4

The supersymmetry algebra

4.1 Rotations

In classical mechanics, rotations of three-vectors can be represented by a rotation matrix $\mathbf{R}$ acting upon vectors such as $\mathbf{x} = (x, y, z)$ as

$$x_i \to x_i' = \mathbf{R}_{ij} x_j. \tag{4.1}$$

In quantum mechanics, rotation transformations are represented by unitary operators $U(\boldsymbol{\theta})$ acting upon state vectors $|\psi\rangle$ such that

$$|\psi\rangle \to |\psi'\rangle = U(\boldsymbol{\theta})|\psi\rangle, \tag{4.2}$$

where the direction of $\boldsymbol{\theta}$ is along the axis about which the rotation occurs, and its magnitude is the rotation angle. For infinitesimal rotations, the operator $U(\boldsymbol{\theta})$ can be written as $U(\boldsymbol{\theta}) \simeq 1 + \mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J}$, where the Hermitian operators $\mathbf{J}$ are the rotation generators. For spinless states, J_i can be represented as differential operators ($J_k = \frac{1}{2}\epsilon_{ijk} J_{ij}$, with $J_{ij} = -\mathrm{i}(x_i \partial_j - x_j \partial_i)$), it is easy to check that the commutation relations

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k \tag{4.3}$$

are satisfied for $i, j = 1, 2, 3 \leftrightarrow x, y, z$. Finite rotations can be built up from an infinite product of infinitesimal ones, so that the operator $U(\boldsymbol{\theta}) = \exp(\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J})$. The operators $U(\boldsymbol{\theta})$ form a representation of the Lie group $SU(2)$, for which the J_i are the group generators, and where Eq. (4.3) defines the Lie algebra associated with the group $SU(2)$. Since the parameters θ_i each run over a compact domain 0 to 2π, we say that $SU(2)$ is a compact Lie group.

A Casimir operator is an operator that commutes with all of the group generators. The eigenvalues of a Casimir operator are unchanged under group transformations, so they serve as a useful tool to classify group representations. The representations of $SU(2)$ can be labelled according to the eigenvalues of the quadratic Casimir operator

https://doi.org/10.1017/9781009289801.005 Published online by Cambridge University Press

$J^2 = \mathbf{J} \cdot \mathbf{J}$, for which $J^2|jm\rangle = j(j+1)|jm\rangle$, with $j = 0, 1/2, 1, 3/2, 2, \ldots$ For the $j = 1/2$ representation, the operators J_i can be represented by the Pauli spin matrices $J_i = \sigma_i/2$, and the state vectors can be represented by 2-component spinors. Higher j representations can be constructed by taking direct products of lower j representations. For higher j representations of $SU(2)$, the J_i's can be represented by $(2j+1) \times (2j+1)$ matrices, and the corresponding state vectors by $2j+1$ component column matrices.

4.2 The Lorentz group

We want to build a quantum theory that is invariant under Lorentz transformations. We restrict our discussion to proper, orthochronous Lorentz transformations, i.e. boosts and rotations, and neglect parity and time reversal. In addition to rotations which mix the spatial coordinates amongst themselves, we now have boost transformations which mix the time co-ordinate x^0 with the spatial co-ordinates; e.g. a boost along the x^1 direction can be written as $x'^0 = x^0 \cosh\phi + x^1 \sinh\phi$, $x'^1 = x^0 \sinh\phi + x^1 \cosh\phi$, $x'^2 = x^2$ and $x'^3 = x^3$. The usual velocity parameter β that characterizes the boost is given in terms of the rapidity ϕ by $\beta = \tanh\phi$. The infinitesimal transformation matrix U that transforms quantum mechanical states can then be augmented to,

$$U(\boldsymbol{\theta}, \boldsymbol{\phi}) \simeq 1 + \mathrm{i}\boldsymbol{\theta} \cdot \mathbf{J} + \mathrm{i}\boldsymbol{\phi} \cdot \mathbf{K}, \tag{4.4}$$

where K_i are the boost generators and $\boldsymbol{\phi}$ points along the direction of the boost.[1] A Lorentz transformation is thus characterized by the six parameters (θ_i, ϕ_j). Since the parameters ϕ_j are not restricted to a compact interval, the Lorentz group, unlike the rotation group, is not compact.

