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Averaging Operators and Martingale Inequalities in Rearrangement Invariant Function Spaces

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Abstract. We shall study some connection between averaging operators and martingale inequalities in rearrangement invariant function spaces. In Section 2 the equivalence between Shimogaki's theorem and some martingale inequalities will be established, and in Section 3 the equivalence between Boyd's theorem and martingale inequalities with change of probability measure will be established.

1 Introduction and Notation

Recently several authors studied, independently in [1], [11], [15], some martingale inequalities in rearrangement invariant function spaces over the unit interval I. Using the Boyd indices they characterized rearrangement invariant function spaces in which the Burkholder-Davis-Gundy inequality is valid. In Section 2, we shall prove the equivalence between their result and Shimogaki's theorem on the boundedness of averaging operator. In Section 3 we shall consider some change of probability measure and extend the weighted norm inequalities established by Izumisawa and Kazamaki [10]. We shall investigate also some relations between Boyd's theorem and martingale inequalities under a change of probability measure.

In this note we shall deal with (local) martingales on complete probability spaces, say $(\Omega, \mathcal{F}, P)$, endowed with a filtration satisfying the *usual conditions* (see [7, p. 183]). We always assume that Ω is not completely atomic, that is, $P(\Omega \setminus \Omega_0) > 0$, where Ω_0 is the union of all atoms in Ω. Furthermore, in Section 3 we assume that Ω contains no atom. Every process $X = (X_t)_{t\geq 0}$ is assumed to be adapted to a given filtration, right continuous, and have left-hand limits. We set $X_{0-} = 0$ and denote by $(\mathcal{M}X_t)_{t\geq 0}$ the maximal process of X; $\mathcal{M}X_t = \sup_{s\leq t} |X_s|$. We use this notation instead of X^* in order to reserve "$*$" for the decreasing rearrangement of random variables. If X is a (local) martingale, $([X, X]_t)_{t\geq 0}$ denotes the quadratic variation process of X. For details of the martingale theory we refer to Dellacherie and Meyer [8].

Now let f be a random variable on $(\Omega, \mathcal{F}, P)$. The *decreasing rearrangement* of f, denoted by f^*, is a right continuous decreasing function on the interval $I = [0, 1]$ such that

$$P\{|f| > \lambda\} = m\{s \in I : f^*(s) > \lambda\}, \quad \lambda > 0,$$

where m stands for the Lebesgue measure on I. An explicit expression of f^* is given by

$$f^*(t) = \inf\{\lambda > 0 : P\{|f| > \lambda\} \leq t\}, \quad t \in I.$$

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For example, if X is a process and T is a stopping time, X_T^* denotes the decreasing rearrangement of the random variable X_T, but not $\mathcal{M}X_T = \sup_{t\le T} |X_t|$. For a function x on I, x^* denotes the decreasing rearrangement with respect to the Lebesgue measure.

For two random variables f and g, we write $f \prec g$ if

$$\int_0^t f^*(s)\,ds \le \int_0^t g^*(s)\,ds \quad \text{for all } t \in I.$$

A Banach space $(B, \|\cdot\|_B)$ consisting of (equivalence classes) of random variables is called a *rearrangement invariant (r.i.) function space* if it has the following properties:

(i) $L^\infty \hookrightarrow B \hookrightarrow L^1$;
(ii) $|f| \le |g|$, $g \in B$ implies $f \in B$ and $\|f\|_B \le \|g\|_B$;
(iii) $0 \le f_n \uparrow f$, $\sup_n \|f_n\|_B < \infty$ implies $f \in B$ and $\|f\|_B = \sup_n \|f_n\|_B$;
(iv) $f^* = g^*$, $g \in B$ implies $f \in B$ and $\|f\|_B = \|g\|_B$.

If B has properties (i)–(iii) and

(iv') $f \prec g$, $g \in B$ implies $f \in B$ and $\|f\|_B \le \|g\|_B$,

then it is called a *universally rearrangement invariant (u.r.i.) function space*. Obviously every u.r.i. function space is r.i., and the converse is true if Ω contains no atom. An important characterization of r.i. function space is the Luxemburg representation theorem: B is u.r.i. if and only if there exists a r.i. function space $\hat{B}$ over I with the Lebesgue measure m such that

$$\|f\|_B = \|f^*\|_{\hat{B}} \quad \text{for all } f \in B \tag{1}$$

(*cf.* [13, p. 121], [2, p. 90]). Our assumption $t_0 := P(\Omega \setminus \Omega_0) > 0$ implies that if both $\hat{B}_1$ and $\hat{B}_2$ satisfy (1), then the norms of these spaces are equivalent. Indeed, if $x \in \hat{B}_2$, then there exists $f \in B$ such that $f^*(t) = x^*(t \wedge t_0) =: x^*_{t_0}(t)$, $t \in I$ and hence we have $\|x\|_{\hat{B}_1} \le \|x^*_{t_0}\|_{\hat{B}_1} = \|x^*_{t_0}\|_{\hat{B}_2} < \infty$. Thus $\hat{B}_2 \subset \hat{B}_1$, and in the same way $\hat{B}_1 \subset \hat{B}_2$. The equivalence of norms of $\hat{B}_i$ follows from the closed graph theorem.

Now let B be a r.i. (or u.r.i.) function space. The *associate space* B' of B is the r.i. (u.r.t.) function space defined by

$$\|f\|_{B'} := \sup\{E[fg] : g \in B, \|g\|_B \le 1\};$$
$$B' := \{f : \|f\|_{B'} < \infty\}.$$

The associate space of B' is equal to B (*cf.* [2, p. 10]).

