $0 \notin \operatorname{Conv Hull}(\mathcal{L}) = \{\sum \alpha_j L_j; \alpha_j \geq 0, \sum \alpha_j = 1\}$ we say that $\mathfrak{g}$ is an NC-algebra.

R-algebras are NC-algebras, and if the algebra $\mathfrak{g}$ is not NC we say that $\mathfrak{g}$ is a C-algebra [in the above definitions, we are exclusively dealing here with *amenable* algebras].

It has been shown in [17] [1] that when G is a C-group we have

$$C_2(g)e^{-c_2 t^{1/3}} \leq \phi_t(g) \leq C_1(g)e^{-c_1 t^{1/3}}; \quad g \in G,\ t \geq 1$$

where $c_1, c_2 > 0$ only depends on G and $C_1, C_2 > 0$ depends also on g but is independent of t.

In this paper I shall prove the analogous result for NC-groups and show that there exists $\nu = \nu(G, \Delta)$ that depends on G and Δ and which will be explicitly computed such that

$$C_2(g)t^{-\nu} \leq \phi_t(g) \leq C_1(g)t^{-\nu}; \quad g \in G,\ t \geq 1 \tag{0.1.1}$$

with $C_1, C_2 > 0$ as before.

0.2 *Some definitions: the Euclidean structure and the cone.* Let $\mathfrak{g}$ be some NC-algebra and let $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ be two ideals such that $\mathfrak{g}/\mathfrak{h}_i$ is an R-algebra. By considering $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}_1 \times \mathfrak{g}/\mathfrak{h}_2$ we see that $\mathfrak{g}/\mathfrak{h}_1 \cap \mathfrak{h}_2$ is also an R-algebra and that therefore we can define $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ the smallest ideal for which $\mathfrak{g}/\tilde{\mathfrak{g}}$ is an R-algebra. We have $\tilde{\mathfrak{g}} \subset \mathfrak{n}$ for

$$\mathfrak{g}/\mathfrak{n} = \mathfrak{q}/\mathfrak{n} \oplus \mathfrak{s} \tag{0.2.1}$$

(where the sum is actually *direct* and not only semi direct; *cf.* [15]). It was shown in [16] that

$$\mathfrak{g} = \tilde{\mathfrak{g}} \rtimes \mathfrak{g}_R$$

for some R-subalgebra $\mathfrak{g}_R$ (in the notation of [16] we had $\tilde{\mathfrak{g}} = \mathfrak{n}_R$). I shall refer the reader to [16] for a detailed analysis of the situation, but to make the paper self contained I shall recall some relevant facts in Section 2.1. What is true in addition (this is evident when G is simply connected) is that the analytic subgroup $\tilde{G} \subset G$ that corresponds to $\tilde{\mathfrak{g}}$ is closed. It is an easy matter to see that $\tilde{G}$ is the intersection of all closed normal subgroups $H \subset G$ s.t. G/H is unimodular. The group $G/\tilde{G}$ can then be defined and the volume growth function (*cf.* [21], [7]) $\gamma(t) \approx t^D$ ($t \geq 1$) can then be defined with $D = 0, 1, \ldots$ some integer that depends only on G. It will be seen in Section 2.1 that

$$G = \tilde{G} \rtimes G_R$$

where $G/\tilde{G} \cong G_R \subset G$ is some closed subgroup.

Now let $N \subset G$ be the closed nilradical of G. When G is simply connected, by (0.2.1) it follows that

$$G/N \cong V \times S$$

where $V \cong Q/N \cong \mathbb{R}^n$ where $Q \subset G$ is the radical of G and S is the corresponding Levi subgroup. We will see in Section 2.1 that we have in general

$$G/N \cong \tilde{V} \times K$$

where $\tilde{V} \cong \mathbb{R}^a$ and K is compact. Observe here that quite generally when some Lie group H ($= G/N$ in our case) admits some decomposition $H = V_1 \times K_1$ where $V_1 \cong \mathbb{R}^a$ and K_1 is compact (or semi simple!) then for any other similar decomposition $H = V_2 \times K_2$ we have $K_1 = K_2$. Indeed if $\pi_i: H \longrightarrow V_i$ $(i = 1, 2)$ are the two canonical projections we have $\pi_i(K_j) = 0$ and therefore $K_i \subset K_j$ for $i, j = 1, 2$. Furthermore $\pi_i|_{V_j}: V_j \longrightarrow V_i$ induces an isomorphism since $\pi_i|_{V_j} = (\pi_j|_{V_i})^{-1}$. From the above it follows that H admits a unique maximal compact subgroup K and that $H = V \times K$ for some $V \cong \mathbb{R}^a$.

The real parts of the roots can be identified to elements of $(\tilde{V})^*$. This is evident when G is simply connected and is easy to see in general [*cf.* Section 2.1]. Let $\pi: G \longrightarrow \tilde{V}$ the composition $G \longrightarrow G/N \cong \tilde{V} \times K \longrightarrow \tilde{V}$ and let $\Delta_0 = d\pi(\Delta)$ the corresponding second other differential operator on $\tilde{V} \cong \mathbb{R}^a$. The operator Δ_0 allows us to define a unique euclidean structure on $\tilde{V}$ which is given by the unique (up to orthogonal transformation) coordinates such that

$$\Delta_0 = -\Sigma \frac{\partial^2}{\partial^2 x_i}.$$

Furthermore if $\tilde{V}_1$ is some other choice of the direct factor $\tilde{V}_1 \times K \approx G/N$ then it is clear that the above defined isomorphism $\theta: \tilde{V} \leftrightarrow \tilde{V}_1$ is orthogonal for the induced euclidean structures and that θ gives a correspondence between the L_i $(i = 1, \ldots, s)$ on $\tilde{V}$ and $\tilde{V}_1$.

Let us assume that $\mathcal{L} \neq \emptyset$; the above construction defines a unique euclidean structure E on $\mathbb{R}^a$ ($\cong V$) and, up to orthogonal transformation, a unique convex non empty cone

$$\Pi(\mathcal{L}) = \{x \in E, L_j(x) > 0, j = 1, \ldots, s\}. \tag{0.2.2}$$

0.3 *The index: and the main theorem.* Now let $\Sigma = \{x \in E, |x| = 1\}$ be the unit sphere in the above Euclidean space and let Δ_Σ be the corresponding spherical Laplacian so that:

$$-\Delta_0 = \Sigma \frac{\partial^2}{\partial x_i^2} = \frac{d^2}{dr^2} + \frac{a-1}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_\Sigma$$

where $a = \dim E$ is assumed $a \geq 2$ and $(r, \sigma) \in \mathbb{R}_+^* \times \Sigma$ are polar coordinates on E.

Let $\Pi_\Sigma = \Pi \cap \Sigma \subset \Sigma$ (we assume as before that $\mathcal{L} \neq \emptyset$) and let λ be the first Dirichlet eigenvalue of the region Π_Σ. Thus explicitly

$$\lambda = \inf\bigl(-(\Delta_\Sigma f, f);\ \|f\|_2 = 1, f \in C_0^\infty(\Pi_\Sigma)\bigr)$$

where the scalar product and the L^2-norm is taken in $L^2(\Sigma)$ for the Euclidean volume element on Σ. We then define $\alpha = \alpha_{\mathcal{L}} > 0$ the unique positive solution of

$$\alpha(\alpha + a - 2) = \lambda.$$

When $\mathcal{L} \neq \emptyset$ and $a = 1$ we set $\alpha = 1$. When $\mathcal{L} = \emptyset$ and G is an R-group we set $\alpha = 0$. We can then state the main theorem of this paper:

THEOREM. *Let G be some amenable* NC*-group and let $D = 0, 1, 2, \ldots$ and $\alpha \geq 0$ be defined as above. Then (0.1.1) holds with $\nu = \alpha + D/2$.*

An alternative way to define the index α is the following: α is the degree of homogeneity of the "reduite" in Π, *i.e.*, of the unique, up to scalar factor, harmonic function $u = u_{\mathcal{L}}$ in $\overset{\circ}{\Pi}$ that is positive and vanishes at $\partial\Pi$ (*cf.* [2], and Section 0.5 below).

From this definition of α it follows that if Δ is identified with an operator Δ_1 on the simply connected cover G_1 of G, then the α that corresponds to Δ on G is the same as the α_1 that corresponds to Δ_1 on G_1. From the original definition, on the other hand, it follows that α is a "continuous function" of Δ, and that if we assume that $\dim\{L_1, \ldots, L_s\} > 1$ then for any $\rho \in]1, +\infty[$ there exists some Δ s.t. $\alpha(\Delta) = \rho$.

0.4 *The index α and the Weyl group.* The index $\alpha = \alpha_{\mathcal{L}}$ that we used in Section 0.3 can not, in general, be "explicitly" computed and varied continuously with $\mathcal{L}$ since the first eigenvalue λ is a "continuous function of the geometry of $\Pi \cap \Sigma$". There are some special cases worth noting however where an explicit formula for α is possible. For instance this is the case when $\mathbb{R}^a = \mathbb{R}^2$, $s = 2$ and when θ the angle between L_1, L_2 comes from a root of unity: $\theta = 2\pi/k$, $k = 1, 2, \ldots$. The harmonic function we considered at the end of the previous section can immediately be obtained from the holomorphic functions z^k ($k = 1, 2, \ldots; z \in \mathbb{C}$).

For our purposes what is more relevant is a type of configuration that naturally arises in the context of semisimple groups and the attached Weyl Chambers. We have then $\mathcal{L} = (L_1, \ldots, L_s) \subset \mathbb{R}^n$ where the L_j's are distinct, and we assume that the set $\mathcal{L}$ is stabilised by every orthogonal reflexion on each hyperplane $[L_i = 0]$. Indeed the above configuration is a natural generalization of our previous example of the roots of unity where $n = 2$. For $n \geq 2$ many natural examples as above rise from abstract root systems (*cf.* [22]). In the above configuration the L_is are not in general linearly independent (*e.g.* for the roots of unity we have $\Sigma L_i = 0$). In all generality the positive harmonic function $u_{\mathcal{L}}$ in $\Pi(\mathcal{L})$ can be explicitly computed (*cf.* [8], Chapter III, Section 3.2) and is

$$u_{\mathcal{L}}(x) = \prod_{i=1}^{s} \langle L_i, x \rangle.$$

It follows that we have then $\alpha_{\mathcal{L}} = s$.

0.5 *Potential theoretic results.* Some of the technical tools needed for the proof of the main theorem are potential theoretic in nature and represent some independent interest. These results will be described in this section.

Let $C \subset E$ be an open convex cone ($\emptyset \neq C \neq E$) as in Section 0.3 where E is some Euclidean space of dimension $n \geq 1$. It is then well known and easy to prove that there exists one and, up to multiplicative constant, only one positive harmonic function $u(x)$ in C s.t. it vanishes at the boundary ∂C. The existence can be seen directly by trying out $u(x)$ to be homogeneous of degree $\alpha = \alpha_C$ and passing to polar coordinates. The uniqueness is more subtle to prove, one could use the Kelvin transformation and [2]. The uniqueness is not really absolutely essential for the understanding of this paper. In what follows, I will restrict my attention to cones of the form (0.2.2):

$$C = \Pi = \Pi(L_1, \dots, L_s) = \Pi(\mathcal{L})$$

and I will then denote the corresponding positive harmonic function:

$$u(x) = u_{\mathcal{L}}(x) \quad x \in \Pi; \; u(x) = 0 \quad x \in \partial\Pi.$$

We shall also set $u(x) = 0$ for $x \notin \Pi(\mathcal{L})$. I shall denote then by $p_t(x,y) = p_t(\Pi; x,y) = p_t(\mathcal{L}; x,y)$ the corresponding Dirichlet heat diffusion kernel in E *i.e.* the fundamental solution of $\frac{\partial}{\partial t} - \Sigma \frac{\partial^2}{\partial x_i^2}$ with Dirichlet (*i.e.* vanishing) boundary conditions.

I shall also denote

$$P_{\mathcal{L}}(t,x) = P(t,x) = \int_\Pi p_t(x,y)\, dy = \mathbb{P}_x[b(s) \in \Pi; 0 < s < t]; \quad x \in E, \; t > 0$$

where $b(s) \in E$ $(s > 0)$ denotes standard Brownian motion in E. The following estimates then hold (and seem to be new!)

$$p_t(x,y) \leq C_\varepsilon \frac{u(x)u(y)}{t^{\alpha+n/2}} \exp\left(-\frac{|x-y|^2}{(4+\varepsilon)t}\right), \quad P(t,x) \leq c\frac{u(x)}{t^{\alpha/2}}; \quad t > 0, \; x, y \in \Pi$$

where $0 < \varepsilon < 1$ is arbitrary, and C_ε also depends on ε. It is of course possible to improve upon this "ε", but this will not be done and is of no use to us. Together with the above upper estimates we also have the following lower estimates:

$$P(t,x) \geq C\frac{u(x)}{t^{\alpha/2}}, \; p_t(x,x) \geq C\frac{u^2(x)}{t^{\alpha+n/2}}; \quad x \in \Pi, \; t \geq C|x|^2.$$

The same estimates hold for an arbitrary convex conical region $C \subset E$ and not just the pyramidal ones that are considered in this paper (*cf.* [20]. The convexity, on the other hand, cannot be relaxed). This, however, will not be needed here.

