Canad. Math. Bull. Vol. **23** (3) 1980

THE RUIN PROBLEM FOR SUMS OF DEPENDENT RANDOM VARIABLES

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ABSTRACT. In this paper we show how to compute the probability that a sequence $Z_n = X_0 + \cdots + X_n$ of partial sums of dependent random variables, each taking on the values ± 1, will first leave an interval (a, b) at a; and how to compute the expected time it takes for the partial sums to leave the interval (a, b).

Suppose that $M(\Omega, \mathcal{F})$ is the linear space of all measures φ on the measurable space $(\Omega, \mathcal{F})$ of all functions ω mapping the non-negative integers into the finite set $\{-1, +1\}$ with $\mathcal{F}$ being the σ-field generated by the events $X_t(\omega) = \omega(t) = \delta = \pm 1$; and that Φ is a linear subspace of $M(\Omega, \mathcal{F})$ which is closed under the operators T and $E(\delta)$, $\delta = \pm 1$, defined by

$$T\varphi(X_0 = \delta_0, \ldots, X_n = \delta_n) = \varphi(X_1 = \delta_0, \ldots, X_{n+1} = \delta_n)$$
$$E(\delta)\varphi(\Lambda) = \varphi(X_0 = \delta, \Lambda).$$

Suppose that a and b are integers with $a < -1$ and $b > +1$ and that $\tau_{ab}(\omega)$ is the first time n for which $\omega(0) + \cdots + \omega(n)$ is not in the interval (a, b). Then we have the following

THEOREM. *For each* $\varphi \in \Phi$,

$$\varphi[X_0 + \cdots + X_{\tau_{ab}} = a] = p^* E(-1)[E(-1)A^{b-a-2}E(-1)]^{-1}A^{b-1}\varphi$$

and

$$\int \tau_{ab}(\omega)\varphi(d\omega) = p^* B(I + \cdots + A^{b-2})\varphi - p^* B(I + \cdots + A^{b-2}) \times [E(-1)A^{b-a-2}E(-1)]^{-1}A^{b-1}\varphi$$

where the linear functional p^* *and the operators* A *and* B *are defined via*

$$p^*\varphi = \varphi(\Omega)$$
$$A = [E(-1) + \lambda TE(+1)][\lambda E(+1) + TE(-1)]^{-1}$$
$$B = [\lambda E(+1) - E(-1)][\lambda E(+1) + TE(-1)]^{-1}$$

where it is assumed that a complex number λ *can be found such that the inverse of* $\lambda E(+1) + TE(-1)$ *exists and where* $[E(-1)A^{b-a-2}E(-1)]^{-1}$ *is the inverse of* $E(-1)A^{b-a-2}E(-1)$ *thought of as an operator mapping* $E(-1)\Phi$ *into* $E(-1)\Phi$.

Received by the editors November 22, 1978 and, in revised form, Febuary 20, 1979.

https://doi.org/10.4153/CMB-1980-047-2 Published online by Cambridge University Press

Proof. Let $f_{a,b}$ be the indicator function of the event $[X_0+\cdots+X_{\tau_{a,b}}=a]$ and let $f^*_{a,b}$ be the linear functional defined by $f^*_{a,b}=\int f_{a,b}(\omega)\varphi(d\omega)$. Then for any $\varphi\in\Phi$,

$$(1)\qquad f^*_{a,b}TE(-1)\varphi=f^*_{a-1,b-1}E(-1)\varphi$$

$$(2)\qquad f^*_{a-1,b-1}TE(+1)\varphi=f^*_{a,b}E(+1)\varphi$$

$$(3)\qquad f^*_{-1,b}E(-1)\varphi=p^*E(-1)\varphi$$

$$(4)\qquad f^*_{a,1}E(+1)\varphi=0$$

since if $\omega_1^+(u)=\omega(1+u)$ we have

$$\begin{aligned} f^*_{a,b}T\varphi &= \int f_{a,b}(\omega)T\varphi(d\omega)\\ &= \int f_{a,b}(\omega_1^+)\varphi(d\omega)\\ &= \int f_{a+\omega(0),b+\omega(0)}(\omega)\varphi(d\omega)\\ &= \int f_{a+\omega(0),b+\omega(0)}(\omega)E(-1)\varphi(d\omega)+\int f_{a+\omega(0),b+\omega(0)}(\omega)E(+1)\varphi(d\omega)\\ &= \int f_{a-1,b-1}(\omega)E(-1)\varphi(d\omega)+\int f_{a+1,b+1}(\omega)E(+1)\varphi(d\omega)\\ &= \int f^*_{a-1,b-1}E(-1)\varphi+f^*_{a+1,b+1}E(+1)\varphi \end{aligned}$$