To describe our results, we use the Boyd indices of B introduced by Boyd [5]: let D_s be the operator defined on $L^1(I)$ by

$$D_s x(t) = \begin{cases} x(st) & \text{if } 0 \le t \le 1 \wedge s^{-1}, \\ 0 & \text{if } 1 \wedge s^{-1} < t \le 1, \end{cases}$$

and set

$$\underline{\alpha}_B = \underline{\alpha}_{\hat{B}} = \inf_{t>1} \frac{\log \|D_{1/t}\|_{\hat{B}}}{\log t} = \lim_{t\to\infty} \frac{\log \|D_{1/t}\|_{\hat{B}}}{\log t};$$
$$\bar{\alpha}_B = \bar{\alpha}_{\hat{B}} = \sup_{0<t<1} \frac{\log \|D_{1/t}\|_{\hat{B}}}{\log t} = \lim_{t\downarrow 0} \frac{\log \|D_{1/t}\|_{\hat{B}}}{\log t},$$

where $\|D_s\|_{\hat{B}}$ denotes the norm of D_s as an operator from $\hat{B}$ into itself. We call $\underline{\alpha}_B$ and $\bar{\alpha}_B$ the *upper* and *lower Boyd index* of B, respectively. Remark that in [1] and [11] the Boyd indices are taken to be reciprocals of ones we use here.

2 Averaging Operator and Martingale Inequalities

Throughout this section let $(\Omega, \mathcal{F}, P)$ be a fixed probability space and I be the interval $[0, 1]$ with the Lebesgue measure m. For a function x on I, Hardy's averaging operator is defined by

$$\mathcal{P}x(t) = \frac{1}{t}\int_0^t x(s)\, ds,$$

and its adjoint $\mathcal{P}'$ is given by

$$\mathcal{P}'x(t) = \int_t^1 \frac{x(s)}{s}\, ds,$$

whenever the defining integrals exist a.e. Shimogaki [17] studied the boundedness of these operators in r.i. function spaces over I. His result, in terms of Boyd indices, is as follows:

Theorem A **(Shimogaki [17]; Boyd [5])** *Let $\hat{B}$ be a r.i. function space over I. Then:*

(i) *$\mathcal{P}$ is a bounded linear operator on $\hat{B}$ into itself if and only if $\underline{\alpha}_{\hat{B}} < 1$;*
(ii) *$\mathcal{P}'$ is a bounded linear operator on $\hat{B}$ into itself if and only if $\bar{\alpha}_{\hat{B}} > 0$.*

In this section we shall prove that Shimogaki's theorem is equivalent to the following theorem on martingale inequalities.

Theorem B *Let B be a u.r.i. function space over Ω. Then:*

(i) *The inequality*

$$\|\mathcal{M}X_\infty\|_B \le C_B \|X_\infty\|_B \tag{2}$$

is valid for every uniformly integrable martingale $X = (X_t)_{t\ge 0}$ with respect to an arbitrary filtration if and only if $\underline{\alpha}_B < 1$.

(ii) *The inequalities*

$$c_B \|[X, X]_\infty^{1/2}\|_B \le \|\mathcal{M}X_\infty\|_B \le C_B \|[X, X]_\infty^{1/2}\|_B \tag{3}$$

are valid for every martingale $X = (X_t)_{t\ge 0}$ with respect to an arbitrary filtration if and only if $\bar{\alpha}_B > 0$.

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This theorem is proved independently by Antipa [1], Johnson and Schechtman [11] and Novikov [15], in the case where $\Omega = I$.

The following theorem shows that each of Theorems A and B is deduced from the other.

Theorem 1 *Let B be a u.r.i. function space over Ω and $\hat{B}$ be the r.i. function space over I satisfying (1). Then:*

(i) $\mathcal{P}$ is a bounded linear operator on $\hat{B}$ into itself if and only if (2) holds for every uniformly integrable martingale $X = (X_t)$ with respect to an arbitrary filtration.
(ii) $\mathcal{P}'$ is a bounded linear operator on $\hat{B}$ into itself if and only if (3) holds for every martingale $X = (X_t)$ with respect to an arbitrary filtration.

To prove this theorem, we need some preliminaries. For each $x \in L^1(I)$, we put $x^\# = \mathcal{P}x - x$. Then, since $\mathcal{P}\mathcal{P}'x = \mathcal{P}x + \mathcal{P}'x$ for every $x \in L^1(I)$, we have

$$(4) \qquad (\mathcal{P}'x)^\# = \mathcal{P}x, \quad x \in L^1(I).$$

Furthermore if $x^\# \in L^1(I)$ and $y^\# \in L^\infty(I)$, then

$$(5) \qquad \int_0^1 x(s)y(s)\,ds = \int_0^1 x^\#(s)y^\#(s)\,ds + \left(\int_0^1 x(s)\,ds\right)\left(\int_0^1 y(s)\,ds\right).$$

In fact, this follows from the identity

$$\mathcal{P}'\mathcal{P}x + \int_0^1 x(s)\,ds = \mathcal{P}x + \mathcal{P}'x,$$

which is valid at least for $x \in L^1$ such that $\mathcal{P}x \in L^1$.

Note that, if $x^\# \le y^\#$ on I, then $x^\# 1_{[0,t]}{}^\# \le y^\# 1_{[0,t]}{}^\#$, and hence from (5) we obtain:

Lemma 2 *Let $x, y \in L^1(I)$ be positive decreasing functions. If $x^\# \le y^\#$ on I and $\int_I x\,ds \le \int_I y\,ds$, then $x \prec y$.*

Lemma 3 *Let B and $\hat{B}$ be as in Theorem 1 and suppose that $\mathcal{P}'$ is a bounded operator on $\hat{B}$ into itself. If $Y \in L^1(\Omega)$ and $A = (A_t)_{t\ge 0}$ is an adapted increasing process satisfying*

$$(6) \qquad E\big[A_\infty - A_{T-} | \mathcal{F}_T\big] \le E\big[Y | \mathcal{F}_T\big]$$

for every stopping time T, then $\|A_\infty\|_B \le \|\mathcal{P}'\|_{\hat{B}} \|Y\|_B$, where $\|\mathcal{P}'\|_{\hat{B}}$ denotes the norm of $\mathcal{P}'$.

