On some theorems in the theory of numbers.

By R. E. Allardice, M.A.

The number of groups of n which may be selected from $2n$ is $2n(2n-1)\ldots(n+1)/n!$ But make the $2n$ into two groups of n, and select r out of the first and $n-r$ out of the second. This gives $[n(n-1)\ldots(n-r+1)/r!]+[n(n-1)\ldots(r+1)/(n-r)!]$ ways of thus making a group of n. Hence

$$* \; 2n(2n-1)\ldots(n+1)/n! = 1 + n^2 + [n(n-1)/2!]^2 + \ldots \quad \ldots \quad (1).$$

$$\therefore 2n(2n-1)\ldots(n+1)/n! - 2$$
$$= n^2\{1^2 + [(n-1)/2!]^2 + [(n-1)(n-2)/3!]^2 + \ldots\}$$
$$= n^2\{P_1^2 + P_2^2 + \ldots + P_{n-1}^2\} \text{ (say)}.$$

We shall now show that $P_1^2 + P_2^2 + - - + P_{n-1}^2$ is divisible by n, if n be prime.

$P_r \equiv P_s$ (mod. n.), if $r+s=n$, but not otherwise. For if $P_r \equiv P_s$ $(r>s)$, then

$$(n-1)(n-2)\ldots(n-r+1)/r! - (n-1)(n-2)\ldots(n-s+1)/s! \equiv 0;$$

$$\therefore \frac{(n-1)(n-2)\ldots(n-s+1)}{s!}\left\{\frac{(n-s)(n-s-1)\ldots(n-r+1)}{(s+1)(s+2)\ldots \quad r} - 1\right\} \equiv 0;$$

$$\therefore (n-s)(n-\overline{s+1})(n-\overline{s+2})\ldots(n-\overline{r-1}) - (s+1)(s+2)\ldots r \equiv 0;$$

$$\therefore \quad \pm s(s+1)(s+2)\ldots(r-1) - (s+1)(s+2)\ldots r \equiv 0;$$

$$\therefore \quad -(s+1)(s+2)\ldots(r-1)(\mp s + r) \equiv 0;$$

and this is true if $r+s=n$ (otherwise obvious) and not in any other case. [If $r+s=n$, then $r-s=n-2s$, which is odd, and the lower sign is to be taken where the double sign is printed.]

It is obvious that $P_r + P_s$ is not divisible by n; and hence if we divide $P_1, P_2, \ldots P_{(n-1)/2}$ by n, we must get for remainders either 1 or $n-1$ and either 2 or $n-2$ and so on.

Now since $(n-r)^2 = n^2 - 2nr + r^2 \equiv r^2$ (mod. n), we must have

$$P_1^2 + P_2^2 + \ldots P_{(n-1)/2}^2 \equiv 1^2 + 3^2 + 5^2 \ldots + (n-2)^2$$
$$= n(n-1)(n-2)/6;$$

which is divisible by n if n be any prime except 2 or 3.

From this, and the identity (1), it follows that

$$(2n-1)(2n-2)\ldots(n+1) - (n-1)(n-2)\ldots 1 \equiv 0 \text{ (mod. } n^3).$$

* The use of this identity was suggested to me by Professor Tait.

https://doi.org/10.1017/S0013091500030467 Published online by Cambridge University Press

We shall next show that $\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)(n-1)!$ is divisible by n^2.*

We have

$$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)(n-1)! = \left(\frac{n}{1.(n-1)}+\frac{n}{2.(n-2)}+\ldots\right)(n-1)!$$

Hence we have to show that $\left(\frac{1}{1.(n-1)}+\frac{1}{2.(n-2)}+\ldots\right)(n-1)!$ is exactly divisible by n.

Assume $(n-1)!/1.(n-1) = a_1$, $(n-1)!/2.(n-2) = a_2$, &c.