A useful motivation for the above results is supplied by the easily verifiable fact that the function

$$u(t,x) = t^{-\alpha-n/2} u(x) \exp\left(-\frac{x^2}{4t}\right)$$

is a heat function in C, *i.e.*, a solution of the equation $\frac{\partial}{\partial t} - \Sigma \frac{\partial^2}{\partial x_i^2} = 0$ and vanishes at the boundary of $C \times]0, \infty[$ except at $(0,0)$.

0.6 *A wedge in a Lie group.* Let G be a connected Lie group and let

$$\pi : G \longrightarrow \mathbb{R}^n \supset \Pi(\mathcal{L}) = \Pi$$

be a surjective homomorphism and Π as in Section 0.2. We shall fix $\Delta = -\Sigma X_j^2$ some sublaplacian on G as in Section 0.1 and fix the unique Euclidean structure on $\mathbb{R}^n = E$ for which $d\pi(\Delta) = -\Sigma \frac{\partial^2}{\partial x_i^2}$ as in Section 0.2. To that Euclidean structure we can then define $\alpha = \alpha_\Pi \geq 1$ as in Section 0.3. We shall denote also by $\{z(t) \in G; t > 0\}$ the paths of the diffusion on G that is generated by Δ so that $\pi\big(z(t)\big) = b(t) \in \mathbb{R}^n$ is standard Brownian motion. We shall abusively also denote

$$\pi^{-1}[\Pi(\mathcal{L})] = \Pi_G = \Pi(\mathcal{L}) = \Pi \subset G$$

the corresponding "wedge" in the group. We can then define

$$P(t,g) = P_G(t,g) = \mathbb{P}_g[z(s) \in \Pi_G; 0 < s < t]; \quad t > 0,\ g \in \Pi.$$

We clearly have:

$$P_G(t,g) = P_{\mathbb{R}^n}\big(t, \pi(g)\big)$$

so that we have for $P_G(t,g)$ the same estimates as before

$$P_G(t,g) \leq C\frac{u(g)}{t^{\alpha/2}} \quad \big[u(g) \stackrel{\text{def}}{=} u\big(\pi(g)\big)\ g \in G\big],\ t > 0$$
$$P_G(t,g) \geq C\frac{u(g)}{t^{\alpha/2}}; \quad t > C|\pi(g)|^2.$$

Now let $p_t(g,h) = p_t(\Pi_G; g,h)$, $(t > 0, g,h \in G)$ be the Dirichlet heat diffusion kernel of the region Π_G (*i.e.*, we kill diffusion when it reaches the boundary) with respect to left Haar measure dg. In terms of $p_t(\cdot\,,\cdot)$. We have in particular:

$$P(t,g) = \int_{\Pi_G} p_t(\Pi_G; g,h)\,dh.$$

What is important is that we have for the above kernel the analogue of the estimates of Section 0.5. To be more precise, let us assume that G is a group of polynomial volume growth $\gamma(t) \approx t^D$ $(t \geq 1)$. We have then

$$p_t(\Pi_G; g,h) \leq C\frac{u(g)u(h)}{t^{\alpha+D/2}}\exp\left(-\frac{d^2(g,h)}{(4+\varepsilon)t}\right); \quad t > 1,\ g,h \in G$$
$$p_t(\Pi_G; g,g) \geq C\frac{u^2(g)}{t^{\alpha+D/2}}; \quad t > 1,\ C^{-1} \leq |\pi(g)| \leq C, g \in G$$

where $d(\cdot\,,\cdot)$ denotes the canonical distance that is induced in G by the fields $X_1,\dots,X_k$ of the sublaplacian Δ (*cf.* [21]).

The following refinement will be needed. We shall assume again that G is a group of polynomial growth. We have then

$$(0.6.1) \qquad \int_{|x|_G \geq At^{1/2}} p_t(\Pi_G, x_0, x)\,dx \leq u(x_0)\varepsilon(A)t^{-\alpha/2}; \quad t \geq 1,\ A > 0, |x_0|^2 \leq Ct$$

where $\varepsilon(A) \xrightarrow[A\to\infty]{} 0$ and $|x|_G = d(e,x)$ where $e \in G$ denotes throughout the neutral element of G.

0.7 *The probabilistic estimates.* Let $\{z(t) \in G; t > 0\}$ and $\Pi \subset G$ be as in Section 0.4. We shall normalise the $L_j \in \mathcal{L} \subset E^*$ so that $L_j\big((1,0,\dots,0)\big) = 1$ and we shall denote by

$$C(\lambda) = (\lambda, 0, \dots, 0) + \Pi(\mathcal{L}) \subset E \quad \lambda \in \mathbb{R}$$

and denote again by $C(\lambda) = \pi^{-1}\big(C(\lambda)\big) \subset G$.

The first refinement has to do with "curved boundaries" in the sense of [14]. Let $f(s)$ $(s > 0)$ some positive increasing function that satisfies $f(s) = 0(s^{\varepsilon})$ for some $\varepsilon > 0$ that is small enough. We shall then show that there exists $C > 0$ such that:

$$\mathbb{P}_e\Big[z(s) \in C\big(-1 + f(s)\big); 0 < s < t\Big] \geq Ct^{-\alpha/2}; \quad t > 1.$$

The above event simply says that as time goes on (and with a "rate $f(s)$") the diffusion penetrates deeper and deeper in the cone Π. In fact, a slightly more precise result will be needed. Let

$$X_j = z(j-1)^{-1}z(j) \in G; \quad j = 1, \ldots$$

(set also $X_0 = e$) be the integer time increments of the diffusion $\{z(t), t > 0\}$. We shall show that there exists a, $C > 0$ such that

$$\mathbb{P}_e\Big[z(j) \in C\big(-1 + f(j)\big),\ |X_j|_G \leq a \log^{1/2}(1 + j); j = 1, \ldots, n\Big] \geq Cn^{-\alpha/2}.$$

This means that in the above event we require in addition that the increments grow very slowly with time. We denote here and throughout $|g| = d(e, g)$ $(g \in G)$.

The other probabilistic refinement that we shall need is in some sense "dual" and has to do with "sampling" the time parameter. Indeed we can sample at integer times $t = 1, \ldots$ and obtain the following estimate:

$$\mathbb{P}_e[z(j) \in C(-\lambda), j = 1, \ldots, n;\ |z(n)|_G \leq 1] \leq C(1 + \lambda)^c n^{-\alpha - D/2}; \quad n, \lambda \geq 1$$

provided that G is a group of polynomial volume growth $\gamma(t) \approx t^D$ $(t \geq 1)$, where $C, c > 0$ are independent of λ and n. We can in fact take $c = 2\alpha$ but this is irrelevant for our applications. Surprisingly enough, even for $\lambda = \lambda_0$ fixed the above estimate is non trivial to extract from the corresponding continuous time estimate. [If however we are prepared to settle with an upper estimate of the form $C(\lambda, \varepsilon)t^{-\alpha - D/2 + \varepsilon}$ then the proof becomes much simpler]. In fact a further refinement will be needed. Let G be, as before, some group of polynomial volume growth $\gamma(t) \approx t^D$ $(t \geq 1)$ and let $e \in A \subset G$ be some (Borel) subset such that

$$\mathbb{P}[z(1) \in A] = 1 - \eta$$

where $0 < \eta \ll 1$ is small. For every $n \geq 1$ we shall denote:

$$J_n = \{j \in \mathbb{Z};\ 0 \leq j \leq n;\ X_j \in A\}$$

which is a random subset of $[0, 1, \ldots, n]$. We shall then "sample" on that subset and show that:

$$\mathbb{P}_e[z(j) \in C(-\lambda), j \in J_n;\ |z(n)| \leq 1] \leq C(1 + \lambda)^c t^{-\alpha - D/2}; \quad \lambda, t \geq 1$$

provided that η is small enough with $C, c > 0$ that depend on η. (We can again take $c = 2\alpha$ but this, being irrelevant, will not be proved here).

1. **Potential and probability theory.**

1.1 *Potential theory on a wedge.* Let $\mathcal{L} = (L_1, \ldots)$ be a finite set of linear functionals on $\mathbb{R}^d$ we still denote by:

$$\Pi = \Pi(\mathcal{L}) = \{x \in \mathbb{R}^d; L(x) > 0 \ \forall L \in \mathcal{L}\}.$$

Π is clearly a convex cone and we shall assume throughout that $\Pi \neq \emptyset$ and that $\mathcal{L}$ is minimal under $\Pi(\mathcal{L}) = \Pi$ (*i.e.*, that there does not exist a smaller set $\mathcal{L}' \subset \mathcal{L}$ such that $\Pi(\mathcal{L}) = \Pi(\mathcal{L}')$). I shall denote by:

$$p_t(x,y) = p_t(\mathcal{L}; x, y); \quad t > 0, \ x, y \in \Pi.$$

The heat diffusion kernel of that region, *i.e.*, the fundamental solution with Dirichlet (: vanishing) boundary conditions on $\partial\Pi$. We shall also denote:

$$P(t,x) = \int p_t(x,y)\,dy = \mathbb{P}_x[b(s) \in \Pi\,;\, 0 < s < t]; \quad x \in \Pi, \ t > 0$$

where $b(s) \in \mathbb{R}^d$ ($s > 0$) is standard Brownian motion. We shall also denote by $u(x)$ ($= u_{\mathcal{L}} = u_\Pi; x \in \Pi$) the unique, up to positive multiplicative constant, strictly positive harmonic function on Π that vanishes on $\partial\Pi$. The function u is positively homogeneous $u(\lambda x) = \lambda^\alpha u(x)$ ($x \in \Pi, \lambda > 0$) with $\alpha = \alpha_{\mathcal{L}}$ so that $u(x) \le C|x|^\alpha$ ($x \in \mathbb{R}^d$). Let $d \ge 2$ and let us denote by Δ_Σ ($\Sigma = \{x \in \mathbb{R}^n; |x| = 1\}$) the spherical Laplacian (*cf.* Section 0) [with the sign convention: $(\Delta_\Sigma f, f) \ge 0, f \in C^\infty(\Sigma)$].

Let us then consider the "first eigenvalue problem" on the region $\Pi \cap \Sigma \subset \Sigma$, *i.e.*, let us consider the unique $\lambda > 0$, $u_0 \in C^\infty$ $(\Pi \cap \Sigma)$ such that:

$$\Delta_\Sigma u_0 = \lambda u_0, \quad u_0 \ge 0 \text{ and } u_0 \equiv 0 \text{ on the boundary.}$$

By the use of the polar coordinates it is then clear that $u(x) = r^\alpha u_0(\sigma)$ ($x = (r, \sigma)$) with $\alpha(\alpha + d - 2) = \lambda$. We clearly have $\alpha = 1$ if $d = 1$. It follows from the above that for $\mathcal{M} \subset \mathcal{L}$ we have $\alpha_{\mathcal{M}} \le \alpha_{\mathcal{L}}$ [4]. We also have:

$$u(x) \le C|x|^{\alpha-1}\,\mathrm{dist}(x, \partial\Pi); \quad x \in \mathbb{R}^d. \tag{1.1.1}$$

Both members of (1.1.1) have the same homogeneity and the above inequality is a standard fact from the theory of the boundary behaviour of harmonic functions on Lipschitz domains [2], [13]. For the convenience of the reader and just to give a flavour of things to come, I will outline an "elementary proof": let $x_0 \in \partial\Pi(\mathcal{L})$, $|x_0| = 1$ and let $\mathcal{M} \subset \mathcal{L}$ be minimal under $x_0 \in \partial\Pi(\mathcal{M})$. Then by the above eigenvalue argument it follows that $u_{\mathcal{M}}(x) = 0[\mathrm{dist}(x, \bigcap_{L \in \mathcal{M}} L^{-1}(0)]^{\alpha_{\mathcal{M}}}$ and we have of course $\alpha_{\mathcal{M}} \ge 1$. The Harnack boundary principle (*cf.* [2] and (1.1.10) below) allows us to compare $u_{\mathcal{L}}(x)$ with $u_{\mathcal{M}}(x)$ and the homogeneity does the rest. The above type of argument, which will be repeated a number of times in what follows, has the advantage that is "elementary" and only relies on the Harnack boundary principle. The drawback however is that it only works for "polygonal boundaries". With a little more knowledge of potential theory we can of course deal with arbitrary convex cones (*cf.* [20]).