Recall that a process is called *increasing* if almost every path is positive and increasing. If A is predictable and $A_0 = 0$, then (6) can be replaced by

$$E\big[A_\infty - A_T | \mathcal{F}_T\big] \le E\big[Y | \mathcal{F}_T\big].$$

Proof Setting $T = \inf\{t \ge 0 : A_t > \lambda\}$ for $\lambda > 0$, we have by (6),

$$E\big[(A_\infty - \lambda)1_{\{A_\infty > \lambda\}}\big] \le E\big[Y 1_{\{A_\infty > \lambda\}}\big], \quad \lambda > 0,$$

Substituting $A^*_\infty(t)$ for λ, we have

$$\int_0^t \big(A^*_\infty(s) - A^*_\infty(t)\big)\,ds \le \sup\{E[Y\,1_F] : P(F) \le t\} \le \int_0^t Y^*(s)\,ds.$$

This, together with (4), implies that $(A^*_\infty)^\#(t) \le \mathcal{P}Y^*(t) = (\mathcal{P}'Y^*)^\#(t)$ for all $t \in I$. Since (6) yields that

$$\int_0^1 A^*_\infty(s)\,ds = E[A_\infty] \le E[Y] = \int_0^1 \mathcal{P}'Y^*(s)\,ds,$$

Lemma 2 gives that $A^*_\infty \prec \mathcal{P}'Y^*$. It then follows that

$$\|A_\infty\|_B = \|A^*_\infty\|_{\hat B} \le \|\mathcal{P}'Y^*\|_{\hat B} \le \|\mathcal{P}'\|_{\hat B}\,\|Y^*\|_{\hat B} = \|\mathcal{P}'\|_{\hat B}\,\|Y\|_B\,,$$

which completes the proof. ■

Lemma 3 will be used for the proof of "only if" part of (ii) in Theorem 1. The "if" part will be proved using the following two lemmas.

Lemma 4 *Let $\hat B$ be a r.i. function space over I and $0 < t_0 \le 1$. Then:*

(i) If the inequality

$$\|(\mathcal{P}y)1_{[0,t_0[}\|_{\hat B} \le c\,\|y\|_{\hat B} \tag{7}$$

holds for every positive $y \in L^1(I)$, then $\mathcal{P}\colon \hat B \to \hat B$ is bounded, where c is a positive constant.

(ii) If the inequality

$$\|(\mathcal{P}y)1_{[0,t_0[}\|_{\hat B} \le c\,\big(\|y^\#\|_{\hat B} + \|y\|_1\big) \tag{8}$$

holds for every $y \in L^1(I)$, then $\mathcal{P}'\colon \hat B \to \hat B$ is bounded.

Proof Without loss of generality, we may assume that $\|1\|_{\hat B} = 1$. Let $x \in L^1(I)$. Since $|\mathcal{P}x(t)| \le t_0^{-1}\,\|x\|_1$ for every $t \in [t_0, 1]$, we have by (7)

$$\begin{aligned}\|\mathcal{P}x\|_{\hat B} &\le \|(\mathcal{P}x)1_{[0,t_0[}\|_{\hat B} + \|(\mathcal{P}x)1_{[t_0,1]}\|_{\hat B}\\ &\le c\,\|x\|_{\hat B} + t_0^{-1}\,\|x\|_1 \le C\,\|x\|_{\hat B}\,,\end{aligned}$$

where the last inequality follows from the fact that $\hat B \hookrightarrow L^1(I)$. Thus the operator $\mathcal{P}\colon \hat B \to \hat B$ is bounded.

We now pass to the proof of the second statement. It suffices to show that $\|\mathcal{P}'x\|_{\hat B} \le C\,\|x\|_{\hat B}$ for every positive $x \in L^1(I)$. Put $y = \mathcal{P}'x - x$. Clearly we have $y \in L^1(I)$ and

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$\|y\|_1 \le \|\mathcal{P}'x\|_1 + \|x\|_1 \le 2\|x\|_1$. Since $\mathcal{P}\mathcal{P}' = \mathcal{P} + \mathcal{P}'$, we get $\mathcal{P}y = \mathcal{P}'x$ and $y^{\#} = x$. As $|\mathcal{P}'x(t)| \le t_0^{-1}\|x\|_1$ for $t \in [t_0, 1]$, (8) gives that

$$\begin{aligned}\|\mathcal{P}'x\|_{\hat{B}} &\le \|(\mathcal{P}'x)1_{[0,t_0[}\|_{\hat{B}} + \|(\mathcal{P}'x)1_{[t_0,1]}\|_{\hat{B}} \\ &\le \|(\mathcal{P}y)1_{[0,t_0]}\|_{\hat{B}} + t_0^{-1}\|x\|_1 \\ &\le c\big(\|y^{\#}\|_{\hat{B}} + \|y\|_1\big) + t_0^{-1}\|x\|_1 \\ &\le c\|x\|_{\hat{B}} + (2c + t_0^{-1})\|x\|_1 \le C\|x\|_{\hat{B}},\end{aligned}$$

which completes the proof. ■

The following lemma is essential for the proof of "if" part of (i), and (ii) in Theorem 1.

Lemma 5 *Let $t_0 = P(\Omega \setminus \Omega_0) > 0$. Then, for each $x \in L^1(I)$, there exists a uniformly integrable martingale $X = (X_t)_{t\ge 0}$ satisfying the following conditions:*

(i) $|X_0| \le t_0^{-1}\|x\|_1$,
(ii) $X_\infty^*(t) = \big(x1_{[0,t_0[}\big)^*(t)$, $t \in I$,
(iii) $\{(\mathcal{P}x)1_{[0,t_0[}\}^*(t) \le (\mathcal{M}X_\infty)^*(t)$, $t \in I$,
(iv) $\{([X, X]_\infty - X_0^2)^{1/2}\}^*(t) = (x^{\#}1_{[0,t_0[})^*(t)$, $t \in I$.