which, upon replacing φ by $E(-1)\varphi$ and $E(+1)\varphi$ give us equations (1) and (2). Equations (3) and (4) are obvious. From equations (1) and (2) we see that

$$f^*_{a,b}[\lambda E(+1)+TE(-1)]\varphi=f^*_{a-1,b-1}[E(-1)+\lambda TE(+1)]\varphi$$

or, in terms of the adjoints of these operators,

$$[\lambda E(+1)+TE(-1)]^*f^*_{a,b}=[E(-1)+\lambda TE(+1)]^*f^*_{a-1,b-1}$$

which now gives us

$$f^*_{a,b}=[\lambda E(+1)+TE(-1)]^{*-1}[E(-1)+\lambda TE(+1)]^*f^*_{a-1,b-1}=A^*f^*_{a-1,b-1}$$

or, by using equation (4) above,

$$(5)\qquad \begin{aligned} f^*_{a,b}=A^{*b-1}f^*_{a-b+1,1}&=A^{*b-1}E(-1)^*f^*_{a-b+1,1}+A^{*b-1}E(+1)^*f^*_{a-b+1,1}\\ &=A^{*b-1}E(-1)^*f^*_{a-b+1,1}. \end{aligned}$$

Using equations (3) and (5), we see that

$$(6)\qquad E(-1)^*p^*=E(-1)^*f^*_{-1,b}=E(-1)^*A^{*b-1}E(-1)^*f^*_{-b,1}$$

so that

$$E(-1)^*f^*_{-b,1}=[E(-1)^*A^{*b-1}E(-1)^*]^{-1}E(-1)^*p^*$$

where we take the inverse of $E(-1)^*A^{*b-1}E(-1)^*$ as an operator of $E(-1)^*\Phi^*$ into $E(-1)^*\Phi^*$. Thus by (5) and (6), with $-b$ replaced by $a-b+1$ in (6), we have

$$f^*_{a,b}\varphi=A^{*b-1}[E(-1)^*A^{*b-a-2}E(-1)^*]^{-1}E(-1)^*p^*\varphi$$

or

$$\varphi[X_0+\cdots+X_{\tau_{ab}}=a]=p^*E(-1)[E(-1)A^{b-a-2}E(-1)]^{-1}A^{b-1}\varphi$$

as was to be proved.

Now let $\tau^*_{a,b}$ be the linear functional on Φ defined by $\tau^*_{a,b}\varphi=\int\tau_{a,b}(\omega)\varphi(d\omega)$. Then the second part of the theorem will follow, as in the first part of the proof, from the equalities:

(7) $$E(-1)^*T^*\tau^*_{a,b}=E(-1)^*\tau^*_{a-1,b-1}-E(-1)^*p^*$$

(8) $$E(+1)^*T^*\tau^*_{a-1,b-1}=E(+1)^*\tau^*_{a,b}-E(+1)^*p^*$$

(9) $$E(-1)^*\tau^*_{-1,b}=0$$

(10) $$E(+1)^*\tau^*_{a,+1}=0.$$

For example, the first two equalities follow from

$$\begin{aligned}T^*\tau^*_{a,b}\varphi&=\tau^*_{a,b}T\varphi\\&=\int\tau_{a,b}(\omega)T\varphi(d\omega)\\&=\int\tau_{a,b}(\omega_1^+)\varphi(d\omega)\\&=\int\tau_{a,b}(\omega_1^+)E(-1)\varphi(d\omega)+\int\tau_{a,b}(\omega_1^+)E(+1)\varphi(d\omega)\\&=\int[\tau_{a-1,b-1}(\omega)-1]E(-1)\varphi(d\omega)+\int[\tau_{a+1,b+1}(\omega)-1]E(+1)\varphi(d\omega)\\&=\tau^*_{a-1,b-1}E(-1)\varphi+\tau^*_{a+1,b+1}E(+1)\varphi-p^*\varphi\\&=E(-1)^*\tau^*_{a-1,b-1}\varphi+E(+1)^*\tau^*_{a+1,b+1}\varphi-p^*\varphi.\end{aligned}$$