For the heat diffusion in Π we shall prove the following upper and lower estimates

$$(1.1.2)\quad p_t(x,y)\le C_\varepsilon \frac{u(x)u(y)}{t^{\alpha+d/2}}\exp\left(-\frac{|x-y|^2}{(4+\varepsilon)t}\right),\quad P(t,x)\le C\frac{u(x)}{t^{\alpha/2}};\quad t>0,\ x,y\in\Pi$$

where $\varepsilon>0$ in the above estimate is arbitrary but $C_\varepsilon>0$ depends on $\varepsilon>0$. Getting rid of the ε is non trivial but unnecessary for us. We also have:

$$(1.1.3)\qquad P(t,x)\ge C\frac{u(x)}{t^{\alpha/2}};\ p_t(x,x)\ge C\frac{u^2(x)}{t^{\alpha+d/2}};\quad x\in\Pi,\ t\ge C|x|^2.$$

We shall need to generalize the above notions and estimates. Let G be a connected real Lie group of polynomial volume growth $\gamma(t)\sim t^D$ $(t>1; D=1,2,\ldots)$ and let $\pi\colon G\longrightarrow\mathbb{R}^d=V$ be some surjective group homomorphism. Further, let $\Delta=-\Sigma X_j^2$ be a subelliptic sublaplacian on G and let us fix coordinates on $\mathbb{R}^d$ (*cf.* Section 0.2) for which $d\pi(\Delta)=-\Sigma\frac{\partial^2}{\partial x_i^2}$. Further, let $\mathcal{L}=(L_1,\ldots)$ a set of linear functions as above on V and $\Pi=\Pi(\mathcal{L})\subset\mathbb{R}^d$ the corresponding wedge. I shall denote by $\Pi_G=\Pi_G(\mathcal{L})=\pi^{-1}\big(\Pi(\mathcal{L})\big)\subset G$ (and more often then not I will drop the index G). I shall denote by $p_t(x,y)=p_t(\Pi_G,x,y)$ $(t>0,x,y\in\Pi_G)$ the corresponding heat diffusion kernel for the sublaplacian Δ (with Dirichlet boundary conditions in Π_G) and by:

$$P_G(t,x)=\int p_t(\Pi_G;x,y)\,dy=P_{\mathbb{R}^d}\big(t,\pi(x)\big)=\mathbb{P}_x[z(s)\in\Pi_G;0<s<t];\quad t>0,\ x\in\Pi$$

where $z(s)\in G$ $(s>0)$ denotes the diffusion generated by Δ. I shall also denote $u_G(x)=u_{\mathcal{L}}\big(\pi(x)\big)$ $(x\in\Pi_G)$. From the definition of P_G and (1.1.2) (1.1.3) we have the following upper and lower estimates

$$(1.1.4)\qquad P_G(t,x)\le C\frac{u_G(x)}{t^{\alpha/2}},\quad t>0,\ x\in G;\ P_G(t,x)\ge C\frac{u_G(x)}{t^{\alpha/2}}|\pi(x)|^2<Ct.$$

We also have:

$$(1.1.5)\qquad p_t(g,h)\le C_\varepsilon\frac{u(g)u(h)}{t^{\alpha+D/2}}\exp\left(-\frac{d^2(g,h)}{(4+\varepsilon)t}\right);\quad g,h\in G,\ t>0$$

$$(1.1.6)\qquad p_t(g,g)\ge Ct^{-\alpha-D/2}u^2(g);\quad g\in G,\ t>1,\ c\le|\pi(g)|\le C.$$

The proof of the above estimates depend on the parabolic boundary Harnack principle (*cf.* [9], [6]). Both $P(t,x)$ and $u(x)$ are positive solutions of $\frac{\partial}{\partial t}-\Sigma\frac{\partial^2}{\partial x_i^2}$ in $\Pi(\mathcal{L})$ and they both vanish on $\partial\Pi$. It follows therefore (*cf.* [9]) that there exists a constant (depending on the normalisation of u) s.t.

$$(1.1.7)\qquad C^{-1}u(x)\le P(1,x)\le Cu(x)\quad |x|\le C.$$

If we apply this to a subset $\mathcal{M}\subset\mathcal{L}$ we obtain also:

$$(1.1.8)\qquad C^{-1}u_{\mathcal{M}}(x)\le P_{\mathcal{M}}(1,x)\le Cu_{\mathcal{M}}(x);\quad x\in\Pi(\mathcal{M}),\ \operatorname{dist}\big(x,\mathcal{M}^{-1}(0)\big)\le 1$$

where we denote $\mathcal{M}^{-1}(0) = \bigcap_{L\in\mathcal{M}} L^{-1}(0)$ and where we use the fact that:

$$u_{\mathcal{M}}(x+\xi) = u_{\mathcal{M}}(x),\ P_{\mathcal{M}}(t,x+\xi) = P_{\mathcal{M}}(t,x);\quad x\in\mathbb{R}^n,\ \xi\in\mathcal{M}^{-1}(0).$$

We shall also make use of the obvious fact that

$$P_{\mathcal{L}}(t,x) \le P_{\mathcal{M}}(t,x) \le 1;\quad t>0,\ x\in\mathbb{R}^d,\ \mathcal{M}\subset\mathcal{L} \tag{1.1.9}$$

and for every subset $\mathcal{M}\subset\mathcal{L}$ we shall fix once and for all some normalisation of $u_{\mathcal{M}}(x)$.

Now let $x_0\in\mathcal{M}^{-1}(0)$ and let us assume that $x_0\notin\mathcal{M}_1^{-1}(0)$ for every $\mathcal{M}\underset{\neq}{\subset}\mathcal{M}_1\subseteq\mathcal{L}$. It is clear then geometrically that there exists $x_0\in\Omega$ some Nhd of x_0 s.t.

$$\Omega\cap\partial\Pi(\mathcal{M}_1) = \Omega\cap\partial\Pi(\mathcal{M});\quad \mathcal{M}\subset\mathcal{M}_1\subset L. \tag{1.1.10}$$

If we apply the (elliptic) Harnack boundary principle it follows therefore that:

$$C^{-1}u_{\mathcal{M}_1}(x) \le u_{\mathcal{M}}(x) \le Cu_{\mathcal{M}_1}(x);\quad x\in\Omega,\ \mathcal{M}\subset\mathcal{M}_1\subset\mathcal{L}. \tag{1.1.11}$$

From this and the fact that $\alpha_{\mathcal{M}}\le\alpha_{\mathcal{L}}$ we deduce that:

$$u_{\mathcal{M}}(x) \le Cu_{\mathcal{L}}(x);\quad x\in\lambda\Omega,\ \lambda\ge 1. \tag{1.1.12}$$

From this we shall deduce by induction on the dimension d and on card($\mathcal{L}$) that there exists $C>0$ s.t.

$$P(1,x) \le Cu_{\mathcal{L}}(x);\quad x\in\Pi(\mathcal{L}). \tag{1.1.13}$$

Indeed we can cover $\partial_\Sigma(\Sigma\cup\Pi)$ by a finite number of $\Omega_1,\ldots,\Omega_\ell$ as in (1.1.10), (1.1.11), and (1.1.12). For every $x\in\Pi$ that lies in $\lambda(\bigcup\Omega_j)$ ($\lambda\ge 1$) the estimate (1.1.13) holds by induction on the dimension d and on Card($\mathcal{L}$). In the complement of that region the estimate (1.1.13) follows from (1.1.8) and (1.1.9).

This together with the obvious scaling property $P(\lambda^2t,\lambda x) = P(x,t)$ ($x\in\Pi,\lambda,t>0$) gives the second estimate (1.1.2). If we combine that estimate with (1.1.9) we can also obtain the following improvement:

$$P(t,x) \le C\inf_{\mathcal{M}\subset\mathcal{L}}\left\{\frac{u_{\mathcal{M}}(x)}{t^{\alpha_{\mathcal{M}}/2}}\right\};\quad x\in\Pi,\ t>0.$$

The first lower estimate (1.1.3) is obtained from (1.1.7) by scaling. Now for every fixed $x\in\Pi$, $h(t,y) = h_x(t,y) = p_t(x,y)$ is also a solution of the heat equation. Let us fix $y_0\in\Pi$ and $0<t_1<t_2<t_3<t_4$ and $A>0$. Then the Harnack boundary parabolic principle says that there exists $C = C(t_i, 1\le i\le 4; y_0, A)$ s.t.

$$C^{-1}\frac{h_x(t,y)}{h_x(t_4,y_0)} \le \frac{u(y)}{u(y_0)} \le C\frac{h_x(t,y)}{h_x(t_1,y_0)};\quad t\in[t_2,t_3],\ |y|\le A,\ x\in\Pi. \tag{1.1.14}$$

These estimates clearly imply:

$$p_t(x,y) \le Cu(y)p_{t_4}^{1/2}(x,x)p_{t_4}^{1/2}(y_0,y_0) \le Cu(y)p_{t_4}^{1/2}(x,x)$$

which, together with the fact that $p_t(x,x)$ is a decreasing function of t, gives for $x = y$

$$p_t^{1/2}(y,y) \leq Cu(y); \quad |y| \leq A, \ t \in [t_2, t_3]. \tag{1.1.15}$$

Similarly, by setting $x = y$ or y_0 on the right hand side inequality of (1.1.14), we have

$$p_{t_1}(y,y_0)u(y) \leq Cp_t(y,y); \quad p_{t_{1/2}}(y_0,y_0)u(y) \leq Cp_{t_1}(y,y_0)$$

which finally gives

$$p_t(y,y) \geq Cu^2(y); \quad |y| \leq A, \ t \in [t_2, t_3].$$

From this the second estimate of (1.1.3) immediately follows by the scaling:

$$p_{\lambda^2 t}(\lambda x, \lambda y) = \lambda^{-d} p_t(x,y); \quad t, \lambda > 0, \ x, y \in \Pi.$$

If we make the same analysis that we made before with the subsets $\mathcal{M} \subset \mathcal{L}$ [*cf.* (1.1.7)–(1.1.13)] and bear in mind that $p_1(x,x) \leq C$ ($x \in \Pi$) we deduce as before from (1.1.15) that

$$p_1(x,x) \leq Cu^2(x); \quad x \in \Pi.$$

This by scaling and polarisation gives

$$p_t(x,x) \leq Ct^{-\alpha-d/2}u^2(x); \ p_t(x,y) \leq Ct^{-\alpha-d/2}u(x)u(y); \quad t > 0, \ x, y \in \Pi.$$

The Gaussian estimate (1.1.2) is then obtained by general (and by now) standard methods (*cf.* [5]). We can also, of course, improve as before that upper estimate by replacing the cofactor of the Gaussian by $\inf_{\mathcal{M}}\big(t^{-\alpha_{\mathcal{M}}-d/2}u_{\mathcal{M}}(x)u_{\mathcal{M}}(y)\big)$.

A new idea is needed for the proof of (1.1.5), (1.1.6). We have:

$$p_t(\Pi_G; x, y) = \iint p_{t/3}(\Pi_G; x, z) p_{t/3}(\Pi_G; z, u) p_{t/3}(\Pi_G; u, y)\, dz\, du$$

and since $p_{t/3}(\Pi_G; z, u) \leq p^G_{t/3}(z,u) \leq Ct^{-D/2}(t > 1)$, where we denote here by $p^G_t(\cdot\,,\cdot)$ the heat diffusion kernel on G, we deduce (for $x = y$) that:

$$p_t(\Pi_G, x, x) \leq Ct^{-D/2}P^2_G(t/3, x) \leq Ct^{-\alpha-D/2}u^2(x); \quad t > 0, \ x \in \Pi_G.$$

But then the same abstract procedure (*cf.* [5]) as before applied this time directly to the operator Δ on G gives the proof of the estimate (1.1.5).

It is of some interest to observe that the above argument applied to a unimodular group G of exponential volume growth gives the estimate (1.1.5) where D is an arbitrarily large number this time. The proof of the lower estimate (1.1.6) is more involved. Observe however that this lower estimate is not needed for our applications. The only thing that is needed is Lemma 2 below. For that lower estimate we shall need some additional technical estimates:

LEMMA 1. *Let* $p_t(\mathcal{L};x,y)$ $x,y\in\Pi(\mathcal{L})\subset\mathbb{R}^d$, $u=u_{\mathcal{L}}$ *be as above. Then there exists* $\delta(\varepsilon)>0$ $(\varepsilon>0)$ s. t. $\delta(\varepsilon)\longrightarrow 0$ $(\varepsilon\longrightarrow 0)$ *and for which we have:*

$$\int_{|x|\le\varepsilon|x_0|\sqrt{t}} p_t(\mathcal{L},x_0,x)\,dx\le\delta(\varepsilon)\frac{u(x_0)}{t^{\alpha/2}};\quad x_0\in\Pi(\mathcal{L}),\ t>0.$$

$$\int_{\operatorname{dist}(x,\partial\Pi)\le\varepsilon|x|} p_t(\mathcal{L},x_0,x)\,dx\le\delta(\varepsilon)\frac{u(x_0)}{t^{\alpha/2}};\quad x_0\in\Pi(\mathcal{L}),\ t\ge 1,\ |x_0|^2\le Ct.$$

LEMMA 2. *Let* $p_t(\Pi_G;x,y)$ *with* $\Pi_G\subset G$ *be as before then there exists* $\delta(A)>0$ $(A>0)$ *such that* $\delta(A)\underset{A\to\infty}{\longrightarrow}0$

$$\int_{|g|\ge A\sqrt{t}} p_t(\Pi_G;g,g_0)\,dg\le\delta(A)\frac{u(g_0)}{t^{\alpha/2}};\quad t>1,\ g_0\in\Pi_G,\ |g_0|^2\le Ct.$$

Finally let $p_t^0(x,y)$, $t>1$, $x,y\in B_e(c_0\sqrt{t})$ be the heat kernel with Dirichlet boundary for the ball $B_t=B(c_0\sqrt{t})=\{g\in G;|g|\le c_0\sqrt{t}\}$ (*i.e.* we "kill" diffusion when it reaches the boundary of the ball). Then for any $c_0>c_1$ there exists $C>0$ (*cf.* [21], 8.3.3) s. t.

$$p_t^0(x,y)\ge Ct^{-D/2},\quad x,y\in G,\ |x|,|y|\le c_1\sqrt{t}. \tag{1.1.16}$$

An alternative way to prove (1.1.16) would be to consider the semigroup T_s^0 attached to p_s^0 and to use $\frac{d}{ds}\langle T_s^0 f_t,f_t\rangle$ where $f_t(\cdot)=\operatorname{dist}(\cdot,\partial B_t)$. This can be estimated immediately from below by $-\|\nabla f_t\|_2^2$.