Then

$$\begin{aligned}(r+1)^2 a_{r+1} - r^2 a_r &= (r+1)^2(\overline{n-1}!)/(r+1)(n-r-1) - r^2(\overline{n-1}!)/r(n-r)\\ &= \{(r+1)/(n-r-1) - r/(n-r)\}(n-1)!\\ &= (n!)/(n-r)(n-r-1) \equiv 0 \text{ (mod. } n).\end{aligned}$$

$$\begin{aligned}\therefore\ (r+1)^2 a_{r+1} &\equiv r^2 a_r\\ &\equiv (r-1)^2 a_{r-1}\\ &\ldots \quad \ldots \quad \ldots\\ &\equiv 1^2.a_1\\ &\equiv 1 \qquad \text{(by Wilson's theorem).}\end{aligned}$$

Hence we may write

$$\begin{aligned}a_1 &= n\mu_1 + 1\\ 2^2 a_2 &= n\mu_2 + 1\\ &\ldots \quad \ldots \quad \ldots\\ m^2 a_m &= n\mu_m + 1 \qquad \text{(where } m = (n-1)/2)\end{aligned}$$

$\therefore\ (m!)^2\Sigma a_r = \text{P}.n + (m!)^2(1/1^2 + 1/2^2 + \ldots + 1/m^2)$.

Now, if we assume $(m\,!)/r \equiv a_r$, we may easily show that $a_r \pm a_s$ is not divisible by n, and hence that

$$a_1{}^2 + a_2{}^2 + \ldots + a_m^2 \equiv 0 \text{ (mod. } n),$$

which proves the theorem.

Consider again the result

$$\text{A} = (2n-1)(2n-2)\ldots(n+1) - (n-1)(n-2)\ldots 1 \equiv 0 \text{ (mod. } n^3).$$

This gives

$$\begin{aligned}\text{A} &= (n+\overline{n-1})(n+\overline{n-2})\ldots(n+1) - (n-1)(n-2)\ldots 1\\ &= n^{n-1} + p_1 n^{n-2} + \ldots + p_{n+3}n^2 + p_{n-2}n,\end{aligned}$$

where

$$\begin{aligned}p_{n-3} &= (\overline{n-1}!)\Sigma(1/rs)(r+s)\\ p_{n-2} &= (\overline{n-1}!)\Sigma(1/r).\end{aligned}$$

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* Compare a paper by Mr Leudesdorf, in the *Proceedings of the Lond. Math. Soc.* for 1889, p. 199—a paper which I did not see till after the above was written.

Now p_{n-2} is divisible by n^2, and hence p_{n-3} is divisible by n.

This theorem may also be proved in the following manner:—

We have $2(\overline{n-1}!)\Sigma(1/rs)$

$$=\{(\overline{n-1}!)/1.2+(\overline{n-1}!)/1.3+\ldots+(\overline{n-1}!)/1.(n-1)\}(=\text{P}_1)$$
$$+\{(\overline{n-1}!)/2.1+(\overline{n-1}!)/2.3+\ldots+(\overline{n-1}!)/2.(n-1)\}(=\text{P}_2)$$
$$+\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

Now consider the terms of P_r, namely,

$$(\overline{n-1}!)/r.1, (\overline{n-1}!)/r.2, \ldots\ldots (\overline{n-1}!)/r.(r-1), (\overline{n-1}!)/r.(r+1), \text{ \&c.}$$

No two of these can be congruent; and

$$(\overline{n-1}!)/r.p+(\overline{n-1}!)r.(n-p)=n!/r.p.(n-p)\equiv 0 \pmod{n}.$$

Hence if we divide each of the terms of P_r by n, we get as remainders all the numbers 1, 2, $3\ldots n-1$, with the exception of that number which is complimentary to a_r where a_r

$$\equiv(\overline{n-1}!)/r.(n-r) \pmod{n}.$$

Hence the sum of all the remainders in $2\Sigma(n-1!)/rs$

$$=(n-1)(1+2+\ldots+\overline{n-1})-2(a_1+a_2+\ldots+a_{(n-1)})$$
$$=(n-1)^2n/2-2\{1/1.(n-1)+1/2.(n-2)+\ldots\}$$

which is divisible by n.