Proof Since $\Omega_1 = \Omega \setminus \Omega_0$ contains no atom, there exists a family of measurable sets $\{A(t) : 0 \le t \le t_0\}$ satisfying the following conditions:

a) $A(t) \subset A(s) \subset \Omega_1$ if $0 \le s \le t \le t_0$;
b) $P\big(A(t)\big) = t_0 - t$ for every $0 \le t \le t_0$.

For the proof, see [6, p. 44]. For each $t \le t_0$, let $\mathcal{F}_t$ be the σ–field generated by all measurable subsets of $\Omega \setminus A(t)$ and P-negligible sets, and for each $t \ge t_0$, set $\mathcal{F}_t = \mathcal{F}_{t_0}$. Clearly $(\mathcal{F}_t)_{t\ge 0}$ satisfies the usual conditions, and for $t \le t_0$, $A(t)$ is an $\mathcal{F}_t$-atom.

Now for each $\omega \in \Omega$, put

$$T(\omega) = \begin{cases} \sup\{s \in [0, t_0] : \omega \in A(s)\} & \text{if } \omega \in A(0), \\ 0 & \text{otherwise.}\end{cases}$$

It is easy to see that $\{T > t\} = A(t)$ a.s. for every $t \in [0, t_0]$. Hence T is an $(\mathcal{F}_t)$-stopping time, and $T^*(s) = (t_0 - s)^+ = (t_0 - s) \vee 0$, $s \in I$. Let $x \in L^1(I)$. Since $x(t_0 - T)1_{\{T>0\}}$ and $x1_{[0,t_0[}$ have the same distribution, $x(t_0 - T)1_{\{T>0\}}$ is integrable over Ω. Let $X = (X_t)$ be the martingale induced by $x(t_0 - T)1_{\{T>0\}}$, that is,

$$X_t = E\big[x(t_0 - T) \mid \mathcal{F}_t\big]1_{\{T>0\}} = x(t_0 - T)1_{\{0<T\le t\}} + \mathcal{P}x(t_0 - t)1_{\{t<T\}}. \tag{9}$$

Note that the processes on both sides of (9) are indistinguishable, that is, (9) holds for every $t \ge 0$ on a set Ω' of probability one.

We show that X satisfies the required conditions. In fact, (i) and (ii) are straightforward consequences of the definition: we have $|X_0| \le |\mathcal{P}x(t_0)| \le t_0^{-1}\|x\|_1$ and $X_\infty^* = \{x(t_0 - T)1_{\{T>0\}}\}^* = (x1_{[0,t_0[})^*$, since $T^*(s) = (t_0 - s)^+$.

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It is easy to see from (9) that $|\mathcal{P}x(t_0 - T)|1_{\{T>0\}} = |X_{T-}| \le \mathcal{M}X_\infty$, which implies (ii).

Now it remains to prove (iv). Again from (9) we see that the path $t \mapsto X_t(\omega)$ is of bounded variation on $[0, \infty[$, continuous on $[0, T(\omega)[$ and constant on $[T(\omega), \infty[$, provided that $\omega \in \Omega'$. Therefore we have $\triangle X_T 1_{\{T>0\}} = -x^\#(t_0 - T)1_{\{T>0\}}$ and the continuous martingale part X^c of X is equal to zero (*cf.* [14, p. 267]). This implies that

$$\big([X, X]_\infty - X_0^2\big)^{1/2} = |x^\#(t_0 - T)|1_{\{T>0\}}.$$

Thus (iv) is obtained and the lemma is established. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1 (i) Suppose that $\mathcal{P}\colon \hat{B} \to \hat{B}$ is bounded. By Doob's inequality we have

$$\lambda \le P(\mathcal{M}X_\infty \ge \lambda)^{-1} \int_{\{\mathcal{M}X_\infty \ge \lambda\}} X_\infty \, dP \le \mathcal{P}X_\infty^*\big(P(\mathcal{M}X_\infty \ge \lambda)\big) \tag{10}$$

for every $\lambda > 0$ and every uniformly integrable martingale $X = (X_t)$, where we have used Hardy's inequality

$$\int_A f\, dP \le \int_0^{P(A)} f^*(s)\, ds, \quad f \in L^1(P), \quad A \in \mathcal{F}.$$

Setting $\lambda = (\mathcal{M}X_\infty)^*(t)$ in (10), we get

$$(\mathcal{M}X_\infty^*)(t) \le \mathcal{P}X_\infty^*(t), \quad t \in I,$$

since $P\big(\mathcal{M}X_\infty \ge (\mathcal{M}X_\infty)^*(t)\big) \ge t$. Hence we have

$$\|\mathcal{M}X_\infty\|_B = \|(\mathcal{M}X_\infty)^*\|_{\hat{B}} \le \|\mathcal{P}X_\infty^*\|_{\hat{B}} \le \|\mathcal{P}\|_{\hat{B}}\, \|X_\infty^*\|_{\hat{B}} \le \|\mathcal{P}\|_{\hat{B}}\, \|X_\infty\|_B\,.$$

Conversely assume that (2) holds for every uniformly integrable martingale. Let $x \in L^1(I)$ be a positive function and, using Lemma 5, choose a uniformly integrable martingale $X = (X_t)$ so as to satisfy (ii) and (iii) of Lemma 5. Then we have

$$\|(\mathcal{P}x)1_{[0,t_0[}\|_{\hat{B}} \le \|(\mathcal{M}X_\infty)^*\|_{\hat{B}} = \|\mathcal{M}X_\infty\|_B \le C_B\, \|X_\infty\|_B \le C_B\, \|x\|_{\hat{B}}\,.$$

Lemma 4 (i) shows that $\mathcal{P}\colon \hat{B} \to \hat{B}$ is bounded. Thus (i) of Theorem 1 is established.

