Can. J. Math., Vol. XXVIII, No. 1, 1976, pp. 186–200

COMPATIBLE TIGHT RIESZ ORDERS

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1. Introduction. N. R. Reilly has obtained an algebraic characterization of the compatible tight Riesz orders that can be supported by certain partially ordered groups [**13**; **14**]. The purpose of this paper is to give a "geometric" characterization by the use of ordered permutation groups. Our restrictions on the partially ordered groups will likewise be geometric rather than algebraic. Davis and Bolz [**3**] have done some work on groups of all order-preserving permutations of a totally ordered field; from our more general theorems, we will be able to recapture their results.

The expression (G, S) will be used to indicate that G is an l-subgroup of $A(S)$. the lattice-ordered group (l-group) of all order-preserving permutations of the totally ordered set S under the point-wise ordering. Any such G has a (unique) natural extension to $(G, \bar{S})$, where $\bar{S}$ is the Dedekind completion of S (without end points). For any $g \in G$, fps $(g, S) = \{s \in S : sg = s\}$ and supp $(g, S) = S \backslash$fps (g, S); fps $(g, \bar{S})$ and supp $(g, \bar{S})$ are defined analogously where g is identified with its extension to an order-preserving permutation of $\bar{S}$. An *o-block* of (G, S) is a convex subset C of S such that for each $g \in G$, $Cg = C$ or $Cg \cap C = \emptyset$. A *convex congruence* of (G, S) is an equivalence relation on S which is respected by G and all of whose classes are convex subsets of S. The classes of a convex congruence are o-blocks and, if (G, S) is transitive, the translates of any o-block of (G, S) by elements of G gives rise to a convex congruence of (G, S). If $\mathscr{B}$ and $\mathscr{C}$ are convex congruences of (G, S), we write $\mathscr{B} \leqq \mathscr{C}$ if and only if $\mathscr{B}$ refines $\mathscr{C}$. If (G, S) is transitive, then the set of convex congruences of (G, S) is totally ordered by $\leqq$, and the set of o-blocks of (G, S) containing any given $s \in S$ is totally ordered by inclusion. In particular, if B and C are o-blocks containing $s \in S$, which give rise to convex congruences $\mathscr{B}$ and $\mathscr{C}$, respectively, then $\mathscr{B} \leqq \mathscr{C}$ if and only if $B \subseteq C$. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are convex congruences of (G, S) such that $\mathscr{B}_1 < \mathscr{B}_2$ and no convex congruence of (G, S) lies between $\mathscr{B}_1$ and $\mathscr{B}_2$, then we say that $(\mathscr{B}_1, \mathscr{B}_2)$ is a *covering pair of convex congruences* of (G, S). The set of convex congruences of (G, S) will be denoted by $\Gamma(G, S)$ and the γth covering pair by $(\mathscr{S}_\gamma, \mathscr{S}^\gamma)$. Observe that if G is transitive on S, then $\Gamma(G, S)$ is a totally ordered set.

A transitive ordered permutation group is said to be *o-primitive* provided its only convex congruences are the two improper convex congruences. Each covering pair $(\mathscr{S}_\gamma, \mathscr{S}^\gamma)$ yields an *o-primitive component* (G_γ, S_γ) in the following way: Choose any $s \in S$ and let $S_\gamma = s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, the $\mathscr{S}^\gamma$ equivalence class of s

Received April 22, 1975 and in revised form, August 16, 1975.

modulo the $\mathcal{S}_\gamma$ classes contained in $s\mathcal{S}^\gamma$. Let G_γ denote the action of G_x on S_γ where $x = \sup s\mathcal{S}^\gamma$ and $G_x = \{g \in G : xg = x\}$ (note that this is not, in general, a faithful representation of G_x on S_γ). Let

$$G_{(s\mathcal{S}^\gamma)} = \{g \in G : (s\mathcal{S}^\gamma)g = s\mathcal{S}^\gamma\} = G_x.$$

To obtain a faithful action on $s\mathcal{S}^\gamma$ it is necessary to factor out a convex normal subgroup of $G_{(s\mathcal{S}^\gamma)}$. The resulting group will be denoted by G_γ. Hence we obtain G_γ from $G_{(s\mathcal{S}^\gamma)}$ in a natural (one-to-one) way. If G is transitive on S, (G_γ, S_γ) is o-primitive and independent (to within isomorphism) of the choice of s.

The set $\Gamma(G, S)$ and the o-primitive components of (G, S) will play a central role in the compatible tight Riesz orders on ordered permutation groups (G, S) for which G is an l-group—henceforth called l-permutation groups—since they are the building blocks of every transitive l-permutation group.

If (G, S) has a minimal o-primitive component (i.e., associated with a minimal covering pair) (G_μ, S_μ), then (G, S) is said to be *locally o-primitive* and the $\mathcal{S}^\mu$ classes are called the *primitive segments*.