The theorem that the sum of the reciprocals of the numbers $1, 2, \ldots \overline{n-1}$, is divisible by n^2, when n is a prime, may be extended to the sum of the m^{th} powers of these numbers, where m is an integer, positive or negative.

Let $\text{S}_m=1^m+2^m+\ldots(n-1)^m$; it being understood that if m is negative $(=-l)$, the sum of the powers is to be multiplied by $(\overline{n-1}!)^l$, so that it may be made integral.

Since, when n is prime the equation

$$(x-1)(x-2)\ldots(x-\overline{n-1})-x^{n-1}+1\equiv 0 \pmod{n}$$

has $(n-1)$ incongruent solutions, each co-efficient is divisible by n. Hence, if m is positive, S_m is divisible by n, unless m is a multiple of $(n-1)$.

Suppose now that m is an odd positive integer and $n \neq 2$; then

$$2\ \text{S}_m=2\Sigma a^m=\Sigma(a^m+\overline{n-a}^m)$$
$$=\Sigma\{a^m+n^m-{}_m\text{C}_1n^{m-1}a+\ldots+{}_m\text{C}_1na^{m-1}-a^m\}$$
$$\equiv n\Sigma_m\text{C}_1a^{m-1}\equiv nm\text{S}_{m-1};$$

and $\quad \text{S}_{m-1}\equiv 0 \pmod{n}$, unless $m-1$ is a multiple of $n-1$;

$\therefore \quad \text{S}_m\equiv 0 \pmod{n^2}$, unless $m-1$ is a multiple of $n-1$;

and the theorem is true even in this last case if m is a multiple of n.

https://doi.org/10.1017/S0013091500030467 Published online by Cambridge University Press

Now consider

$$S_{-m} = \{1 + 1/2^m + 1/3^m + \ldots + 1/(n-1)^m\}(\overline{n-1}!)^m.$$

We have

$$\begin{aligned} 2\,S_{-m} &= (\overline{n-1}!)^m \Sigma\{1/r^m + 1/(n-r)^m\} \\ &= (\overline{n-1}!)^m \Sigma \frac{(n-r)^m + r^m}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{n^m - {}_mC_1 n^{m-1} r \ldots + {}_mC_1 n r^{m-1}}{r^m(n-r)^m} \\ &= (\overline{n-1}!)^m \Sigma \frac{Pn^2 + {}_mC_1 n}{r(n-r)^m}. \end{aligned}$$

We have thus to show that ${}_mC_1(\overline{n-1}!)^m\Sigma\{1/r(n-r)^m\}$ is divisible by n. We shall suppose, for the sake of clearness, that m is less than n; but the following method will be applicable, even if m be greater than n.

Assume $\quad (\overline{n-1}!)^m/\{(n-r)r^m\} \equiv a_r \pmod{n}$

$\therefore \quad (\overline{n-1}!)^m \equiv a_r(n-r)r^m$

$\therefore \quad a_r(n-r)r^m \equiv -1 \quad$ (by Wilson's theorem).

Now since $\quad (n-r)^{n-1} \equiv r^{n-1} \equiv 1$, we get

$\quad a_r \equiv -r^{n-m-1}(n-r)^{n-2}$

$\therefore \quad a_r \equiv r^{n-m-1}r^{n-2} \pmod{n}$

$\quad \equiv r^{n-m-2}$

Hence $\quad \Sigma a_r \equiv \Sigma r^{n-m-2}$

$\quad \equiv 0$, if $n-m-2 \neq 0$.

It follows that S_{-m} is divisible by n_2, m being subject to the restriction $n-m-2$ be not zero. If we remove the condition that m is to be less than n, we shall easily find that the general restriction as to the value of m, is that $m+1$ must not be a multiple of $n-1$.

In the paper referred to before in a footnote, Mr Leudesdorf considers the case where n is not prime and S_m denotes the sum of the m^{th} powers of the numbers less than n and prime to it. His method however cannot be considered rigorous, as it involves the use of divergent series.

Note on normals to conics.

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1. The following condition may be new; it does not appear in any of the books:—

The condition that the straight line

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