For the proof of Lemma 1 we can scale and assume that $t=1$. The first estimate follows then from (1.1.2) and the elementary estimate

$$\int_{|x|\le\varepsilon|x_0|} u(x)\exp(-c|x_0-x|^2)\,dx\le C\varepsilon^A|x_0|^A\exp(-c|x_0|^2).$$

For the second estimate we have $|x_0|\le C$ and if we use (1.1.2) we see that it suffices to show that

$$\int_{d(x,\partial\Pi)\le\varepsilon|x|} u(x)\exp(-c|x-x_0|^2)\,dx=\int_{|x|\le 2}+\int_{|x|\ge 2}=I_1+I_2\le\delta(\varepsilon).$$

On I_1 we replace $\exp(\cdots)$ by 1 and on I_2 we replace it by $\exp(-c|x|^2)$. The result follows.

A simple use of the estimate (1.1.5) shows that Lemma 2 is a consequence of the fact that uniformly in $t>1$ we have:

$$t^{-\alpha/2-D/2}\int_{|g|\ge A\sqrt{t}} u\big(\pi(g)\big)\exp\Big(-c\frac{|g|^2}{t}\Big)\,dg=o(1)\quad\text{as } A\longrightarrow\infty.$$

To see this we recall that $u\big(\pi(g)\big)\le C|g|^\alpha$ and we integrate along "spherical shells" in G. This reduces the integral to

$$\int_{A\sqrt{t}}^{\infty}\lambda^{\alpha+D-1}\exp\Big(-\frac{\lambda^2}{ct}\Big)\,d\lambda$$

and the estimate follows at once.

If we put together (1.1.4) and the two lemmas it follows that for ε small enough there exist $C, c > 0$ such that:

$$\int_{R_\varepsilon(t,g_0)} p_t(\Pi_G; g_0, g)\,dg \geq C\frac{u\big(\pi(g_0)\big)}{t^{\alpha/2}}; \quad t \geq 1,\ c \leq |g_0| \leq C$$

where now $R_\varepsilon(t, g_0)$ is the subregion of Π_G obtained by requiring that:

$$|\pi(g)| > \varepsilon\sqrt{t}|\pi(g_0)|; \ d(g, \partial\Pi_G) > \varepsilon|\pi(g)|, \quad |g| \leq 1/\varepsilon\sqrt{t}$$

where we shall assume, by changing if necessary g_0 to some other point on $g_0 \ker\pi$, that $|g_0| \sim |\pi(g_0)|$. By obvious volume considerations it is clear then we can cover $R_\varepsilon(t, g_0)$ by C metric balls

$$B_\nu(c_0) = \{g; d(g, g_\nu) \leq c_0\sqrt{t}\} \subset \Pi_G(\mathcal{L}) \quad \nu = 1, \ldots, C$$
$$R_\varepsilon(t, g_0) \subset \bigcup_{\nu=1}^{C} B_\nu(c_1)$$

where c_0, c_1 are chosen small enough and so that (1.1.16) is satisfied. The conclusion is that we can find some ν_0 (depending on t and $|g_0|$) such that $B_0 = B_{\nu_0}(c_1)$ satisfies:

$$\int_{B_0} p_t(\Pi_G; g_0, g)\,dg \geq C\frac{u\big(\pi(g_0)\big)}{t^{\alpha/2}}.$$

We are finally in a position to prove (1.1.6). We have:

$$\begin{aligned} p_t(\Pi_G; g_0, g_0) &\geq \int_{B_0}\int_{B_0} p_{t/3}(\Pi_G; g_0, h)p_{t/3}(\Pi_G; h, k)p_{t/3}(\Pi_G; k, g_0)\,dh\,dk \\ &\geq C\frac{u^2\big(\pi(g_0)\big)}{t^\alpha} \inf_{h,k\in B_0} p_{t/3}(\Pi_G; h, k) \end{aligned}$$

where we clearly have $p_t(\Pi_G; h, k) \geq p_t^0(h, k), p^0$ being the heat kernel for the ball $B_0(c_0)$. It suffices therefore to apply (1.1.16) and (1.1.6) follows.

1.2 *The lower probabilistic estimate.* We preserve all the notation of Section 1.1 and we normalise and rotate the $L_j \in \mathcal{L}$ so that $L_j(\mathbf{1}) = L_j(1, 0, 0, \ldots, 0) = 1, j = 1, \ldots, s$. We shall also set:

$$L(x) = \inf_j L_j(x), \quad x \in \mathbb{R}^n; \ \Lambda(t) = L\Big[\pi\big(z(t)\big)\Big], \quad t > 0$$

where $\{z(s) \in G; s > 0\}$ is a diffusion induced on the Lie group G by the sublaplacian Δ and $\pi\colon G \longrightarrow \mathbb{R}^n$ is some (surjective) group homomorphism. Observe that here, contrary to what was done in Section 1.1, G is not assumed to be of polynomial volume growth. The estimate (1.1.3) has the following equivalent probabilistic formulation:

$$\mathbb{P}[\Lambda(s) \geq -1; 0 < s < t] \geq Ct^{-\alpha/2}$$

where we denote throughout $\mathbb{P} = \mathbb{P}_e$ (or $\mathbb{P}_0$ when $G = \mathbb{R}^n$). This is the key to the lower estimate of our theorem. Before it can be used, however, it has to be somewhat refined. The first type of refinement refers to the "curved boundary" (*cf.* [14]). So that we shall prove

(1.2.1)

$$\mathbb{P}[\Lambda(s) \geq -1 + f(s), 0 < s < t] \approx \mathbb{P}\big[z(s) \in \pi^{-1}C\big(-1 + f(s)\big), 0 < s < t\big] \geq Ct^{-\alpha/2}$$

where $f(s) \geq 0$ will be assumed increasing but "slowly enough" with $f(0) = 0$ and where we shall adopt the notation

$$C(\lambda) = (\lambda, 0, 0, \ldots) + \Pi(\mathcal{L}) \subset \mathbb{R}^n; \quad \lambda \in \mathbb{R}$$
$$T_h = \inf\{t; z(t) \notin C(-h)\}; \quad h \in \mathbb{R}.$$

With the above notation the estimates (1.1.2), (1.1.3) say that:

$$C^{-1} \min[t^{-\alpha/2}, 1] \leq \mathbb{P}[T_1 > t] \leq C \min[t^{-\alpha/2}, 1]; \quad t > 0$$

where to obtain the lower estimate for $0 < t < 1$ we use the additional (trivial) fact that $\mathbb{P}[T_1 > 1] = c > 0$. If we use the dilatation properties of $\pi\big(z(t)\big) = b(t) \in \mathbb{R}^n$ that take $h \longrightarrow 1$, $t \longrightarrow t/h^2$ we deduce that

$$\mathbb{P}[T_h > t] \approx \min[(ht^{-1/2})^\alpha, 1]; \quad t, h > 0. \tag{1.2.2}$$

We shall first consider the event:

$$W_m = [|b(s)| \leq 2^{m/2}\lambda(m); 2^{m-1} \leq s \leq 2^m]$$

for some appropriate $\lambda(m) \longrightarrow \infty$. We shall show that:

$$\mathbb{P}[W_m^c \parallel T_1 \geq 4N] \leq p(m); \quad N \geq 2^{m-1},\ m \geq 1,$$

where the $p(m)$ can be made to go to zero so fast that

$$\sum_{m \geq 1} p(m) < +\infty. \tag{1.2.3}$$

To avoid unnecessary complications let $\lambda(m) = 2^{\lambda m}$ for some $0 < \lambda \ll 1$ and let

$$U_{m,n} = \Big[\sup_{2^{m-1} \leq s \leq 2^m} |b(s)| \in [2^{(1/2+\lambda)(n-1)}, 2^{(1/2+\lambda)n}] = I_\lambda(n)\Big].$$

So that:

$$W_m^c = \bigcup_{n \geq m+1} U_{m,n}.$$

It clearly is only a matter of showing that for $\lambda > 0$ small enough we can find an estimate

$$\mathbb{P}[U_{m,n} \parallel T_1 > 4N] = p(m,n); \quad n > m,\ N \geq 2^{m-1}$$

that is independent of N and such that:

$$(1.2.4) \qquad \sum_{\substack{n,m\\ n>m}} p(m,n) < +\infty.$$

To see this let

$$\tau = \inf\big[t \in [2^{m-1}, 2^m]; b(t) \in I_\lambda(n)\big],$$

and note that $\tau < +\infty$ on $U_{m,n}$ We then have:

(1.2.5)

$$\mathbb{P}[U_{m,n} \cap (T_1 > 4N)] \leq \mathbb{P}[(\tau < +\infty) \cap (T_1 > 4N)] = \mathbb{E}\mathbb{P}[(\tau < +\infty) \cap (T_1 > 4N) \parallel \mathcal{F}_\tau]$$
$$\leq \mathbb{P}(\tau < +\infty) \sup\big[\mathbb{P}_\xi[T_1 > 4N]; |\xi| \in I_\lambda(n)\big].$$

For the first factor of the right hand side of (1.2.5) we use the standard Gaussian estimate for the maximal Brownian motion [11]:

$$\mathbb{P}(\tau < +\infty) \leq C \exp(-c2^{2\lambda n + (n-m)}).$$

For the second factor we use the estimate (1.2.2) that gives

$$2^{c\alpha n} N^{-\alpha/2};$$

(1.2.4) clearly follows.

We shall now consider the event:

$$(1.2.5) \qquad V_m = \big\{z(s) \in C\big(-1 + f(2^{m-1})\big); 2^{m-1} \leq s \leq 2^m\big\}$$

and for $N \geq 2^{m-1}$ we shall show that:

$$(1.2.6) \qquad \mathbb{P}[V_m^c \cap W_m \parallel T_1 \geq 4N] \leq Cf(2^{m-1}) 2^{-m/2} \lambda(m)^{\alpha-1}.$$

Let us consider:

$$\Omega_m = C(-1) \setminus C\big(-1 + f(2^{m-1})\big); \quad \tau = \inf[2^{m-1} \leq t \leq 2^m; z(t) \in \Omega_m \cap W_m].$$

We can condition then on $\mathcal{F}_\tau$ and obtain:

$$\mathbb{P}_0[W_m \cap V_m^c \cap (T_1 > 4N)] = \mathbb{E}\mathbb{P}[W_m \cap V_m^c \cap (T_1 > 4N) \parallel \mathcal{F}_\tau],$$

which by strong Markov gives (since $\tau < +\infty$ on the event in question)

$$\leq \mathbb{P}[T_1 > 2^{m-1}] \operatorname*{Sup}_{x \in \Omega_m \cap W_m} \mathbb{P}_x[z(s) \in C(-1); 0 < s < N].$$

The estimate (1.2.6) is therefore an immediate consequence of the fact that because of (1.1.1) and (1.1.2) the above sup can be estimated by:

$$\Big(\operatorname*{Sup}_{x \in \Omega_m \cap W_m} u_{\mathcal{L}}(x)\Big) N^{-\alpha/2} \leq C[\lambda(m)]^{\alpha-1} f(2^{m-1}) 2^{m/2(\alpha-1)} N^{-\alpha/2}.$$

We shall now assume that f satisfies

$$\sum_{m\geq 0} 2^{-m/2} f(2^m)\big(\lambda(m)\big)^{\alpha-1} < +\infty.$$

It follows then from (1.2.3) and (1.2.6) that for all $\varepsilon > 0$ there exists m_0 [depending on f and λ among other things] such that for all $M \geq m_0$ we have:

$$\sum_{m=m_0}^{M} \mathbb{P}(V_m^c \parallel T_1 \geq 2^{M+1}) \leq \varepsilon \tag{1.2.7}$$

which implies that:

$$\mathbb{P}\Big(\bigcap_{m=m_0}^{M} V_m \parallel T_1 \geq 2^{M+1}\Big) \geq 1-\varepsilon.$$

This, by (1.2.2), in turn implies that:

$$\mathbb{P}\big(\Lambda(s) \geq -1 + f(s); 2^{m_0} < s < t\big) \geq ct^{-\alpha/2}; \quad t > 1.$$

The above argument is an adaptation to higher dimensions of an argument in [14] for random walks. We almost but not quite have a proof of the estimate (1.2.1).

Before we go any further we shall go back to the original diffusion $z(t) \in G$ and prove some technical estimates. For $s < t/2$ we have:

$$\begin{aligned} \mathbb{P}[a \leq |z(s)|_G \leq a+1 \parallel T_1 \geq t] &= \frac{\mathbb{P}[|z(s)| \in [a, a+1]; T_1 \geq t]}{\mathbb{P}[T_1 \geq t]} \\ &\leq \mathbb{P}\big[|z(s)| \in [a, a+1]\big] \frac{\sup_\xi \mathbb{P}_\xi[T_1 \geq t-s]}{\mathbb{P}[T_1 \geq t]} \end{aligned} \tag{1.2.8}$$

by the Markov property where the sup runs through $[\xi \in \mathbb{R}^n; |\xi| \leq a+1]$. Because $s < t/2$ from our estimate (1.2.2) we see that the second factor in the right hand side of (1.2.8) is bounded by $C(a+1)^C$. We can use the standard Gaussian estimate for the heat diffusion on G (*cf.* [21]) to estimate the first factor and we finally obtain the estimate

$$C \exp\Big(-c\frac{a^2}{s} + ca\Big)$$

for the left hands side of (1.2.8). The factor e^{ca} comes from the modular function and the, at most exponential, volume growth of the group G.