The Lorentz group generators satisfy

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k, \quad [K_i, J_j] = \mathrm{i}\epsilon_{ijk} K_k, \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk} J_k. \tag{4.5}$$

The first of these relations shows that rotation generators form a closed sub-algebra, so that the rotation group forms a subgroup of the Lorentz group. The commutator of two boost generators is a rotation generator (this is the origin of Thomas precession) so that the boosts, by themselves, do not form a sub-algebra.

The Lorentz algebra that we have introduced above can be written in a manifestly covariant form by writing the generators as the six components of an antisymmetric second rank tensor generator $M_{\mu\nu}$, with $M_{ij} = \epsilon_{ijk} J_k$ and $M_{0i} = -M_{i0} = -K_i$. The commutators for the Lorentz group generators can then be recast into covariant

[1] For infinitesimal boosts, notice that $|\boldsymbol{\phi}| = |\boldsymbol{\beta}|$.

form

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma} + g_{\nu\sigma}M_{\mu\rho}). \tag{4.6}$$

To find the finite-dimensional unitary representations of the Lorentz group, the generators can be alternatively written by defining $S_i = \frac{1}{2}(J_i + \mathrm{i}K_i)$ and $T_i = \frac{1}{2}(J_i - \mathrm{i}K_i)$. In this case, it is easy to check that the commutators of the generators become

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k, \quad [T_i, T_j] = \mathrm{i}\epsilon_{ijk}T_k, \quad [S_i, T_j] = 0, \tag{4.7}$$

i.e. the algebra decomposes into the product of two independent $SU(2)$ groups, for which we know the representations. For the Lorentz group, there are thus *two* Casimir operators, S^2 and T^2, with eigenvalues $s(s+1)$ and $t(t+1)$, again with $s, t = 0, 1/2, 1, \ldots$ (Note that J^2 is no longer a Casimir operator since it no longer commutes with all the group generators, e.g. $[J^2, K_1] \neq 0$.) The irreducible representations can be categorized according to values of (s, t). A Lorentz scalar transforms as the $(0, 0)$ representation while a four-vector transforms as $(1/2, 1/2)$ representation. There are *two* distinct fundamental representations $(1/2, 0)$ and $(0, 1/2)$, each of which corresponds to two-spinors. The $(1/2, 0)$ object, as we will soon see, transforms as a left-handed Weyl two-spinor whose components are usually denoted by $\psi_{\mathrm{L}A}$, with $A = 1, 2$. The $(0, 1/2)$ object transforms as a right-handed two-spinor with components $\psi_{\mathrm{R}}^{\dot{A}}$, where the dot on the index calls attention to the fact that the spinor transforms under the second of the two $SU(2)$ groups.

Exercise *Verify that the boost transformation for a state transforming as the* $(1/2, 0)$ *and* $(0, 1/2)$ *representations of the Lorentz group are respectively given by* $\psi'_{\mathrm{L,R}} = (\cosh\frac{\phi}{2} \mp \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\sinh\frac{\phi}{2})\psi_{\mathrm{L,R}}$, *where* $\hat{\mathbf{p}}$ *is a unit vector along the direction of* $\mathbf{p}$. *Recalling that* $\tanh\phi = \beta$, *show that the spinors for states with momentum* $\mathbf{p}$ *can be obtained from the corresponding rest frame states as,*

$$\psi_{\mathrm{L,R}}(\mathbf{p}) = \frac{E + m \mp \boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{2m(E+m)}}\psi_{\mathrm{L,R}}(0).$$

Noting that $\psi_{\mathrm{L}}(0) = \psi_{\mathrm{R}}(0)$ *because there is no preferred direction in the rest frame to define the particle's handedness, show that*

$$(E \pm \boldsymbol{\sigma}\cdot\mathbf{p})\psi_{\mathrm{L,R}}(\mathbf{p}) = m\psi_{R,L}(\mathbf{p}).$$

This is just the Dirac equation in two-component notation. Notice that for $m = 0$, *we have* $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\psi_{\mathrm{L,R}}(\mathbf{p}) = \mp\psi_{\mathrm{L,R}}(\mathbf{p})$, *which justifies the use of our labels left and right for the states transforming as the* $(1/2, 0)$ *and* $(0, 1/2)$ *representations of the Lorentz group.*

---

https://doi.org/10.1017/9781009289801.005 Published online by Cambridge University Press

