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A NOTE ON STOCHASTIC BOUNDS FOR QUEUEING NETWORKS

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Abstract

Recently, Massey [1] proved that the vector of queue lengths of some queueing networks is stochastically dominated at any given time by that of a corresponding system of parallel $M/M/1$ queues. This result is interesting, even though the bounds are generally quite conservative, in that the transient behavior of independent parallel $M/M/1$ queues is considerably easier to analyze than that of a network.

This note provides an alternative proof of a generalized form of that result.

1. Notation and basic lemma

For $a := (a^1, \cdots, a^d)$ and $b \in R^d$ $(d \geqq 1)$, $a \geqq b$ will indicate that $a^i \geqq b^i$ for $i = 1, \cdots, d$. Let X, Y be two R^d-valued random variables. One writes

$$X \underset{s}{\geqq} Y \quad \text{if} \quad \Pr\{X \geqq a\} \geqq \Pr\{Y \geqq a\}, \quad \text{for all} \quad a \in R^d,$$

$$X \underset{s}{=} Y \quad \text{if} \quad X \underset{s}{\geqq} Y \quad \text{and} \quad Y \underset{s}{\geqq} X.$$

(Thus $X \underset{s}{\geqq} Y$ indicates that X and Y have the same probability distribution function.)

For $x, y \in R$, $x \wedge y := \min\{x, y\}$ and $x^+ = \max\{x, 0\}$. Let $Z_+ = \{0, 1, 2, \cdots\}$.

Lemma. Let X, Y, M, N be Z_+^2-valued random variables such that $X \underset{s}{\geqq} Y$, $\{X, Y\}$ and M are independent, and given Y,

$$N = (N^1, N^2) \underset{s}{=} (N^2, N^1), \quad \text{and} \quad (-M^1, M^2) \underset{s}{\geqq} (-N^1, N^2).$$

Then

$$((X^{-1} - M^1)^+, X^2 + M^2) \underset{s}{\geqq} W = ((Y^1 - N^1)^+, Y^2 + N^1 \wedge Y^1).$$

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Proof. By conditioning on M one finds that

$$((X^1-M^1)^+, X^2+M^2) \underset{s}{\geqq} ((Y^1-M^1)^+, Y^2+M^2) =: V.$$

To show $V \underset{s}{\geqq} W$ one must prove that for all $m^1, m^2 \in Z_+$,

(1) $$\Pr\{V^1 \geqq m^1, V^2 \geqq m^2\} \geqq \Pr\{W^1 \geqq m^1, W^2 \geqq m^2\}.$$

For $m^1 = 0$, (1) reads $V^2 \underset{s}{\geqq} W^2$ and follows from

$$Y^2 + M^2 \underset{s}{\geqq} Y^2 + N^2 \underset{s}{=} Y^2 + N^1 \underset{s}{\geqq} Y^2 + N^1 \wedge Y^1.$$

For $m^1 > 0$, (1) is implied by the following inequality:

(2) $$(Y^1 - M^1, Y^2 + M^2) \underset{s}{\geqq} (Y^1 - N^1, Y^2 + N^1).$$

To see why (2) holds, observe that

$$(Y^1 - M^1, Y^2 + M^2) \underset{s}{\geqq} (Y^1 - N^1, Y^2 + N^2),$$

so that it suffices to show that

$$(Y^1 - N^1, Y^2 + N^2) \underset{s}{\geqq} (Y^1 - N^1, Y^2 + N^1).$$

By conditioning on Y it remains only to prove that for all $m^1, m^2 \in Z_+$

$$\Pr\{N^1 \leqq m^1, N^2 \geqq m^2\} \geqq \Pr\{N^1 \leqq m^1, N^1 \geqq m^2\}.$$

But this last inequality is immediate from $(N^1, N^2) \underset{s}{=} (N^2, N^1)$.

2. Stochastic networks

Let $\{x_t, y_t, t \geqq 0\}$ be two Z_+^d-valued processes corresponding to the vectors of queue lengths in two networks of d queues. Alternatively, each $i \in \{1, \cdots, d\}$ could be some $(k, c) \in \{1, \cdots, K\} \times \{1, \cdots, C\}$ where c would indicate a customer class and k a node number. Other variations are possible.