The next technical estimate has to do with the increments $X_s = z(s)^{-1}z(s+1) \in G$ of the diffusion. Let $s+1 \leq t$, we have then:

$$\mathbb{P}[|X_s|_G \geq a \parallel T_1 \geq t] \leq \frac{\mathbb{P}[|X_s| \geq a]\mathbb{P}[T_1 \geq s-1]}{\mathbb{P}[T_1 \geq t]}$$

by the definition of the conditional probability and by conditioning on $[z(r), r \leq s-1]$. If we assume that $s \geq ct$ for some $c > 0$ then because of (1.2.2) the above conditional probability can be estimated by

$$\mathbb{P}[|X_s| \geq a] \leq Ce^{-ca^2+ca}$$

where the factor e^{ca} comes again from (the possibly exponential) volume growth of G.

The same estimate holds in general, *i.e.*, for the case $s < 10^{-10}t$ but the proof is less direct. Indeed, with the notation of Section 1.1 what we have to show is:

$$\int_{d_G(g,h)\geq a}\int p_s(\Pi_G; g_0, g)p_1(g,h)P_G(h, t-s)\,dg\,dh \leq Ce^{-ca^2}t^{-\alpha/2} \tag{1.2.9}$$

where $p_t(\cdot,\cdot)$ denotes the heat diffusion kernel on G and $g_0 \in G$ is some fixed point s.t. that $\pi(g_0) = \mathbf{1} = (1,0,\ldots,0) \in \mathbb{R}^n$. Because of (1.1.4) the estimate (1.2.9) is a consequence of

$$\int_{d(g,h)\geq a}\int p_s(\Pi_G, g_0, g)p_1(g,h)|\pi(h)|^\alpha\,dg\,dh \leq Ce^{-ca^2}.$$

We shall first prove the estimate:

$$\int_{d(g,h)\geq a} p_1(g,h)|\pi(h)|^\alpha\,dh \leq Ce^{-ca^2}\big(|\pi(g)|^\alpha + 1\big). \tag{1.2.10}$$

We can clearly find some $\tilde{g} \in G$ s.t. $\pi(\tilde{g}) = \pi(g)$ and $|\tilde{g}|_G = |\pi(g)|$ so using the left translation by the elements of $\operatorname{Ker}\pi$ we see that it suffices to show that

$$\int_{d(\tilde{g},h)\geq a} p_1(\tilde{g},h)|\pi(h)|^\alpha\,dh \leq Ce^{-ca^2}(|\tilde{g}|^\alpha + 1).$$

If we now use the standard Gaussian estimate

$$p_1(g,h) \leq \exp\big(-cd^2(g,h)\big); \quad g,h \in G,$$

we see that we can replace $p_1(\tilde{g},h)$ by $\exp\big(-ca^2 - cd^2(\tilde{g},h)\big)$ and integrate over G. It suffices therefore to verify that:

$$\int \exp\big(-cd^2(g,h)\big)|h|^\alpha\,dh = \int_{|h|\leq 10|g|} + \int_{|h|\geq 10|g|} \leq C(|g|^\alpha + 1).$$

In the first integral we replace $|h|^\alpha$ by $|g|^\alpha$ and in the second we replace $d(g,h)$ by $|h|$ and the estimate (1.2.10) is a consequence of the obvious estimate

$$\int e^{-c|g|^2}(|g|^\alpha + 1)\,dg \leq C.$$

To finish the proof of (1.2.9) we are therefore left with proving the estimate:

$$\int_G p_s(\Pi_G; g_0, g)\big(|\pi(g)|^\alpha + 1\big)\,dg \leq C.$$

This clearly reduces to the corresponding estimate on $\mathbb{R}^n$ namely:

$$\int_{\pi(\mathcal{L})} p_s(\mathcal{L}; \mathbf{1}, x)|x|^\alpha\,dx \leq C$$

which is clearly an easy consequence of the estimate (1.1.2).

By the above considerations we have finally proved that we have in general:

$$\mathbb{P}[|X_s| = |z(s)^{-1}z(s+1)| \geq a \parallel T_1 \geq t] \leq Ce^{-ca^2} \quad 0 < s < t-1 \tag{1.2.11}$$

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with constants $C, c > 0$ that are uniform in a, s, t.

If we combine the considerations that led to (1.2.7) together with the above estimate of (1.2.8) and (1.2.11) it follows that for all $\varepsilon > 0$ we can find $m, a > 0$ s.t.

$$\mathbb{P}\Big[\bigcap_{p=m}^{M} V_p; |z(2^{m-1})| < a; \Big|\big(z(j)\big)^{-1} z(j+1)\Big| \leq a \log^{1/2} j, 2^{m-1} \leq j \leq 2^M \parallel T_1 \geq 2^{M+1}\Big] \geq 1 - \varepsilon; M \geq m.$$

If we rewrite this estimate in terms of $\mathbb{E}\big(\cdot \parallel z(2^{m-1})\big)$ we obtain:

(1.2.12)
$$\int_{h \in G, |h| \leq a} \mathbb{P}[z(2^{m-1}) \in dh] \cdot \mathbb{P}[\Lambda(j) \geq -1 + f(j), |z(j)^{-1} z(j+1)| \leq a \log^{1/2} j; 2^{m-1} \leq j \leq 2^M \parallel z(m) = h] \geq CN^{-\alpha/2}; \quad N = 2^{M-1}.$$

The final observation is that for all $n, a, b > 0$ there exists $C > 0$ such that:

$$\mathbb{P}[z(n) \in dh] \leq C\mathbb{P}[z(n) \in dh; \Lambda(j) \geq -1 + f(j); \big|\big(z(j)\big)^{-1} z(j+1)| \leq b, j = 0, 1, \ldots, n-1]; \quad h \in G, \ |h|_G \leq a.$$

If we insert that last estimate in the integral (1.2.12) we finally deduce that:

$$\mathbb{P}\Big[\Lambda(j) \geq -1 + f(j), \Big|\big(z(j)\big)^{-1} z(j+1)\Big| \leq \rho \log^{1/2}(j+1); j = 1, 2, \ldots, N\Big] \geq CN^{-\alpha/2}, \quad N \geq 1 \tag{1.2.13}$$

for some appropriate $\rho > 0$.

1.3 *Technical lemmas.* Let G be some Lie group of polynomial growth $\big(\gamma(t) \sim t^D\big)$ and as before let $\{z(t) \in G; t > 0\}$ be the diffusion generated by the differential operator Δ. We have then:

$$\mathbb{P}[\sup_{0<s<t} |z(s)|_G \geq \lambda] \leq C \exp\Big(-c\frac{\lambda^2}{t}\Big); \quad \lambda \geq c\sqrt{t} \geq C > 0. \tag{1.3.1}$$

The proof is clear: the above event is the union of the following two events:

$$[|z(t)|_G \geq \varepsilon_0 \lambda]; \quad [|z(t)|_G \leq \varepsilon_0 \lambda\,;\ \operatorname*{Sup}_{0<s<t} |z(s)|_G > \lambda]. \tag{1.3.2}$$

The probability of the first event (1.3.2) has the correct bound (*cf.* [21]). To estimate the probability of the second event (1.3.2) we condition on $z(\tau) \in \partial B_e(\lambda)$ the first exit point from the ball of radius λ and obtain the estimate

$$\int_{g \in B(\varepsilon_0 \lambda)} \int_{\xi \in \partial B(\lambda)} \operatorname*{Sup}_{0<s<t} p_s(\xi, g)\, d\sigma(\xi)\, dg \tag{1.3.3}$$

where $\sigma(\xi) \in \mathbb{P}(\partial B)$ is the harmonic measure on ∂B. The $p_s(x,y)$ can be estimated by $s^{-D/2} \exp(-\frac{d^2(x,y)}{cs})$ (*cf.* [21]). For $\lambda \gg \sqrt{t}$ it is clear that the sup inside the integral

(1.3.3) is obtained for $s = t$. This gives the estimate $(\lambda^2/t)^{D/2}\exp(-c\lambda^2/t)$ for the expression in (1.3.3) and finishes the proof of (1.3.1). Observe that this type of argument applies to groups (or more generally to manifolds) of exponential volume growth, but the information that one then obtains is less precise.

We shall define now:

$$\mathbb{P}_x\Big[z(t+1)\in dy;\ \inf_{t<s<t+1} d\big(z(s),\xi\big)\leq 1\Big] = P_t(x,y;\xi)\,dy;\quad x,y,\xi\in G$$

and we shall prove the estimate:

$$P_t(x,y;\xi)\leq C_t\exp\Big(-c\frac{d^2(x,\xi)}{t}-cd^2(y,\xi)\Big);\quad t\geq 1 \tag{1.3.4}$$

where $C_t \leq C$ (in fact $C_t = O(t^{-D/2})$ but this will not be proved).

Let B be the unit ball centered at ξ and let τ be the first entry time in B after t:

$$B=\{g\in G; d(g,\xi)\leq 1\};\quad \tau=\inf[s; s\geq t, z(s)\in B].$$

With this notation it is clear that

$$P_t(x,y,\xi)=\mathbb{E}_x\Big[p_{t+1-\tau}\big(z(\tau),y\big);\tau<t+1\Big]$$

where $p_s(\cdot\,,\cdot)$ is as before the heat kernel on G, and we obtain the estimate

$$\Big(\mathbb{E}_x\Big[p^2_{t+1-\tau}\big(z(\tau),y\big)\Big]\Big)^{1/2}(\mathbb{P}_x[\tau<t+1])^{1/2}. \tag{1.3.5}$$

The second factor can be estimated by $\exp(-\frac{d^2(x,\xi)}{ct})$ because of (1.3.1). To estimate the first factor in (1.3.5) we distinguish two cases:

CASE (i). $d(y,\xi) > 10^{10}$. Then clearly $d\big(z(\tau),y\big)\geq\frac{1}{2}d(\xi,y)$ and we can estimate $p_{t+1-\tau}\big(z(\tau),y\big)\leq C\exp\big(-cd^2(\xi,y)\big)$. The estimate (1.3.4) follows

CASE (ii). $d(y,\xi)\leq 10^{10}$. We have in general:

$$P_t(x,y;\xi)\leq p_{t+1}(x,y)\leq\exp\Big(-c\frac{d^2(x,y)}{t}\Big);\quad t\geq 1$$

and in our case we can replace $d(x,y)$ by $d(x,\xi)$ by the triangle inequality. Our estimate (1.3.4) follows again.

With Π as in Section 0.5 and $\pi\colon G\longrightarrow\mathbb{R}^d$ as in Section 0.6 we shall now introduce the notation

$$D(g)=D\big(\pi(g)\big)=\big(|\pi(g)|^{\alpha-1}+1\big)\Big(\operatorname{dist}\big(\pi(g),\partial\Pi\big)+1\Big),\quad g\in G.$$

The use of that function lies in the following estimates (*cf.* (1.1.1)):

$$u(g)\leq CD(g),\quad g\in G.$$

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It is furthermore clear that we have

$$D(hg) = D(gh) \leq C(1 + |h|^{\alpha})D(g); \quad g, h \in G. \tag{1.3.6}$$

For further use we shall need the estimate

(1.3.7)

$$\int_G D(x) \exp\left(-\frac{d^2(g,x)}{Ct} - Cd^2(h,x)\right) dx$$
$$\leq C\left[D(g)\exp\left(-cd^2(g,h)\right) + D(h)\exp\left(-c\frac{d^2(g,h)}{t}\right)\right]; \quad t \geq 1,\ g,h \in G.$$

To see this we split the integration in the following two regions: $R = [x; d(g,x) \leq \varepsilon d(g,h)]$ and $R^c = G \setminus R$ for some appropriately small ε. The integral on R can clearly be estimated by:

$$C\exp\left(-cd^2(g,h)\right)[1 + d(g,h)]^D \sup_{x \in R} D(x)$$

and since by (1.3.6) the above sup can be estimated by $\left(1 + d^{\alpha}(g,h)\right)D(g)$ we obtain one of the two contributions on the right hand side of (1.3.7). The contribution of the integral on R^c can be estimated by:

$$\exp\left(-c\frac{d^2(g,h)}{t}\right)\int_G D(hx)\exp(-c|x|^2)\, dx$$
$$\leq \exp\left(-c\frac{d^2(g,h)}{t}\right)D(h)\int_G (1 + |x|^{\alpha})\exp(-c|x|^2)\, dx$$
$$\leq CD(h)\exp\left(-c\frac{d^2(g,h)}{t}\right)$$

and this gives the second contribution on the right hand side of (1.3.7).