(ii) Suppose that $\mathcal{P}'\colon \hat{B} \to \hat{B}$ is bounded. By Davis's inequality we have

$$E\big[\mathcal{M}X_\infty - \mathcal{M}X_{T-} \mid \mathcal{F}_T\big] \le cE\big[[X, X]_\infty^{1/2} \mid \mathcal{F}_T\big];$$

$$E\big[[X, X]_\infty^{1/2} - [X, X]_{T-}^{1/2} \mid \mathcal{F}_T\big] \le cE\big[\mathcal{M}X_\infty \mid \mathcal{F}_T\big]$$

for every stopping time T. For the proof *e.g.* see [14, p. 349]. Therefore (3) follows from Lemma 3.

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Next suppose that (3) holds for every martingale. For each $x \in L^1$ there exits a martingale $X = (X_t)$ which satisfies (i), (iii) and (iv) of Lemma 5. Assuming $\|1\|_B = 1$ for simplicity, we have

$$\begin{aligned}
\|(\mathcal{P}x)1_{[0,t_0[}\|_{\hat{B}} &\le \|\mathcal{M}X_\infty\|_B \le C_B \|[X, X]_\infty^{1/2}\|_B \\
&\le C_B \big\| \big([X, X]_\infty - X_0^2\big)^{1/2} \big\|_B + C_B \|X_0\|_B \\
&\le C_B \|x^\# 1_{[0,t_0[}\|_{\hat{B}} + C_B t_0^{-1} \|x\|_1 \\
&\le C_B t_0^{-1} \big(\|x^\# 1_{[0,t_0[}\|_{\hat{B}} + \|x\|_1 \big).
\end{aligned}$$

Lemma 4 (ii) implies the boundedness of $\mathcal{P}' \colon \hat{B} \to \hat{B}$. The theorem is established. ■

3 Change of Probability Measure

In this section we shall prove that there exists a close relation between Boyd's theorem on the boundedness of the averaging operators, and some martingale inequalities relative to some equivalent probability measures. We first recall Boyd's theorem: for $p \ge 1$ define the operators $\mathcal{P}_p$ and $\mathcal{P}'_p$ by

$$\mathcal{P}_p x(t) = t^{-1/p} \int_0^t x(s) s^{-1/p'}\, ds,$$

$$\mathcal{P}'_p x(t) = t^{-1/p} \int_t^1 x(s) s^{-1/p'}\, ds,$$

whenever the integrals exist, where p' stands for the exponent conjugate to p.

Theorem C **(Boyd [5])** *Let $\hat{B}$ be a r.i. function space over I. Then $\mathcal{P}_p$ (resp. $\mathcal{P}'_p$) is a bounded linear operator from $\hat{B}$ into itself if and only if $\underline{\alpha}_{\hat{B}} < 1/p$ (resp. $\bar{\alpha}_{\hat{B}} > 1/p$).*

We consider equivalent probability measures P and Q on $(\Omega, \mathcal{F})$. For the sake of simplicity, we assume that the probability space $(\Omega, \mathcal{F}, P)$ (or $(\Omega, \mathcal{F}, Q)$) is nonatomic throughout this section.

Let W_∞ denote the Radon-Nikodym derivative dQ/dP, and $W = (W_t)_{t \ge 0}$ the martingale $W_t = E_P[W_\infty \mid \mathcal{F}_t]$, $t \ge 0$, where and in what follows E_P and E_Q denote the (conditional) expectations relative to P and Q respectively. We denote by $\mathfrak{M}\big(P, (\mathcal{F}_t)\big)$ the family of all uniformly integrable martingales on the system $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), P\big)$.

Now let $1 < p < \infty$. We say that $W_\infty = dQ/dP$ satisfies (A_p) with respect to P and $(\mathcal{F}_t)$ if

$$(A_p) \qquad \sup_T \big\| E_P\big[(W_T / W_\infty)^{\frac{1}{p-1}} \mid \mathcal{F}_T \big] \big\|_\infty < \infty,$$

where the supremum is taken over all $(\mathcal{F}_t)$-stopping times T. We write $W_\infty \in A_p\big(P, (F_t)\big)$ when W_∞ satisfies (A_p) with respect to P and $(\mathcal{F}_t)$. Condition (A_p) is introduced by Izumisawa and Kazamaki [10]. It was proved by Tsuchikura [18] and Uchiyama [19] (also see

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Kazamaki [12, p. 74]) that $W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$ if and only if

$$(W_p) \qquad \lambda^p Q(\mathcal{M}X_\infty \geq \lambda) \leq \int_{\{\mathcal{M}X_\infty \geq \lambda\}} |X_\infty|^p \, dQ, \quad \lambda > 0,$$

holds for all $X \in \mathfrak{M}\bigl(P,(F_t)\bigr)$.

Let $\hat{B}$ be a r.i. function space over I. We define the space $B(Q)$ by

$$B(Q) := \{f : \|f\|_{B(Q)} < \infty\};$$
$$\|f\|_{B(Q)} := \|f_Q^*\|_{\hat{B}},$$

where f_Q^* denotes the decreasing rearrangement of f relative to Q. Our main results in this section are the following.

Theorem 6 *Let P, Q, W, $\hat{B}$ and $B(Q)$ be as above and $1 < p < \infty$.*

(i) If $dQ/dP = W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$ and $\underline{\alpha}_{\hat{B}} < 1/p$, then the inequality

$$(11) \qquad \|\mathcal{M}X_\infty\|_{B(Q)} \leq c\,\|X_\infty\|_{B(Q)}$$

holds for every $X \in \mathfrak{M}\bigl(P,(\mathcal{F}_t)\bigr)$.