A four-component Dirac spinor can be built out of the direct sum of two-component spinors $(1/2, 0) \oplus (0, 1/2)$ so that

$$\psi_a^D = \begin{pmatrix} \psi_{\mathrm{L}A} \\ \chi_{\mathrm{R}}^{\dot{A}} \end{pmatrix}. \tag{4.8}$$

The two spinors ψ_{L} and χ_{R} are independent. It is simple to check that the two-component spinor $-\mathrm{i}\sigma_2\psi_{\mathrm{L}}^*$ transforms as a $(0, 1/2)$ representation of the Lorentz group, i.e. it transforms as χ_{R}. We can thus construct a different four-component spinor whose right-handed piece is completely determined by its left-handed pieces via $\chi_{\mathrm{R}} = -\mathrm{i}\sigma_2\psi_{\mathrm{L}}^*$. This four-spinor would transform as the $(1/2, 0) \oplus (0, 1/2)$ representation of the Lorentz group, but would have just half as many independent components as ψ^D above. It can be expressed as

$$\psi_a = \begin{pmatrix} \psi_{\mathrm{L}A} \\ (-\mathrm{i}\sigma_2\psi_{\mathrm{L}}^*)^{\dot{A}} \end{pmatrix}. \tag{4.9}$$

This object is the Majorana spinor that we have already encountered in Chapter 3, and the relation $\chi_{\mathrm{R}} = -\mathrm{i}\sigma_2\psi_{\mathrm{L}}^*$ is simply (3.3).[2] Since the Dirac spinor contains twice as many independent components as a Majorana spinor, it can be thought of as a combination of two Majorana spinors, in much the same way that we can think of a complex number as a combination of two real numbers.

Many textbooks and review articles use the more fundamental two-component spinor notation. Here, we formulate everything in terms of four-component spinors, which are perhaps more familiar to particle physicists interested in performing phenomenological calculations.

4.3 The Poincaré group

In addition to rotations and boosts, the other spacetime transformations include translations in space and time. Translations in space and time are generated by the energy–momentum operator P_μ, which can be represented by the differential operator $P_\mu = \mathrm{i}\partial_\mu$. The Poincaré group is formed by combining rotations, boosts, and translations. We then have *ten* independent generators: the six $M_{\mu\nu}$ plus the four P_μ. It is then straightforward to work out the commutation relations for the

[2] In Chapter 3 and elsewhere, $\psi_{\mathrm{L,R}}$ is also used to denote the four-component spinor $P_{\mathrm{L,R}}\psi$ which has only two non-vanishing components in the representation where the matrix γ_5 is diagonal. These non-vanishing components are just the components of the two-spinor $\psi_{\mathrm{L,R}}$ discussed in this chapter. Although this is an abuse of notation, it should be clear from the context whether we are using $\psi_{\mathrm{L,R}}$ to denote four- or two-component spinors.

https://doi.org/10.1017/9781009289801.005 Published online by Cambridge University Press

generators, using their representation as differential operators. One finds:

$$[P_\mu, P_\nu] = 0, \tag{4.10a}$$

$$[M_{\mu\nu}, P_\lambda] = \mathrm{i}(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu), \tag{4.10b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma} + g_{\nu\sigma} M_{\mu\rho}). \tag{4.10c}$$

To classify the representations of the Poincaré group, we again look for Casimir operators. One of these is the operator P^2, which certainly commutes with all the group generators. Its eigenvalue operating on particle state vectors is just the squared mass $P^2|\psi\rangle = m^2|\psi\rangle$. The other Casimir operator is obtained from the Pauli–Lubanski four-vector $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$, with $W^\mu P_\mu = 0$. The square of the Pauli–Lubanski vector, W^2, can be shown to commute with all the generators of the Poincaré group. Notice also that in the rest frame (of a massive state) W^i is proportional to the rotation generator J^i. The various representations of the Poincaré group were first worked out by Wigner. The physically realized unitary representations which are of interest to us are:

- $P^2 \equiv m^2 > 0$, with $W^2 = -m^2 s(s+1)$, where s denotes the spin quantum number $s = 0, \frac{1}{2}, 1, \ldots$ Thus, these states correspond to particles of definite mass and discrete spin values.
- $P^2 = 0$, $W^2 = 0$ so that $W_\mu = \lambda P_\mu$. Here λ is the state helicity value, and $\lambda = \pm s$, for $s = 0, \frac{1}{2}, 1, \ldots$ These correspond e.g. to single particle states of massless particles such as photons with $\lambda = \pm 1$ or the graviton with $\lambda = \pm 2$.
- Finally, $P^\mu \equiv 0$, corresponding to the vacuum state which is invariant under Poincaré transformations.