From (7) and (8) we now have

(11) $$\begin{aligned}\tau^*_{a,b}&=A^*\tau^*_{a-1,b-1}+B^*p^*=\cdots\\&=A^{*b-1}E(-1)^*\tau^*_{a-b+1,1}+(I^*+\cdots+A^{*b-2})B^*p^*.\end{aligned}$$

But by (9) and (11) we have

$$0=E(-1)^*\tau^*_{-1,b}=E(-1)[A^{*b-1}E(-1)^*\tau^*_{-b,1}+(I^*+\cdots+A^{*b-2})B^*p^*]$$

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so that

$$E(-1)^*\tau^*_{-b,1}=-[E(-1)^*A^{*b-1}E(-1)^*]^{-1}[I^*+\cdots+A^{*b-2}]B^*p^*. \tag{12}$$

Replacing $-b$ by $a-b+1$ in (12) and using (11) now gives us

$$\tau^*_{a,b}=(I^*+\cdots+A^{*b-2})B^*p^*-A^{*b-2}[E(-1)^*A^{*b-a-2}E(-1)^*]^{-1}(I^*+\cdots +A^{*b-2})B^*p^*$$

or, equivalently,

$$\int \tau_{ab}(\omega)\varphi(d\omega)=p^*B(I+\cdots+A^{b-2})\varphi-p^*B(I+\cdots+A^{b-2}) \times[E(-1)A^{b-a-2}E(-1)]^{-1}A^{b-1}\varphi$$

which completes the proof of the theorem.

As an example, suppose that each probability measure $\varphi\in\Phi$ represents a temporally homogeneous Markov chain on the integers $\{-1,+1\}$. Then each $\varphi\in\Phi$ can be identified with its initial distribution so that Φ can be thought of as R^2 via $\varphi\to(\varphi(X_0=-1),\ \varphi(X_0=+1))$. The operators T^*, $E(-1)^*$, $E(+1)^*$ and the linear functional p^* now become, in matrix form, using $(1,0)$ and $(0,1)$ as a basis of R^2,

$$T^*=\begin{pmatrix}p_{-1,-1} & p_{-1,+1}\\ p_{+1,-1} & p_{+1,+1}\end{pmatrix},\qquad E(-1)^*=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad E(+1)^*=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad p^*=\begin{pmatrix}1\\1\end{pmatrix}.$$

In the case of the standard random walk, the measure φ represents a sequence of independent, identically distributed random variables so that

$$T^*=\begin{pmatrix}q&p\\q&p\end{pmatrix},\qquad q=\varphi(X_k=-1),\qquad p=\varphi(X_k=+1).$$

In this latter case, the operator A^* is given by

$$A^*=[\lambda E(+1)^*+E(-1)^*T^*]^{-1}[E(-1)^*+\lambda E(+1)^*T^*]=q^{-1}\begin{pmatrix}1-pq & -p^2\\ q^2 & pq\end{pmatrix}$$

so that, after some computation, we have the well known result

$$\varphi(X_0+\cdots+X_{\tau_{ab}}=a)=A^{*b-1}[E(-1)^*A^{*b-a-2}E(-1)^*]^{-1}E(-1)^*p^*\varphi$$

$$=\begin{cases}\dfrac{\left(\frac{q}{p}\right)^{-1-a}-\left(\frac{q}{p}\right)^{b-q}}{1-\left(\frac{q}{p}\right)^{b-a}} & \text{if } \varphi=(1,0),\text{ i.e. } X_0=-1\\[2ex] \dfrac{\left(\frac{q}{p}\right)^{1-a}-\left(\frac{q}{p}\right)^{b-a}}{1-\left(\frac{q}{p}\right)^{b-a}} & \text{if } \varphi=(0,1),\text{ i.e. } X_0=+1.\end{cases}$$

https://doi.org/10.4153/CMB-1980-047-2 Published online by Cambridge University Press

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https://doi.org/10.4153/CMB-1980-047-2 Published online by Cambridge University Press