(ii) If (11) holds for all $X \in \mathfrak{M}\bigl(P,(\mathcal{F}_t)\bigr)$ whenever $dQ/dP = W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$, then $\underline{\alpha}_{\hat{B}} \leq 1/p$.

If $\hat{B} = L^q(I)$, $q > p$, then (i) of the above theorem yields the weighted norm inequalities established by Izumisawa and Kazamaki. Theorem 6 will be proved using Theorem C. On the other hand, we have the following.

Theorem 7 *Theorem C is deduced from the assertion of Theorem 6.*

Combing this with a result of Doléans-Dade and Meyer, we have the following.

Corollary 8 *If $\underline{\alpha}_{\hat{B}} \leq 1/p$, $dQ/dP = W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$ and*

$$(S^-) \qquad W_{T-} \leq K W_T$$

holds for every $(\mathcal{F}_t)$-stopping time T, where $W = (W_t)$ denotes the martingale $W_t = E_P[W_\infty \mid \mathcal{F}_t]$, $t \geq 0$, then (11) holds for every $X \in \mathfrak{M}\bigl(P,(\mathcal{F}_t)\bigr)$.

We begin with some lemmas.

Lemma 9 *If x is a positive decreasing function on I, then*

$$(12) \qquad \left\{\int_0^t x(s)^p \, ds\right\}^{1/p} \leq p^{-1} \int_0^t x(s) s^{-1/p'} \, ds \quad \text{for every } t \in I.$$

Proof Suppose first that x is of the form

$$(13) \qquad x(t) = \sum_{k=1}^n a_k 1_{[0,t_k[}(t), \quad a_k \geq 0, \quad 0 \leq t_1 < t_2 < \cdots < t_n \leq 1.$$

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Then by Minkowski's inequality we have

$$\left\{\int_0^t x(s)^p\,ds\right\}^{1/p} \le \sum_{k=1}^n a_k(t\wedge t_k)^{1/p} = p^{-1}\int_0^t x(s)s^{-1/p'}\,ds.$$

For an arbitrary decreasing function x, we can find a sequence of the functions (x_n) of the form (13) such that $0 \le x_n \uparrow x$ a.e. Hence by the monotone convergence theorem, we have (12). ∎

Lemma 10 *If* $1 < p < \infty$ *and* $dQ/dP = W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$, *then* $(\mathcal{M}X_\infty)^*_Q \le p^{-1}\mathcal{P}_p(X_\infty)^*_Q$ *on* I *for every* $X \in \mathfrak{M}\bigl(P,(\mathcal{F}_t)\bigr)$.

Proof Since $dQ/dP = W_\infty \in A_p\bigl(P,(\mathcal{F}_t)\bigr)$, we have by (W_p) that

$$\begin{aligned}\lambda^p &\le Q(\mathcal{M}X_\infty \ge \lambda)^{-1}\int_{\{\mathcal{M}X_\infty\ge\lambda\}} |X_\infty|^p\,dQ\\ &\le Q(\mathcal{M}X_\infty \ge \lambda)^{-1}\int_0^{Q(\mathcal{M}X_\infty\ge\lambda)} (X_\infty)^*_Q(s)^p\,ds, \quad \lambda > 0,\end{aligned}$$

for every $X \in \mathfrak{M}\bigl(P,(\mathcal{F}_t)\bigr)$. We set $\lambda = (\mathcal{M}X_\infty)^*_Q(t)$. In view of Lemma 9, we obtain that

$$(\mathcal{M}X_\infty)^*_Q(t) \le \left\{\frac{1}{t}\int_0^t (X_\infty)^{*p}_Q(s)\,ds\right\}^{1/p} \le p^{-1}\mathcal{P}_p(X_\infty)^*_Q(t), \quad t \in I. \qquad ∎$$

The following lemma is a key result to the proof of Theorem 6 (ii) and Theorem 7.

Lemma 11 *Let* $1 < p < \infty$ *be fixed. If* $q > p$ *and* $x \in L^1(I)$ *is positive, then we can construct equivalent probability measures* P *and* Q, *a filtration* $(\mathcal{F}_t)_{t\ge0}$ *and a martingale* $X = (X_t)_{t\ge0}$ *so that*

(i) $dQ/dP = W_\infty \in A_q\bigl(P,(\mathcal{F}_t)\bigr)$,
(ii) $(X_\infty)^*_Q = x^*$ *on* I,
(iii) $(\mathcal{M}X_\infty)^*_Q = p^{-1}\mathcal{P}_p x^*$ *on* I.

Proof Let Q be a probability measure such that Ω contains no Q-atom. There exists a random variable V such that $V^*_Q(s) = p^{-1}s^{-1/p'}$ [6, p. 44]. Let P be the probability measure $dP := V\,dQ$. Clearly P and Q are equivalent and Ω contains no P-atom. For each $t \in I$, put

$$A(t) := \{\omega \in \Omega : V(\omega) > p^{-1}(1-t)^{-1/p'}\}.$$

Then $A(t)$ decreases with t and

$$Q\bigl(A(t)\bigr) = 1 - t, \quad t \in I, \tag{14}$$

$$P\bigl(A(t)\bigr) = (1-t)^{1/p}, \quad t \in I. \tag{15}$$

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For each $t \in I$, let $\mathcal{F}_t$ denote the σ-field generated by the measurable subsets of $\Omega \setminus A(t)$ and the negligible sets. For $t > 1$ we set $\mathcal{F}_t = \mathcal{F}_1$. Then $(\mathcal{F}_t)$ satisfies the usual conditions (relative to both P and Q).