In the 1960s, a number of papers were written about the possibility of embedding the spacetime symmetries (i.e. the Poincaré group) into some larger *master* group such as $SU(6)$ that would serve as a more general framework for the symmetry of the laws of physics. These efforts culminated in several no-go theorems, the most general of which was the Coleman–Mandula theorem.[3] It states the following.

Theorem Let G be a connected symmetry group of the S matrix (its generators commute with the S matrix), and assume the following.

- G contains a subgroup which is locally isomorphic to the Poincaré group (Poincaré invariance).
- All particle types correspond to positive energy representations of the Poincaré group. For any finite mass m, there are only a finite number of types of particles with mass less than m.

[3] S. Coleman and J. Mandula, *Phys. Rev.* **159**, 1251 (1967).

- Elastic scattering amplitudes are analytic functions of the Mandelstam variables s and t in some neighborhood of the physical region, except at normal thresholds, and the S matrix is non-trivial in the sense that essentially any two one-particle momentum states scatter, except perhaps for isolated values of s.
- Finally, a technical assumption: the generators of G, considered as integral operators in momentum space, have distributions as their kernels.

Coleman and Mandula asserted that if these conditions hold, *G is locally isomorphic to the direct product of a compact symmetry group and the Poincaré group.*

Stated more simply, under a number of physically reasonable assumptions, it is not possible to form a non-trivial merger of the Poincaré symmetry with other symmetries of the S matrix into a bigger group. It is not possible to have a larger spacetime symmetry and, further, internal symmetries such as local gauge symmetries or additional global symmetries (e.g. isospin) can only be realized as a direct product of these symmetry groups with the Poincaré group. It is intriguing that all the Poincaré group symmetries of the S matrix are in fact realized in nature. It is important to recognize that Coleman and Mandula did not envisage the possibility of anticommuting spinorial charges in their analysis. It is precisely the inclusion of these that allows us to enlarge the spacetime symmetry group to include supersymmetry, as we have already seen in Chapter 3.

4.4 The supersymmetry algebra

Our investigation of the Wess–Zumino model in Chapter 3 shows that it is possible to construct a relativistic quantum field theory that is invariant under supersymmetry transformations, for which the generators are anticommuting spinorial charges Q_a. We saw that the algebra of the Q_a's (this involved anticommutators, which is how the Coleman–Mandula theorem is circumvented) closes to yield P_μ, so that the supersymmetry is, in effect, a spacetime symmetry. In this sense, supersymmetry can be looked upon as a generalization of the special theory of relativity.

We had already worked out the algebra of the spinorial generators Q_a amongst themselves and with the translation generators in Chapter 3. Since this involves anticommutators of the super-charges, it is called a graded Lie algebra. The commutator of the Lorentz generators with the super-charges Q_a is simply given by the fact that these are spin $\frac{1}{2}$ objects. We can thus write the supersymmetric extension of the Poincaré algebra, known as the super-Poincaré algebra, as

$$[P_\mu, P_\nu] = 0, \tag{4.11a}$$

$$[M_{\mu\nu}, P_\lambda] = \mathrm{i}(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu), \tag{4.11b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho} M_{\nu\sigma} - g_{\mu\sigma} M_{\nu\rho} - g_{\nu\rho} M_{\mu\sigma} + g_{\nu\sigma} M_{\mu\rho}), \tag{4.11c}$$

$$[P_\mu, Q_a] = 0, \tag{4.11d}$$

$$[M_{\mu\nu}, Q_a] = -(\tfrac{1}{2}\sigma_{\mu\nu})_{ab} Q_b, \tag{4.11e}$$
$$\{Q_a, \bar{Q}_b\} = 2(\gamma^\mu)_{ab} P_\mu. \tag{4.11f}$$

Since Q is a Majorana spinor charge, we can use the last of these relations to work out the anticommutators between two Q's or $\bar{Q}$'s.