If (G, S) is an o-primitive l-permutation group, then by [**6**; **7**; **9** and **10**], there are just these four possibilities:

(i) (G, S) is *regular* and *archimedean*; $G_{\bar{s}} = \{e\}$ for each $\bar{s} \in \bar{S}$, G is isomorphic to S as an ordered set, and is o-isomorphic to a subgroup of the real numbers [**12**] (e is the group identity).

(ii) (G, S) is *periodic*; there exists $e < f_0 \in A(\bar{S})$ such that for all $g \in G$. $f_0 g = g f_0$, and for each $\bar{s} \in \bar{S}$, $G_{\bar{s}}$ fixes only the points of the coterminal subset $\{\bar{s}f_0^m : m = 0, \pm 1, \pm 2, \ldots\}$, and $G_{\bar{s}}$ is o-2 transitive on the interval $(\bar{s}, \bar{s}f_0)$ and contains an element whose support in $(\bar{s}, \bar{s}f_0)$ is non-empty and bounded. The permutation f_0 is the *period* of G and $G \subseteq C_{A(\bar{S})}(f_0) \cap A(S)$, where $C_{A(\bar{S})}(f_0)$ is the centralizer of $\{f_0\}$ in $A(\bar{S})$. Either there exists a positive integer n such that for $s \in S$, $sf_0^m \in S$ if and only if n divides m—in which case (G, S) is said to have Config (n)—or $sf^m \in S$ if and only if $m = 0$, and (G, S) is said to have Config (∞).

(iii) (G, S) is *o-2 transitive* (if $x < y$ and $z < t$ in S, there exists $g \in G$ such that $xg = z$ and $yg = t$) and contains a non-identity element of bounded support.

(iv) (G, S) is *pathological*; (G, S) is o-2 transitive and contains no non-identity element of bounded support.

The *wreath product* of two l-permutation groups $(G, S)\mathrm{Wr}(H, T)$ is the l-group of all order-preserving permutations of $S \times T$ of the form $(\{g_t : t \in T\}, h)$ where $h \in H$, $g_t \in G$ and $(s, t)(\{g_t\}, h) = (sg_t, th)$. This can be generalized to the wreath product of infinitely many factors indexed by a totally ordered set Γ, written as $\mathrm{Wr}\{(H_\gamma, T_\gamma) : \gamma \in \Gamma\}$ (see [**8**]).

If $\mathcal{L}$ is a lattice with minimal element 0, then $\mathcal{F} \subseteq \mathcal{L}$ is a *filter on* $\mathcal{L}$ if

(i) $\mathcal{F} \neq \emptyset$,

(ii) $x, y \in \mathcal{F}$ imply $x \wedge y \in \mathcal{F}$, and

https://doi.org/10.4153/CJM-1976-024-4 Published online by Cambridge University Press

(iii) $x \in \mathscr{F}$, $z \in \mathscr{L}$ and $x \leqq z$ imply $z \in \mathscr{F}$.
If $0 \notin \mathscr{F}$, then $\mathscr{F}$ is said to be a *proper filter*.

Let G be an l-group and suppose $T \subseteq G^+$ such that G is a tight Riesz group under the order $<$ defined by: $g < h$ if and only if $hg^{-1} \in T$ ($g, h < f, k$ implies there exists $z \in G$ such that $g, h < z < f, k$). If $T \cup$ {pseudo-positive elements of $(G, <)$} $\cup \{e\} = G^+$, then T is called a *compatible tight Riesz order* on G. In [**15**], Andrew Wirth proved that T is a compatible tight Riesz order on G if and only if

(T1) T is a proper filter on G^+.
(T2) $T \cdot T = T$.
(T3) T is a normal subset of G (written $T \lhd G$), and
(T4) $\inf T = \{e\}$.

If conditions (T1), (T2) and (T3) obtain, we will say that T is a *pseudo-compatible tight Riesz order.*

In [**5**], W. Charles Holland proved that every l-group can be l-embedded as the cardinal product of transitive l-permutation groups. So all compatible tight Riesz orders on an l-group G can be obtained from hybrid products if we know all the compatible tight Riesz orders on transitive l-permutation groups (see [**14**]). From now on, therefore, we will assume that (G, S) is a transitive l-permutation group.

In order to understand the results of this paper and the need for certain restrictions, we will first examine some examples.

Example 1. Let S be an Ohkuma set not isomorphic to $\mathbf{Z}$, the totally ordered set of integers. $A(S)$ is an o-group with compatible tight Riesz order $A(S)^* = \{g \in A(S) : g > e\}$, but $A(S)$ is not divisible (see [**4**, Theorem 3.2.4]).

Example 2. Let $S = \mathbf{R}$, the totally ordered set of real numbers. Let $G = \{g \in A(S) : (\forall r \in \mathbf{R})((r+1)g = rg + 1)\}$. Then (G, S) is a periodic group with period $f_0 : x \mapsto x + 1$. Let $T = \{g \in G^+ : \text{fps}\ (g, S) \cap [0, 1]$ has measure $0\}$. Then T is a compatible tight Riesz order.