We now prove that $W_\infty := 1/V = dQ/dP \in A_q\big(P, (\mathcal{F}_t)\big)$ for each $q > p$. It suffices to show that

$$E_P\big[(W_t/W_\infty)^{\frac{1}{q-1}} \mid \mathcal{F}_t\big] \le C \quad \text{a.s.},$$

for every $t \in I$, since every stopping time is the decreasing limit of a sequence of stopping times with values in the set of rationals. By (14) and (15) we have

$$W_t = W_\infty 1_{\Omega\setminus A(t)} + P\big(A(t)\big)^{-1} Q\big(A(t)\big) 1_{A(t)} = W_\infty 1_{\Omega\setminus A(t)} + (1-t)^{1/p'} 1_{A(t)}$$

for each $t \in I$, and on $A(t)$ we have

$$E_P\big[W_\infty^{-\frac{1}{q-1}} \mid \mathcal{F}_t\big] = (1-t)^{-1/p} \int_0^{1-t} V_Q^*(s)^{q'}\, ds = p^{-q'} \cdot \frac{p'}{p'-q'}(1-t)^r,$$

where $r = 1 - p^{-1} - p'^{-1} q' = -\{p'(q-1)\}^{-1}$. It then follows that

$$\begin{aligned} E_P\big[(W_t/W_\infty)^{\frac{1}{q-1}} \mid \mathcal{F}_t\big] &= 1_{\Omega\setminus A(t)} + p^{-q'} \cdot \frac{p'}{p'-q'}(1-t)^{r+\{p'(q-1)\}^{-1}} 1_{A(t)} \\ &\le p^{-q'} \cdot \frac{p(q-1)}{q-p}, \end{aligned}$$

Thus $dQ/dP = W_\infty \in A_q\big(P, (\mathcal{F}_t)\big)$.

Now let τ be the random variable defined by

$$\tau(\omega) = \sup\{t \in I : \omega \in A(t)\}.$$

It is easy to see that $\{\tau > t\} = A(t)$, $t \in I$, P-a.s. and Q-a.s., and therefore τ is an $(\mathcal{F}_t)$-stopping time. Moreover by (14) and (15), we have $\tau_P^*(t) = 1 - t^p$ and $\tau_Q^*(t) = 1 - t$ for all $t \in I$. It follows that if $y \in L^1(I)$, then

$$\text{(16)} \qquad \int_{A(t)} y(1-\tau)\, dP = \int_0^{(1-t)^{1/p}} y(s^p)\, ds = p^{-1} \int_0^{1-t} y(s) s^{-1/p'}\, ds.$$

Assume that $x \in L^1(I)$ is positive and let $X_t = E_P[x^*(1-\tau) \mid \mathcal{F}_t]$, $t \ge 0$. Then $X = (X_t)$ satisfies (ii) and (iii) of the statement. Indeed, (ii) follows immediately from the equality $\tau_Q^*(t) = 1 - t$. Hence it remains only to prove (iii). Observe that

$$X_t = x^*(1-\tau) 1_{\{t \ge \tau\}} + p^{-1} \mathcal{P}_p x^*(1-t) 1_{\{t < \tau\}},$$

which follows from (16). This expression shows that each path is increasing on $[0, \tau(\omega)[$, constant on $[\tau(\omega), \infty[$, and has a jump at $\tau(\omega)$. Therefore, from the fact that $X_{\tau-} = p^{-1}\mathcal{P}_p x^*(1-\tau) \ge x^*(1-\tau) = X_\tau$, we obtain

$$\mathcal{M}X_\infty = p^{-1} \mathcal{P}_p x^*(1-\tau)$$

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As $\tau_Q^*(t) = 1 - t$, this implies (iii) and the lemma is established. ∎

The last lemma, due to Boyd [4], is for the proof of Theorem 7. The assertion follows directly from Theorem C. For the proof of Theorem 7, however, we cannot use Theorem C and must prove the following lemma without using Theorem C.

Lemma 12 *Let $\hat{B}$ be a r.i. function space over I. If $1 < p < \infty$ and $\mathcal{P}_p$ is a bounded linear operator from $\hat{B}$ into itself, then for $q > p$ sufficiently close to p, $\mathcal{P}_q$ is a bounded linear operator from $\hat{B}$ into itself.*

Proof According to Lemma 2 of [4], we have

$$\mathcal{P}_p^n x(t) = \frac{1}{(n-1)!}\int_0^1 \Big(\log\frac{1}{s}\Big)^{n-1} x(st)s^{-1/p'}\,ds, \quad t \in I,$$

where $\mathcal{P}_p^n$ stands for the n-th iterate of $\mathcal{P}_p$. From this it follows that

$$\mathcal{P}_q x(t) = \sum_{n=0}^{\infty}\Big(\frac{1}{q'} - \frac{1}{p'}\Big)^n \mathcal{P}_p^{n+1} x(t)$$

for $q > p$ and for positive x. Taking $q > p$ so close to p that $(1/q' - 1/p')\,\|\mathcal{P}_p\|_{\hat{B}} < 1$, we have

$$\begin{aligned}\|\mathcal{P}_q x\|_{\hat{B}} &= \lim_{N\to\infty}\Big\|\sum_{n=0}^{N}(1/q' - 1/p')^n \mathcal{P}_p^{n+1} x\Big\|_{\hat{B}}\\ &\le \sum_{n=0}^{\infty}(1/q' - 1/p')^n\,\|\mathcal{P}_p\|_{\hat{B}}^{n+1}\,\|x\|_{\hat{B}} = C\,\|x\|_{\hat{B}},\end{aligned}$$

where $\|\mathcal{P}_p\|_{\hat{B}}$ denotes the norm of $\mathcal{P}_p \colon \hat{B} \to \hat{B}$. This completes the proof. ∎

Now we give the proof of Theorems 6 and 7, and Corollary 8.