In actual fact what will be needed is not the estimate of the above integral but a corresponding discrete variant that we shall now describe. We shall consider $Z = \{g_1, \ldots, g_n, \ldots\} \subset G$ some discrete subset that has the following two properties:

$$d(g_j, g_k) \geq 10, \quad j \neq k; \ d(g, Z) < 10, \quad g \in G. \tag{1.3.8}$$

The existence of such a subset is ensured by taking a maximal subset under the first condition (1.3.8). The estimate (1.3.7) will be used in the following obviously equivalent form:

$$\sum_{x \in Z} D(x)\exp\left(-\frac{d^2(g,x)}{Ct} - Cd^2(h,x)\right) \leq \cdots. \tag{1.3.7$'$}$$

Let us now consider the region

$$R_\Lambda = \left[g \in G; \operatorname{dist}\left(\pi(g), \partial\Pi\right) \leq \Lambda\right] \subset G$$

for some fixed Λ and let

$$\gamma^{\#}(t) = \operatorname{Card}\{g \in Z; |g|_G \leq t; g \in R_\lambda\}.$$

In the next section we shall need the estimate

$$\gamma^{\#}(t) \le C(1+\Lambda)^C t^{D-1}, \quad t \ge 1.$$

This estimate, because of the first condition in (1.3.8), is a consequence of the following estimate:

$$V_t = \mathrm{Vol}_G[R_\Lambda \cap B_t] \le C(1+\Lambda)^C t^{D-1} \qquad (1.3.9)$$
$$B_t = [g; |g| \le t].$$

To see (1.3.9) let us denote by:

$$\ell(x,t) = \text{ Haar measure } [\pi^{-1}(x) \cap B_t]; \quad x \in \mathbb{R}^d,\ t > 0$$

where the Haar measure is of course taken on the unimodular group $\operatorname{Ker}\pi \subset G$ that is identified with coset $\pi^{-1}(x)$. We then have (*cf.* [3])

$$V_t = \int_{|x| \le t} \ell(x,t) I[x; \operatorname{dist}(x, \partial\Pi) \le \Lambda]\, dx, \quad t > 0$$

and our estimate (1.3.9) is a consequence of the estimate

$$\ell(x,t) \le Ct^{D-d}; \quad x \in \mathbb{R}^d,\ t > 0. \qquad (1.3.10)$$

To prove (1.3.10) observe that $\ell(x,t) = 0$ if $|x| > t$. So it suffices to prove that for $|x| \le t$ we have $\ell(x,t) \le \ell(x,10t) \le Ct^{D-d}$. On the other hand, by the triangle inequality we have:

$$C^{-1}\ell(x_2, 10t) \le \ell(x_1, 10^5 t) \le C\ell(x_2, 10^{10}t); \quad x_1, x_2 \in \mathbb{R}^d,\ |x_1|, |x_2| \le t$$

and since:

$$\int_{|x| \le t} \ell(x,t)\, dx \le Ct^D$$

(1.3.10) follows.

1.4 *Upper probabilistic estimate.* Let $\Pi = \Pi(\mathcal{L})$ where we normalise as before by $L_j(1,0,\dots,0) = 1$. We shall fix also $[0,t]$ a time interval and we shall consider the subset of Brownian paths

$$\Omega_0 = \{\omega(s) \in \mathbb{R}^d;\ 0 < s < t,\ \omega(0) = 0, |\omega(t)| \le 1\}.$$

We shall define the following two random variables

$$\Lambda = \Lambda(\omega) = \sup\{\lambda \in \mathbb{R}; (\lambda, 0, \dots, 0) + \Pi \supset \omega\}$$
$$\tau = \tau(\omega) = \inf\{s;\ 0 < s < t,\ \omega(s) \in \partial[(\Lambda, 0, \dots, 0) + \Pi]\}.$$

Clearly $\Lambda \le 0$ on Ω_0. Let us define $\nu = 0, 1, \dots$; $\xi \in \mathbb{Z}^d$ (resp.: $a, b \in \mathbb{Z}^d$) by the conditions that $[\nu \le \tau < \nu + 1]$ and the fact ξ (resp.: a, b) is the (or one of the) closest lattice points to $\omega(\tau)$ [resp.: $\omega(\nu)$, $\omega(\nu+1)$]. The above "parameters" give a partition (or to be more precise a covering) of Ω_0 into the subsets

$$S(\nu_0, \xi_0, a_0, b_0) = \{\omega \in \Omega_0 \,;\, \nu(\omega) = \nu_0,\ \xi(\omega) = \xi_0,\ a(\omega) = a_0,\ b(\omega) = b_0\}.$$

The reader should observe geometrically that ξ determines $\omega(\tau)$ up to bounded distance and therefore it determines $\Lambda(\omega)$ up to a finite additive constant.

For every $\xi \in \mathbb{Z}^d$ there exists a unique $\lambda(\xi) = \lambda$ s.t. $\xi \in \partial[(\lambda, 0, \ldots, 0) + \Pi]$. We shall then set $\Lambda_\xi = (\lambda - C_d, 0, \ldots, 0) \in \mathbb{R}^d$, so that $\Lambda_\xi + \Pi$ is essentially the "largest" (*i.e.* more to the right) translate of Π containing all the sets $S(\cdot\,, \xi, \ldots)$. Here and in what follows I denote by $C_d > 0$ a constant that only depends on the dimension d and on the shape of the cone C.

We shall condition now on $\omega(\nu)$ and $\omega(\nu+1)$. The outcome is that we can estimate $P[S(\nu, \xi, a, b)]$ by the product of three probabilities $P_1P_2P_3$ (*cf.* Section 1.1). We have:

$$P_1 \le C\nu^{-\alpha-d/2} u(-\Lambda_\xi)\, u(a - \Lambda_\xi) \exp\left(-\frac{|a|^2}{c\nu}\right); \quad \nu \ge 1$$

$$P_3 \le C(t-\nu)^{-\alpha-d/2} u(b - \Lambda_\xi)\, u(-\Lambda_\xi) \exp\left(-\frac{|b|^2}{c(t-\nu)}\right)$$

which are respectively the probabilities that starting at 0 at time 0, we are close to a at time ν while staying in the Wedge $\Lambda_\xi + \Pi$ and starting at b (or somewhat close) at time $\nu+1$ we are close to 0 at time t while staying on the Wedge. When $\nu = 0$ we have $a = 0$ and then we make the trivial estimate $P_1 \le 1$. When $|\nu - t| \le 2$ the above estimate (or even the definition of $S(\cdots)$ or of P_3) does not always make sense; we shall set $t - \nu = 1$ in the above formula and as we shall see in the considerations that follow, "all will be well." We have to multiply P_1P_3 with P_2 which is the probability that starting at a (or somewhere close) at time ν we are close to b at time $\nu+1$ and that for some $\nu \le s < \nu+1$, $z(s)$ has been very close to ξ, *i.e.*,

$$P_2 = \mathbb{P}_a[\inf_{0<s<1} |\omega(s) - \xi| \le 1, |\omega(1) - b| \le 1].$$

As we have already seen in Section 1.3 P_2 can be estimated by:

$$P_2 \le C \exp(-c|\xi - a|^2) \exp(-c|\xi - b|^2).$$

Let us now assume that $t = n$ is an integer and consider the subset of Ω_0 that consists of the paths which satisfy

$$(1.4.1) \qquad \Lambda_{\omega(j)} \ge -\Lambda; \quad j = 0, 1, \ldots, n$$

for some fixed $\Lambda \ge 0$. What condition (1.4.1) essentially amounts to is that for the discrete times $t = 0, 1, \ldots, n$ we have

$$\omega(0), \omega(1), \cdots, \omega(n) \in -\Lambda + \Pi.$$

Our aim is to estimate the probability of the above paths by:

$$\Lambda^{2\alpha} n^{-\alpha-d/2}. \tag{1.4.2}$$

This estimate simply means that we do not lose anything in estimating the probability (1.1.2) of Section 1.1 if we restrict ourselves to discrete integer times. The way that we shall obtain the estimate (1.4.2) is the following: we shall multiply the above estimates for P_1, P_2, P_3 and sum over the following range of the parameters

$$\begin{gathered} 0 \le \nu \le n-1; \quad a, b \in -\Lambda + \Pi \\ \Lambda_a, \Lambda_b \ge \Lambda_\xi - C_d; \quad \Lambda_\xi \le C_d \end{gathered} \tag{1.4.3}$$

We shall then verify that above summation can be estimated by (1.4.2). This verification is an exercise in summing infinite series and it will be carried out presently in a more general setting.

The generalisations of the above estimate that we shall need are twofold. First we shall consider $\pi\colon G \longrightarrow \mathbb{R}^d$ where G is a Lie group of polynomial growth $\gamma(t) \sim t^D$ $(t \longrightarrow \infty)$ as in Section 1.1. We shall generalise the above estimate in that setting. The other complication is more subtle, and we already had to deal with it in [19]. To be specific let $e \in A \subset G$ be some subset that satisfies $P[X_j \in A] \ge 1 - \eta$ (for some small $\eta > 0$) where $X_j = \big(z(j)\big)^{-1} z(j+1) \in G$ and $\{z(t) \in G; t > 0\}$ is the diffusion on G that is controlled by the sublaplacian Δ. The estimate that we shall need is the following:

(1.4.4)

$$\mathbb{P}\Big[z(\nu+1) \in \pi^{-1}\big(-\Lambda + \Pi(\mathcal{L})\big) \text{ for all } 0 \le \nu \le n-1 \text{ that satisfy } X_\nu \in A; |z(n)|_G \le 1\Big] \le C(1+\Lambda)^c n^{-\alpha-D/2}.$$

We can even set $c = 2\alpha$ but this refinement will not be proved. Here, *a priori*, C, c depend on $\eta > 0, \Pi, G$ *etc.* but are independent of Λ and t. The proof of this new estimate follows the same lines as before, but before we give the proof we have to refine somewhat the notions involved. We shall consider as before:

$$\Omega_0 = \{z(s) \in G, 0 < s < t, z(0) = e, |z(t)|_G \le 1\}$$

and we shall define $\Lambda = \Lambda(\omega) = \Lambda(\omega')$, $\tau = \tau(\omega) = \tau(\omega')$ and $\nu \le \tau < \nu + 1$ exactly as before on $\omega' = \pi(\omega)$ where $\pi\colon G \longrightarrow \mathbb{R}^d$ and ω' is the projected path on $\mathbb{R}^d$. Together with ν we shall need to use another integer time parameter $0 \le j \le \nu$ defined by the condition

$$X_j \in A; \quad X_{j+1}, \dots, X_\nu \notin A.$$

If $X_\nu \in A$ we set $j = \nu$. If for all $j = 0, \dots, \nu$ we have $X_j \notin A$ we agree to set $j = 0$. Instead of the lattice points $\mathbb{Z}^d \subset \mathbb{R}^d$, we have to use $Z = \{g_1, g_2, \dots\} \subset G$ the "net" in G that was defined in Section 1.3 [*cf.* (1.3.8)]: $\xi, a, b, \in Z$ are defined as (one of) the nearest elements in Z to $\omega(\tau)$, $\omega(j)$, $\omega(\nu+1)$ respectively (*i.e.* $\xi \sim \omega(\tau)$, $a \sim \omega(j)$, $b \sim \omega(\nu+1)$). The "partition" (or more exactly the covering) of Ω_0 is now done by the set $S(\nu, j, \xi, a, b)$

where we specify the corresponding values of the above parameters. The probability of $S(\nu, j, \xi, a, b)$ is again estimated by conditioning on $\omega(j)$, $\omega(\nu+1)$ and we obtain the estimate $P[S(\nu, j, \xi, a, b)] \le P_1P_2P_3$ where clearly as before, by the use of Section 1.1, we have

$$P_1 \le Cj^{-\alpha-D/2}u(-\Lambda_\xi)\,u(a-\Lambda_\xi)\exp\Big(-\frac{|a|^2}{Cj}\Big) \tag{1.4.5}$$

$$P_3 \le C(t-\nu)^{-\alpha-D/2}u(b-\Lambda_\xi)\,u(-\Lambda_\xi)\exp\Big(-\frac{|b|^2}{c(t-\nu)}\Big)$$

with the convention that a, b and ξ inside $u(\cdots)$ actually are $\pi(a)$, $\pi(b)$ and $\pi(\xi)$ (the same convention as before is used when $j=0$ or $|t-\nu| \le 2$). What is more subtle is the estimate of P_2 that comes from the trajectory between time j and time $\nu+1$. Section 1.3 gives us already an estimate of that probability but we also have $P_2 \le C\eta^{\nu-j}$ for obvious reasons. We obtain therefore:

$$P_2 \le C\,\mathrm{Min}\Big[\eta^{\nu-j}, \exp\Big(-c\frac{d^2(\xi,a)}{(\nu-j)} - cd^2(\xi,b)\Big)\Big]$$
$$\le \eta^{\frac{\nu-j}{2}}\exp\Big(-c\frac{d^2(\xi,a)}{\nu-j} - cd^2(\xi,b)\Big) \le C\eta^{\nu-j}\exp\big(-cd(\xi,a) - Cd^2(\xi,b)\big)$$

where in the last inequality and also in further considerations we have changed η to a new (larger, but still very small) η'.