Exercise: *Verify that*

$$\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu,$$
$$\{\bar{Q}_a, \bar{Q}_b\} = 2(C^{-1}\gamma^\mu)_{ab} P_\mu.$$

An extension of the Coleman–Mandula type analysis that allows for spinorial charges was worked out by Haag, Lopuszanski, and Sohnius who showed that the super-Poincaré algebra above is indeed the most general extension of the Poincaré algebra, provided we have just a single spinorial charge Q.[4] These authors also showed that theories with more than one spinorial generator are possible. These are referred to as extended supersymmetry theories. Such theories do not allow chiral representations which, as we know, are crucial for phenomenology. Only theories with a single spinorial generator Q_a, known as $N = 1$ supersymmetry theories, allow chiral representations. For this reason, we restrict our attention only to $N = 1$ supersymmetry.[5] To sum up, the Haag–Lopuszanski–Sohnius theorem tells us that the most general symmetry of the S matrix is the direct product of some internal symmetry with super-Poincaré invariance.

The irreducible representations of the super-algebra can be worked out as usual by finding the relevant Casimir operators. For the SUSY algebra above, the operator P^2 again commutes with all generators, so that all particles occurring in a supermultiplet will have the same mass. However, the square of the Pauli–Lubanski pseudovector W^2 is no longer a Casimir invariant, so that supermultiplets can now contain particles of differing spins. We will not discuss the construction of a new Casimir operator for this case, but instead focus on the particle supermultiplets that furnish representations of the super-Poincaré algebra.

For the massive case $P^2 \equiv m^2 > 0$, the representations are labeled by (m, j) with $j = 0, 1/2, 1, \ldots$ For fixed m, the complete supermultiplet contains a state each corresponding to spin $s = j \pm 1/2$, and two states with spin $s = j$ (except for the case $j = 0$ where the state with $s = j - 1/2$ is absent), all of which have the same mass. Notice that the number of helicity states for the two objects with spin j $[2(2j + 1)]$ is exactly balanced by the corresponding number of helicity

[4] R. Haag, J. Lopuszanski and M. Sohnius, *Nucl. Phys.* **B88**, 257 (1975).

[5] The underlying fundamental theory could be an extended supersymmetric theory, but all but one (or none!) of the supersymmetries must somehow be broken at much higher scales, leaving an $N = 1$ supersymmetric theory as the extension of the Standard Model that could have phenomenological relevance.

states for the two states with spins $j \pm \frac{1}{2}$ $[2(j + 1/2) + 1 + 2(j - 1/2) + 1]$. This is just the statement that the number of bosonic and fermionic helicity states are the same. If $j = 0$, we have the multiplet of the Wess–Zumino model – two spin zero states and two spin half states – as we discussed in Chapter 3. If $j = 1/2$, there are four bosonic degrees of freedom (three spin 1 and one spin zero) balanced by four fermionic degrees of freedom corresponding to two Majorana spin half fermions as we will see when we study spontaneously broken gauge theories.

For the massless case, one can show that if j is the state with largest helicity in a supermultiplet, it is always accompanied by another state with helicity $j - 1/2$. Furthermore, a Lorentz invariant field theory always contains these states together with their CPT conjugates which have opposite helicities, $-j$ and $-j + 1/2$ and which are also massless. These states constitute a complete massless supermultiplet. If $j = 1/2$, this multiplet consists of two fermionic states with helicities $\pm 1/2$, and a pair of spin zero bosonic states. This multiplet occurs in the massless Wess–Zumino model. Such a multiplet would also describe a massless neutrino and antineutrino together with its supersymmetric partners which would be two spin zero states (which can be regarded as quanta of one massless *complex* scalar field). For $j = 1$, the bosonic states would correspond to a massless gauge boson (helicities ± 1); the fermionic partner states, which have helicity $\pm 1/2$, then correspond to a Majorana fermion referred to as a *gaugino*. This "gauge multiplet" is a crucial ingredient of supersymmetric gauge theories, which form the basis of all supersymmetry phenomenology. Finally, if $j = 2$, we see that the two bosonic states have helicities ± 2. These are thought to correspond to a graviton, the massless spin two quantum that mediates gravity. The fermionic partners of these $j = 2$ states have helicities $\pm 3/2$ and describe a spin $\frac{3}{2}$ massless Majorana fermion referred to as a *gravitino*. This "gravity multiplet" is essential for theories in which the parameter α of SUSY transformations depends on x^μ. These locally supersymmetric theories necessarily involve Einsteinian gravity, and are referred to as *supergravity* theories.