Assume that those processes admit the following representation. For $t \geqq 0$ and $i \in \{1, \cdots, d\}$,

$$dx_t^i = -\sum_{j=0}^{d} 1\{x_{t-}^i > 0\}\, dS_t^{ij} + \sum_{j=0}^{d} dA_t^{ji}$$
$$dy_t^i = -\sum_{j=0}^{d} 1\{y_{t-}^i > 0\}\, dR_t^{ij} + \sum_{j=0}^{d} 1\{y_{t-}^j > 0\}\, dR_t^{ij}.$$

In these expressions, the differential equations are in the Lebesgue–Stieltjes sense, $y_t^0 \equiv 1$, and the processes R^{ij}, A^{ij}, S^{ij} are point processes with the following properties. The processes S_{ij}, A^{ij}, $\tilde{A}^{ij}$ (see below) are Poisson processes; $\{S^{ij}, \tilde{A}^{ij}\}$, A^{ij} for $0 \leqq i, j \leqq d$ are independent; for every (i, j), A^{ij} and $\tilde{A}^{ij}$ have the same rate; almost surely one has

$$\Delta S_t^{ij} := S_t^{ij} - S_{t-}^{ij} \leqq \Delta R_t^{ij} \leqq \Delta \tilde{A}_t^{ij} \quad \text{for} \quad t \geqq 0, 0 \leqq i, j \leqq d.$$

Thus x_t corresponds to a system of parallel $M/M/1$ queues while y_t corresponds to a network of interconnected queues and need not be Markov. The routing and the service rates in y_t may depend on the 'state' of the complete network; the basic assumption is that the service rate from queue i to queue j is bounded from above and from below when the queue is not empty.

The following proposition extends a result of Massey [1].

Proposition. Assume that $x_0 \underset{s}{\geqq} y_0$. Then $X_T \underset{s}{\geqq} y_t$ for all $t \geqq 0$.

Proof. Let $T_1, T_2, \cdots$ be the successive jump times of $\sum_{ij} (S_t^{ij} + A_t^{ij} + \tilde{A}_t^{ij})$ and define $X_n = x_{T_n}$, $Y_n = y_{T_n}$ for $n \geqq 1$.

It suffices to show that given $\{T_m, m \geqq 1\}$, $X_n \underset{s}{\geqq} Y_n$. Assume that this is true for n.

Notice that given

$$\Psi := \{X_n, Y_n, T_m, m \geqq 1, \Delta A^{ij}_{T_{n+1}} + \Delta \tilde{A}^{ij}_{T_{n+1}} = 1\},$$

the following random variables

$$N^1 = \Delta R^{ij}_{T_{n+1}}, M^1 = \Delta S^{ij}_{T_{n+1}}, M^2 = \Delta A^{ij}_{T_{n+1}},$$
$$X^1 = X^i_n, X^2 = X^j_n, Y^1 = Y^i_n, Y^2 = Y^j_n$$

satisfy the conditions of the lemma with N^2 being an i.i.d. copy of N^1 (given Ψ). Hence, given Ψ, $X_{n+1} \underset{s}{\geqq} Y_{n+1}$, and this concludes the proof.

For instance, if network Y consists of d queues with exogenous arrival rates λ_i, service rates in $[a_i, b_i]$ in queue i when it is non-empty, and routing probabilities r_{ij}, then y_t is stochastically dominated by the vector of queue lengths of d parallel queues with arrival rates $\lambda_i + \sum_j b_j r_{ji}$ and service rates a_i. The proposition shows that this result holds in a more general context.

Remarks.

(1) The result extends to deterministic, and therefore to arbitrary arrivals, by applying the argument to the processes between arrival times.

(2) The idea of considering the Markov chain $\{Y_n\}$ shows that if a network of $M/M/s$ queues and arbitrary arrivals is started with stochastically more customers, then that ordering is preserved at all times.

(3) It can be shown that it is not possible to construct x_t and y_t on the same probability space in such a way that $\Pr\{x_t \geqq y_t, \text{ for all } t \geqq 0\} = 1$. That is, the domination is not pathwise. (See [2].)

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References

[1] MASSEY, W. (1984) Open networks of queues: their algebraic structure and estimating their transient behavior. *Adv. Appl. Prob.* **16,** 176–201.

[2] MASSEY, W. (1984) An operator-analytic approach to the Jackson network. *J. Appl. Prob.* **21,** 379–393.