To obtain the required estimate (1.4.4) we have to sum $\sum P_1P_2P_3$ as before over the values of the parameters

$$0 \le j \le \nu \le n-1; \quad a \in -\Lambda + \Pi, \Lambda_1 \ge \Lambda_\xi - C_d, \Lambda_\xi \le C_d$$

with the same tacit understanding that $\Lambda_\xi = \Lambda_{\pi(\xi)}\cdots$. What remains then to be done is to use the above estimates to show that this summation can be estimated by the right hand side of (1.4.4) and this will clearly complete the proof of (1.4.4). In the summation range (1.4.3) it is crucial to observe that:

$$|\Lambda_\xi| \le C(1+r)(1+\Lambda); \quad r = d(\xi,a).$$

It follows that *modulo* factors of the form $(1+\Lambda)^C$, that will tacitly be dropped in everything that follows, we have (*cf.* Section 1.3)

$$u(-\Lambda_\xi) \le C(1+r)^C; \quad u(b-\Lambda_\xi) \le CD(b)(1+r)^C; \tag{1.4.6}$$
$$D(\xi) \le C|\xi|^{\alpha-1}(1+r).$$

If we use these estimates and sum P_2P_3 over b we obtain (*cf.* Section 1.3)

(1.4.7)

$$\sum_b P_2P_3 \le C(t-\nu)^{-\alpha-D/2}\eta^{\nu-j}\exp\big(-cd(\xi,a)\big)(1+r)^C$$
$$\cdot\sum_b D(b)\exp\Big(-\frac{|b|_G^2}{C(t-\nu)} - cd^2(\xi,b)\Big)$$
$$\le C(1+r)^C(t-\nu)^{-\alpha-D/2}\eta^{\nu-j}\exp\big(-cd(\xi,a)\big)|\xi|^{\alpha-1}\exp\Big(-c\frac{|\xi|^2}{t-\nu}\Big).$$

We shall multiply the right hand side of (1.4.7) with the estimate of P_1 in (1.4.5) and sum over ξ, a. We must therefore estimate

$$(1.4.8)\quad [(t-\nu)j]^{-\alpha-D/2}\eta^{\nu-j}\sum_{a,\xi} D(a)|\xi|^{\alpha-1}\exp\bigl(-cd(\xi,a)\bigr)\exp\Bigl(-c\frac{|\xi|^2}{t-\nu}-c\frac{|a|^2}{j}\Bigr).$$

To analyse the summation that appears in (1.4.8) we shall split it over the following two ranges:

RANGE (i). $|\xi|, |a| \le 10^{10}r \sim 10^{10}d(\xi,a)$.

RANGE (ii). $|\xi|, |a| \ge 10r \sim 10d(\xi,a)$
where r denotes here the closest integer to $d(\xi,a)$.

IN THE RANGE (i). $\sum_{a,\xi}$ can clearly be estimated by $\sum r^{C_0}e^{-cr} \le C$ for some large enough $C_0 > 0$ and we are left with estimating:

$$\sum_{1\le j\le \nu<t} \eta^{\nu-j}[(t-\nu)j]^{-\alpha-D/2}.$$

IN THE RANGE (ii). We can replace $D(a)$ by $(1+r)^C|\xi|^{\alpha-1}$ (*cf.* (1.4.6)) and since we have the factor e^{-cr} at our disposal it suffices to estimate

$$\sum_{\xi}|\xi|^{2\alpha-2}\exp\Bigl(-\frac{|\xi|^2}{cj}-\frac{|\xi|^2}{c(t-\nu)}\Bigr)$$

and multiply it with another factor $(1+r)^C$ to account for the a summation with fixed r. In terms of the growth function $\gamma^\#$ (*cf.* Section 1.3) we can therefore estimate the above sum by

$$\sum_{\lambda}\sum_{|\xi|\sim 2^\lambda}|\xi|^{2\alpha-2}\exp\Bigl(-C|\xi|^2\frac{t-\nu+j}{j(t-\nu)}\Bigr)$$
$$=\sum_{\lambda} C2^{(D+2\alpha-3)\lambda}\exp\Bigl(-C\frac{t-\nu+j}{j(t-\nu)}2^{2\lambda}\Bigr)\sim\Bigl[\frac{t-\nu+j}{j(t-\nu)}\Bigr]^{-\alpha-D/2+3/2}.$$

The final summation over j, ν gives then the estimate

$$\sum_{1\le j\le\nu<t}\eta^{\nu-j}\frac{(t-\nu+j)^{-\alpha-D/2+3/2}}{\bigl(j(t-\nu)\bigr)^{3/2}}$$

which by setting $\nu-j=k$ can be rewritten:

$$\sum_k \eta^k(t-k)^{-\alpha-D/2+3/2}\sum_{0<j<t-k}[j(t-k-j)]^{-3/2}\le C\sum_k\eta^k(t-k)^{-\alpha-D/2}\sim t^{-\alpha-D/2}.$$

Before we finish the proof of the estimate we still have to examine the trivial cases $\nu\ge 0$, $j=0$. In that case we estimate $P_1\le 1$. The estimate of P_2P_3 does not change nor does (1.4.7). The expression (1.4.8) is then replaced by:

$$(t-\nu)^{-\alpha-D/2}\eta^{\nu}\sum_{\xi}|\xi|^{2\alpha-2}\exp\Bigl(-c|\xi|-c\frac{|\xi|^2}{t-\nu}\Bigr)$$

where now $a = 0$. By an even simpler argument than before, the above can be estimated by $t^{-\alpha - D/2}$ and the proof of (1.4.4) is complete.

1.5 *The integrated upper estimate.* Let $L_1, L_2, \ldots$ be as before and we shall use here all the notation of Section 1.4. Let us denote

$$A_j(t) = \inf_{0<s<t} \exp\big(-L_j[b(s)]\big).$$

What will be needed for our applications are estimates of the form:

$$\mathbb{E}[\{A_1(t)A_2(t)\cdots\}^r; |b(t)| \leq 1] = 0(t^{-\alpha-d/2})r > 0; \tag{1.5.1}$$

or more generally, for the diffusion $z(t) \in G$ in a group of volume growth $\gamma(t) \approx t^D$ and $\pi\big(z(t)\big) = b(t)$ ($\pi\colon G \longrightarrow \mathbb{R}^d$ as before), we shall need an estimate of the type:

$$\mathbb{E}[\{A_1(t)A_2(t)\cdots\}^r; |z(t)|_G \leq 1] = 0(t^{-\alpha-D/2}). \tag{1.5.2}$$

These estimates follow immediately from what we have. Indeed observe that since $b(t) \approx -b(t)$ we can replace in (1.5.1) (1.5.2) all the L_j's by $-L_j$'s and consider the corresponding $A_j^-(t)$ but we clearly have

$$A_1^-(t)A_2^-(t)\cdots \leq e^{-c\Lambda} = e^{-c\Lambda(t)}$$

where Λ $(= \Lambda(\omega))$ is as in Section 1.4. But then the estimates (1.5.1) and (1.5.2) follow immediately by integrating:

$$\int_0^\infty e^{-c\lambda}\mathbb{P}[\Lambda \leq \lambda]\, d\lambda = 0(t^{-\alpha-D/2}). \tag{1.5.3}$$

The estimate (1.5.3) (at least for $G = \mathbb{R}^d$, $D = d$) does not need all the machinery developed in Section 1.4, it simply follows (by scaling as in (1.2.2)) from the estimates in Section 1.1. A little more work is needed for the general case $\pi\colon G \longrightarrow \mathbb{R}^d$, but again the argument is straight forward (*cf.* proof of (1.1.5)).

Where the argument becomes more involved is when we sample the inf in the definition of $A_j(t)$ *only* on integer times $t = 1, 2, \ldots$ or even the random set J_n of Section 0.6, Section 2.4 so as to have instead

$$\tilde{A}_n(L_i) = \inf_{0\leq j\leq n} \exp\big(-L_i(b_j)\big); \quad A_n^\#(L_i) = \inf_{j\in J_n} \exp\big(-L_i(b_j)\big),$$

where we denote $b_j = b(j)$, $(j = 0, 1, \ldots)$. To obtain the corresponding estimates for these new expressions of A's the full thrust of Section 1.4 seems to be needed.

Observe however that if we are prepared to settle with a "loss of an ε" *i.e.* have on the right hand side of (1.5.2) $0(t^{-\alpha-D/2+\varepsilon})$ for an arbitrarily small ε then the proof becomes *very simple*. Here is how one can go about giving that simplified proof: First of all, if we are prepared to "lose an ε", already (1.5.1) (1.5.2) can be obtained directly with an elementary construction that does *not* use the boundary Harnack principles. The reader with some knowledge of elementary probability theory can write this down for himself. To obtain from this the corresponding estimates for $\tilde{A}_n$ or $A_n^\#$ one can then use a standard "maximal oscillation" technique as in [17].

2. **Lie groups.**

2.1 *The structure of* NC-*groups and the roots.* To read this section the reader will have to refer to [16] Chapter I. I shall preserve here all the notation of [16] where in particular $\tilde{\mathfrak{g}}$ was denoted by $\mathfrak{n}_R = \tilde{\mathfrak{g}}$. Let us assume first that G is a simply connected NC-group Lie group. We can then write:

$$G = N_R \rtimes (Q_R \rtimes S) = N \rtimes G_R$$

where I denote by N_R, N, N_0, Q, S the analytic subgroups that correspond to $\mathfrak{n}_R, \mathfrak{n}, \ldots, \mathfrak{s}$ of [16]. Q (resp.: N) is in particular the radical (resp.: nilradical) and S a Levi subgroup. I shall denote by $[N_R, N_R]$ the analytic subgroup that corresponds to $[\mathfrak{n}_R, \mathfrak{n}_R]$ and by $H = N_R/[N_R, N_R] \cong \mathbb{R}^n$ so that the above semi direct product induces canonically (by inner automorphisms)

$$\pi: Q_R \rtimes S = G_R \longrightarrow \mathrm{Aut}(H) \cong \mathrm{GL}_n(\mathbb{R})$$
$$d\pi: \mathfrak{q}_R \rtimes \mathfrak{s} = \mathfrak{g}_R \longrightarrow \mathfrak{gl}_n(\mathbb{R}).$$

One should recall here some other facts and notation from [16]. We have $\mathfrak{n}_R = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ and $\mathfrak{q}/\mathfrak{n} \cong \mathfrak{q}_R/\mathfrak{n}_0 = V(\cong \mathbb{R}^d)$ and the subspaces $\mathfrak{n}_j$ $(1 \leq j \leq k)$ correspond to the distinct non zero real parts of the roots $L_1, \ldots, L_k \in \mathrm{Hom}_{\mathbb{R}}[V; \mathbb{R}]$. This decomposition of $\mathfrak{n}_R$ is stable by the action of $\mathfrak{q}_R \rtimes \mathfrak{s}$.

We shall denote by N_0 the (clearly closed *cf.* [15]) analytic subgroup that corresponds to $\mathfrak{n}_0$ and identify $V \cong Q_R/N_0 \cong Q/N$. The above (direct) decomposition of $\mathfrak{n}_R$ induces a decomposition

$$\mathfrak{n}_R/[\mathfrak{n}_R, \mathfrak{n}_R] = \bar{\mathfrak{n}}_1 + \cdots + \bar{\mathfrak{n}}_s$$

which is also direct. Indeed each subspace $\bar{\pi}$ is stable by $d\pi(x)$ $(x \in \mathfrak{q}_R \rtimes \mathfrak{s})$ and the real parts of the roots of $d\pi(q)$ $(q \in \mathfrak{q}_R)$ on $\bar{\mathfrak{n}}_j$ is L_j. By the standard Jordan-Hölder argument [12], it follows therefore that

$$(\bar{\mathfrak{n}}_{\alpha_1} + \cdots + \bar{\mathfrak{n}}_{\alpha_p}) \cap (\bar{\mathfrak{n}}_{\beta_1} + \cdots + \bar{\mathfrak{n}}_{\beta_q}) = \{0\}; \quad \alpha_j \neq \beta_i,\ i,j = 1,2,\ldots.$$

We obtain therefore a direct decomposition that is stable by the action of $\pi(G_R)$.

Now let $z = ng \in Z(G)$ be an element in the center of G. We shall show that $z \in G_R$. Indeed with the standard notation $x^y = y^{-1}xy$ $(x, y \in G)$ we have

$$z = z^h = (ng)^h = n^h g^h; \quad h \in G_R$$

and $n^h \in N_R, g^h \in G_R$; this implies that $n = n^h$ (and $g = g^h$). But the real parts of the roots of the action of G_R on $\mathfrak{n}_R$ are non zero and we have: $n = \mathrm{Exp}(\nu) = \mathrm{Exp}\big(\mathrm{Ad}(h)\nu\big) = n^h$ for some unique $\nu \in \mathfrak{n}_R$ (*cf.* [15]). This implies that $\nu = 0$ and $n = e$. Our assertion follows.