Proof of Theorem 6 (i) Suppose that $dQ/dP = W_\infty \in A_p\big(P, (\mathcal{F}_t)\big)$ and $\underline{\alpha}_{\hat{B}} < 1/p$. According to Lemma 10 and Theorem C, we have

$$\|\mathcal{M}X_\infty\|_{B(Q)} = \|(\mathcal{M}X_\infty)_Q^*\|_{\hat{B}} \le p^{-1}\|\mathcal{P}_p(X_\infty)_Q^*\|_{\hat{B}} \le p^{-1}\,\|\mathcal{P}_p\|_{\hat{B}}\,\|X_\infty\|_{B(Q)}$$

for every $X \in \mathfrak{M}\big(P, (\mathcal{F}_t)\big)$.

(ii) Now assume that (11) holds for every $X \in \mathfrak{M}\big(P, (\mathcal{F}_t)\big)$ whenever $dQ/dP \in A_p\big(P, (\mathcal{F}_t)\big)$. Let $1 < q < p$. By Lemma 11, for each positive $x \in L^1(I)$ we can find equivalent measures P, Q and a martingale $X = (X_t) \in \mathfrak{M}\big(P, (\mathcal{F}_t)\big)$ such that $dQ/dP \in A_p\big(P, (\mathcal{F}_t)\big)$, $(X_\infty)_Q^* = x^*$ and $(\mathcal{M}X_\infty)_Q^* = q^{-1}\mathcal{P}_q x^*$. Then by hypothesis, we get

$$q^{-1}\,\|\mathcal{P}_q x\|_{\hat{B}} \le q^{-1}\,\|\mathcal{P}_q x^*\|_{\hat{B}} = \|\mathcal{M}X_\infty\|_{B(Q)} \le c\,\|X_\infty\|_{B(Q)} = c\,\|x\|_{\hat{B}}.$$

Thus $\mathcal{P}_q \colon \hat{B} \to \hat{B}$ is a bounded linear operator. It follows from Theorem C that $\underline{\alpha}_{\hat{B}} < 1/q$. Letting $q \uparrow p$, we obtain $\underline{\alpha}_{\hat{B}} \le 1/p$. Theorem 6 is proved. ∎

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Proof of Theorem 7 We use Theorem 6. We shall prove the assertion of Theorem C for $\mathcal{P}_p$ only. Suppose that $\underline{\alpha}_{\hat{B}} < 1/q < 1/p$. Choose P, Q and $X = (X_t)$ so as to satisfy (i)–(iii) of Lemma 11 for a given $x \geq 0$ in $L^1(I)$. Since we have assumed that Theorem 6 is true, we may use (11) to get

$$p^{-1} \left\|\mathcal{P}_p x\right\|_{\hat{B}} \leq p^{-1} \left\|\mathcal{P}_p x^*\right\|_{\hat{B}} = p^{-1} \|\mathcal{M}X_\infty\|_{B(Q)} \leq c \|X_\infty\|_{B(Q)} = c \|x\|_{\hat{B}} .$$

Thus $\mathcal{P}_p \colon \hat{B} \to \hat{B}$ is bounded.

Now assume that $\mathcal{P}_p \colon \hat{B} \to \hat{B}$ is bounded. Then by Lemma 12, there exists $q > p$ such that $\mathcal{P}_q$ is a bounded operator from $\hat{B}$ into itself. Suppose that $dQ/dP \in A_q\big(P, (\mathcal{F}_t)\big)$. Then Lemma 10 gives that $(\mathcal{M}X_\infty)_Q^* \leq q^{-1}\mathcal{P}_q(X_\infty)_Q^*$; hence (11) is valid for all $X \in \mathfrak{M}\big(P, (\mathcal{F}_t)\big)$. Thus we have proved that $dQ/dP \in A_q\big(P, (\mathcal{F}_t)\big)$ implies (11). From Theorem 6 (ii), we obtain $\underline{\alpha}_{\hat{B}} \leq 1/q < 1/p$, which completes the proof. ■

Proof of Corollary 8 In [9] Doléans-Dade and Meyer proved that if $dQ/dP = W_\infty \in A_p\big(P, (\mathcal{F}_t)\big)$ and $W = (W_t)$ satisfies (S^-), then $dQ/dP \in A_q\big(P, (\mathcal{F}_t)\big)$ for some $q < p$. Hence Theorem 6 gives that (11) is valid for all $X \in \mathfrak{M}\big(P, (\mathcal{F}_t)\big)$ if $\underline{\alpha}_{\hat{B}} \leq 1/p$. ■

Finally we mention the Burkholder-Davis-Gundy type inequality without proof. Sekiguchi [16] (and independently Bonami and Lépingle [3]) proved that if $dQ/dP = W_\infty \in A_p\big(P, (\mathcal{F}_t)\big)$ for some $p > 1$ and $W = (W_t)$ satisfies

$$(S) \qquad 0 < k \leq W_{T-}/W_T \leq K$$

with some constants k and K, then

$$cE_Q\big[\Phi(\mathcal{M}X_\infty)\big] \leq E_Q\big[\Phi\big([\,X\,,\,X\,]_\infty^{1/2}\big)\big] \leq CE_Q\big[\Phi(\mathcal{M}X_\infty)\big]$$

hold for all local martingales $X = (X_t)_{t\geq 0}$ with respect to P and $(\mathcal{F}_t)$, where Φ is a Young function satisfying the Δ_2-condition. Using this inequality with $\Phi(t) = t$ and Lemma 3, we can prove that if $\bar{\alpha}_{\hat{B}} > 0$, then the inequalities

$$c \|\mathcal{M}X_\infty\|_{B(Q)} \leq \|[\,X\,,\,X\,]_\infty^{1/2}\|_{B(Q)} \leq C \|\mathcal{M}X_\infty\|_{B(Q)}$$

holds for all local martingales $X = (X_t)$ with respect to P and $(\mathcal{F}_t)$, provided $dQ/dP \in A_p\big(P, (\mathcal{F}_t)\big)$ and $W = (W_t)$ satisfies (S).

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