Now let G be a general connected, but not necessarily simply connected, Lie group and let:

$$\theta: \bar{G} = \bar{N}_R \rtimes \bar{G}_R \longrightarrow G$$

be the simply connected covering group and the covering map. It follows from the above that $\ker\theta \subset \bar{G}_R$ and that therefore

$$G = N_R \rtimes G_R \tag{2.1.1}$$

where N_R is the analytic subgroup that corresponds to $\mathfrak{n}_R = \tilde{\mathfrak{g}}$ which is thus simply connected and closed. Let us now consider G/N where $N \subset G$ is the (closed) nilradical. The Lie algebra of that group is $\mathfrak{q}/\mathfrak{n} \oplus \mathfrak{s}$ (*cf.* [15]). Let $\bar{N}$, $\bar{Q}$, $\bar{S}$ denote the simply connected groups that correspond to $\mathfrak{n}$, $\mathfrak{q}$ and $\mathfrak{s}$. Further, let $\bar{V} = \bar{Q}/\bar{N}$. We have then covering maps

$$\alpha\colon \bar{V} \times \bar{S} \longrightarrow G/N; \quad \bar{V} \longrightarrow Q/N = \bar{V}/(\bar{V} \cap \operatorname{Ker}\alpha)$$

where $Q \subset G$ is the (closed) radical of G. It follows in particular that $Q/N \cong \mathbb{R}^a \times \mathbb{T}^b$ and that we can factor α:

$$\bar{V} \times \bar{S} \longrightarrow \bar{V}/(\bar{V} \cap \operatorname{Ker}\alpha) \times \bar{S} \xrightarrow[\beta]{} G/N$$

where now we can identify

$$\beta\colon \mathbb{R}^a \times \mathbb{T}^b \times \bar{S} \longrightarrow G/N; \quad \operatorname{Ker}\beta \cap (\mathbb{R}^a \times \mathbb{T}^b) = \{0\}. \tag{2.1.2}$$

Since, however, $\operatorname{Ker}\beta$ is central (β being a covering transformation), it projects by $\bar{V}/(\bar{V} \cap \operatorname{Ker}\alpha) \times \bar{S} \longrightarrow \bar{S}$ to a subgroup of the *finite* center of $\bar{S}$; this together with (2.1.2) implies that $\operatorname{Ker}\beta$ is finite. Let us finally consider the canonical projection

$$\mathbb{R}^a \times \mathbb{T}^b \times \bar{S} \longrightarrow \mathbb{R}^a.$$

By that projection $\operatorname{Ker}\beta$ (being finite) has to go to 0 and therefore it follows that:

$$\operatorname{Ker}\beta \subset \mathbb{T}^b \times \bar{S}.$$

We finally conclude that we can identify:

$$G_R/(G_R \cap N) = G/N = V \times K = \mathbb{R}^a \times \left[(\mathbb{T}^b \times \bar{S})/\operatorname{Ker}\beta\right].$$

As already observed in Section 0.2, the above factorisation uniquely determines K. On the group G_R we can therefore define

$$\pi\colon G_R \longrightarrow V \cong \mathbb{R}^a.$$

The final thing to observe is that the inner automorphisms induce

$$G_R \longrightarrow \operatorname{Aut}(N_R) \longrightarrow \operatorname{Aut}(H)$$

and that the real parts of the roots of that action ($\in \operatorname{Hom}_{\mathbb{R}}(\mathfrak{q}_R/\mathfrak{n}; \mathbb{R})$) with the obvious identification factor through π and can thus be identified with elements of $(V)^*$. To see this last point we shall show that the roots that are defined on $\bar{V}$ are all purely imaginary on $\Xi = \bar{V} \cap \operatorname{Ker}\alpha$ and therefore all the L_i, $i = 1, 2, \ldots, s$, vanish on the subspace spanned by Ξ. Indeed the elements in $\bar{V} \cap \ker \alpha$ come from central elements of G that induce a trivial $\mathrm{Ad}(\cdot)$ action on $\mathfrak{n}_R$. It follows that

$$\exp\bigl(L_i(\xi)\bigr) = 1 \quad \xi \in \bar{V} \cap \operatorname{Ker} \alpha = \Xi$$

and our assertion follows (the actual roots need not of course reduce to zero on Ξ; they could take the values $2n\pi i$, $n \in \mathbb{Z}$).

2.2 *The lower estimate.* The best way to present the proof of the lower estimate is through probabilistic considerations on the diffusion ($z(t) \in G$, $t > 0$) that is generated by Δ (*cf.* [16],18]). Indeed let $A \subset G$ be arbitrary. We then have

$$\phi_{2t}(e) = \int_G \phi_t(g)\phi_t(g^{-1})\,d^r g = \int \phi_t(g)\phi_t(g^{-1})\,dg = \int_G \phi_t^2(g)\,d^r g$$
$$\mathbb{P}_e[z(t) \in A] = \int_A \phi_t(g)\,d^r g.$$

A simple use of the Hölder inequality then gives

$$\phi_{2t}(e)^{1/2} \geq \mathbb{P}_e[z(t) \in A]\,|A|_r^{-1/2}$$

where $|\ |_r$ denotes the right Haar measure of the set. To obtain the lower estimate in the theorem it suffices therefore for every integer valued $t > 1$ to find the appropriate $A = A_t \subset G$ for which

$$\mathbb{P}_e[z(t) \in A] \geq Ct^{-\alpha/2}; \quad |A|_r \leq Ct^{D/2}.$$

We shall follow very closely the argument of [18] Section 3. Let us write

$$z(t) = Z_t = \gamma_1\gamma_2 \cdots \gamma_t;$$
$$\gamma_j = n_j\,\dot{\gamma}_{j+1} \in G; \quad \dot{\gamma}_j \in G_R; \quad n_j \in N_R$$
$$\dot{Z}_t = \dot{\gamma}_1\dot{\gamma}_2 \cdots \dot{\gamma}_{t+1}, \quad Z_t = \Lambda_t \dot{Z}_t$$
$$\Lambda_t = n_1 n_2^{\dot{\gamma}_2} \cdots n_j^{\dot{\gamma}_1 \cdots \dot{\gamma}_j} \cdots; \quad j = 1, 2, \ldots \ (\text{set } \dot{\gamma}_1 = e).$$

Here $G = N_R \rtimes G_R$ is as in (2.1.1) and we use the standard notation $x^y = yxy^{-1}$ $(x, y \in G)$. Because of the standard exponential distortion of the distance between N_R and G (*cf.* [18], [16]) we have:

$$|n_j|_{N_R} \leq Ce^{C_0|\gamma_j|_G}, \quad j = 1, 2, \ldots.$$

Let us further denote by

$$\ell_j = \inf_{1 \leq i \leq k} L_i(\dot{Z}_j), \quad j = 1, 2, \ldots$$

where, as explained in Section 2.1, we use the canonical projection $\pi = G_R \longrightarrow V$ to identify the real parts of the roots with functions on G_R. We shall consider then the event:

$$\text{(2.2.1)} \qquad |\dot{Z}_t| \leq A_1 t^{1/2}; \quad \ell_j \geq -1 + j^{\varepsilon},$$
$$\text{(2.2.2)} \qquad |\gamma_j|_G \leq \rho \log^{1/2}(1 + j); \quad j = 1, \ldots, t$$

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where the ρ is as in (1.2.13). We shall show that the probability of this event is $\geq Ct^{-\alpha/2}$. If we use the methods and the results of [18] (The reader will have to carry the details out himself), we see that on the event (2.2.1), (2.2.2) we have

$$Z_t \in [x \in N_R; |x| \leq C] \cdot [g \in G_R; |g|_{G_R} \leq Ct^{1/2}] = A_t$$

and that therefore $|A_t|_r \leq Ct^{D/2}$. This will complete the proof of the theorem. The proof of the above probabilistic estimate is a consequence of (1.2.13) and (0.6.1). Indeed we consider the projections

$$G \xrightarrow[\pi_1]{} G_R \xrightarrow[\pi_2]{} V$$

so that $\pi\big(z(s)\big) = z_R(s) \in G_R$ $(s > 0)$ and we shall consider the event in Section 1.2 [*cf.* 1.2.13] that is obtained "at the $G \longrightarrow V$ level" with $f(s) \sim s^\varepsilon$ (ε small enough). Then delete the event (0.6.1) that is obtained "at the $G_R \longrightarrow V$ level" for some large enough A. We thus obtain an event on which all the estimates (2.2.1), (2.2.2) hold and which has the required probability.

The probability theory needed to obtain these estimates in Section 1.2 was quite involved. That probability theory simplifies considerably if we are prepared to "settle" with the "weak lower estimate" $\phi_t(e) \geq C_\varepsilon t^{-\alpha-D/2-\varepsilon}$ (for an arbitrary small $\varepsilon > 0$). Indeed we can then replace C_i, $(i = 1,2)$ in (2.2.1), (2.2.2) with $C_i t^\varepsilon$ and it is then very easy to see that the probability of the corresponding event is $\geq Ct^{-\alpha/2-\varepsilon}$.

2.3 *A general estimate for the heat kernel.* Let us preserve all the notation introduced in Sections 2.1 and 2.2 and let us denote by $\tilde{G} = G/[N_R, N_R] = H \rtimes G_R$. Let us fix Δ some driftless Laplacian on G and let $\tilde{\Delta}$ the Laplacian on $\tilde{G}$ obtained by canonical projection. We shall denote by $\tilde{\phi}_t$ the corresponding heat diffusion Kernel on $\tilde{G}$. Here I shall shift to the estimates and the notation of [17]. By following closely the argument in [17] (this the reader has to do for himself) we obtain the basic estimate:

$$\tilde{\phi}_{n-1}(e) \leq C\mathbb{E}_e\Big[A_n(L_1)\cdots A_n(L_s); d\big(z_R(x), e\big) \leq 1\Big] \tag{2.3.1}$$

where $[z_R(t) \in G_R; t > 0]$ is the diffusion on G_R induced by Δ_R (= the projection of Δ by $G \longrightarrow G_R$) with:

$$A_n(L_i) = \inf_{1\leq j\leq n} \exp\Big(c|z_R(j-1)^{-1} z_R(j)|_{G_R} - L_i(b_j)\Big); \quad i = 1,2,\ldots,s,\ n \geq 1 \tag{2.3.2}$$

where we use the same notation as in [17]. To be more explicit we recall that $L_i \in V^*$ and we denote by $b_j = b(j)$ $(j = 1,2,\ldots)$ the canonical projection of $z_R(j) \in G_R$ on V which is standard Brownian notion (at integer times) with respect to the standard Laplacian $\Delta_V = \sum \frac{\partial^2}{\partial x_\alpha^2}$ (*cf.* Section 0.2). We must also recall that $G/N = G_R/N_0 = Q/N \times K = V \times K$ a direct product so that if we denote by $s_j \in G_R/N_0$ the canonical projection of $z_R(j)$ we have

$$s_j = (b_j, \sigma_j), \quad g_j \approx (X_j, \tilde{\sigma}_j) \in V \times K; \quad s_j = g_1 \cdots g_j, \quad b_j = X_1 + \cdots + X_j \quad (j \geq 1)$$

which is the same notation as in [17]. Finally the constant $c > 0$ that appears in the definition of $A_n(L_i)$ in (2.3.2) *can be assumed* arbitrary small. As explained in [17] this last fact relies on the strict Gaussian nature (*cf.* [16] for definitions) of the heat diffusion kernel and is highly non trivial to prove. Fortunately as we have already explained in [16] and as we shall see here also, we do not have to make essential use of this fact.

2.4 *The upper estimate.* In this section we shall preserve all the notation of the previous section. Under the NC-condition on the real roots $L_1, \ldots, L_s$ we shall estimate the right hand side of (2.3.1). Towards that let:

$$\tilde{A}_n(L_i) = \inf_{0 \leq j \leq n} \exp\bigl(-L_i(b_j)\bigr); \quad B_n = \sup_{1 \leq j \leq n} \exp(C|z_R(j-1)^{-1} z_R(j)|^2).$$

If we bear in mind that

$$\mathbb{P}\Bigl[\sup_{0 \leq j \leq n} |z_R(j-1)^{-1} z_R(j)| > \lambda\Bigr] \leq Cn \exp(-C_1\lambda^2); \quad n = 1, 2, \ldots, \ \lambda > 0$$

where $C, C_1 > 0$ are numerical constants, we see that, as long as the $c > 0$ in (2.3.2) has been chosen small enough we have:

$$\|B_n\|_p = 0(n^{1/p})$$

for any preassigned $p \geq 1$.

It is clear on the other hand that

$$A_n(L_i) \leq B_n \tilde{A}_n(L_i).$$

An obvious use of Hölder inequality shows therefore that we can estimate the right hand side of (2.3.1) by:

$$n^{1/p}\Bigl(\mathbb{E}\bigl[\{\tilde{A}_n(L_1) \cdots \tilde{A}_n(L_s)\}^q; d\bigl(z_R(n), e\bigr) \leq 1\bigr]\Bigr)^{1/q}$$

where $1/p + 1/q = 1$, $1 < p, q < +\infty$ are conjugate Hölder indices. As we saw in Section 1.5, for every $q > 1$ the above expectation is $0(n^{-\alpha-D/2})$. For any $\varepsilon > 0$ if we chose $p > 1$ large enough we obtain therefore the estimate $0(n^{-\alpha-D/2+\varepsilon})$ for the right hand side of (2.3.1). This is the required estimate with a loss of an ε. To "get rid of the ε" we have to use the full thrust of Section 1.4. Let

$$J = J_n = \{j; 1 \leq j \leq n, |z_R(j-1)^{-1} z_R(j)| \leq C\}.$$

J is of course a random subset of $\{1, 2, \ldots, n\}$ that depends on $C > 0$ which is some appropriate positive constant. Let us denote

$$A_n^{\#}(L_i) = \inf_{j \in J_n} \exp\bigl(-L_i(b_j)\bigr).$$

We can estimate again the right hand side of (2.3.1) by:

$$E\bigl[A_n^{\#}(L_1) \cdots A_n^{\#}(L_s); d\bigl(z_R(n), e\bigr) \leq 1\bigr].$$

We shall now invoke the much harder estimate of Section 1.5 to deduce that the right hand side of (2.3.1) can be estimated by $0(n^{-\alpha-D/2})$.

This finishes the proof of our upper estimate for the group $\tilde{G}$ and therefore also by the standard local Harnack estimate for the original group G.

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