Example 3. Let $S = \mathbf{R}$ and

$$G = \{g \in A(S) : (\exists\, n \in \mathbf{Z}^+)(\forall r \in \mathbf{R})((r+n)g = rg + n)\}$$

where $\mathbf{Z}^+ = \{n \in \mathbf{Z} : n > 0\}$. (G, S) is pathological with compatible tight Riesz order $T = \{g \in G^+ : \text{fps}\ (g, S)$ has measure $0\}$.

Example 4. Let $S = \mathbf{R} \overleftarrow{\times} \mathbf{R}$ and consider $(A(S), S)$. Let $T = \{g \in A(S)^+ : g = (\{g_r\}, \bar{g})$ and fps $(\bar{g}, \mathbf{R})$ has measure $0\}$. Then T is a compatible tight Riesz order and the intersection of T with any local component is order-isomorphic to $A(\mathbf{R})^+$—which is not a compatible tight Riesz order for $A(\mathbf{R})$.

Example 5. Let $(G, S) = \text{Wr}\{(A(\mathbf{R}), \mathbf{R}) : \gamma \in \mathbf{Z}^-\}$. Let

$$T = \{g \in G : sg > s \text{ for all } s \in S\}.$$

Then T is a compatible tight Riesz order for (G, S).

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Example 6. Let $(G, S) = \mathrm{Wr}\{(A(\mathbf{R}), \mathbf{R}) : \gamma \in \mathbf{Z}^-\}$. Let

$$T = \{g \in G : (\exists \gamma \in \mathbf{Z}^-)(s\mathscr{S}_\gamma)g > s\mathscr{S}_\gamma \text{ for all } s \in S\}.$$

Then T is a compatible tight Riesz order for (G, S) which is contained in the compatible tight Riesz order of Example 5.

For further background information see the references already cited together with [**2**; **11** and **1**].

2. Existence of compatible tight Riesz orders.

Example 7. Let $S = \mathbf{R}$ and G be the l-group of all elements of $A(\mathbf{R})$ which have bounded support. Then (G, S) is an o-2 transitive simple l-permutation group that is divisible (therefore, dense). However, (G, S) has no compatible tight Riesz orders even though S is a totally ordered field. For if T were a proper normal filter on G^+, let $e \neq g \in T$. There exist $a < b$ in $\mathbf{R}$ such that supp $(g, S) \subseteq [a, b]$. Let $r > b$ and $f \in G$ be such that $af = r$. Now supp $(f^{-1}gf, S) \subseteq [r, bf]$ so $f^{-1}gf \wedge g = e$. Since $g \in T, f^{-1}gf \in T$ and so $e \in T$, a contradiction.

Indeed, we have shown:

THEOREM 2.1. *If (G, S) is a transitive l-permutation group and T is a compatible tight Riesz order on G, then T contains no element of bounded support. In particular, if (G, S) is a transitive l-permutation group, $G \neq \{e\}$ and (G, S) contains no elements of unbounded support, then G has no compatible tight Riesz order.*

The converse is false since $(\mathbf{Z}, \mathbf{Z})$ has no non-identity element of bounded support but $\mathbf{Z}$ has no compatible tight Riesz order. Also see Theorem 3.7.

Let (G, S) be an l-permutation group. $\{g_i : i \in I\}$ a set of elements of G is said to be *strongly pairwise disjoint* if there exists a convex congruence $\mathscr{C}$ on S and $\{s_i : i \in I\} \subseteq S$ such that $s_i\mathscr{C} \neq s_j\mathscr{C}$ if $i \neq j$ and supp $(g_i, S) \subseteq s_i\mathscr{C}$ for all $i \in I$. If (G, S) is an l-permutation group such that every strongly pairwise disjoint subset of (G, S) has a supremum in G, then (G, S) is said to be *weakly laterally complete*. (G, S) is *weakly depressible* if for any o-block Δ of (G, S) and $g \in G$, if $\Delta g = \Delta$, there exists $f \in G$ such that

$$sf = \begin{cases} sg \text{ if } s \in \Delta \\ s \text{ if } s \notin \Delta \end{cases}.$$

If Δ is a γ-class for a minimal $\gamma \in \Gamma(G, S)$ and γ is static, then Δ is said to be a *dead segment* of (G, S). If for every natural o-block Δ of (G, S) that is not dead or a singleton, there exists $e \neq g \in G$ such that supp $(g, S) \subseteq \Delta$, we say that (G, S) enjoys the *support property*. If (G, S) is transitive and weakly depressible, it clearly has the support property. Observe that if $(A(S), S)$ is transitive, it is weakly depressible and weakly laterally complete.

THEOREM 2.2. *Let (G, S) be an l-permutation group that is weakly depressible and weakly laterally complete. Further suppose that (G, S) is locally o-primitive but not o-primitive and let T_0 be a pseudo-compatible tight Riesz order on G_μ, where μ is the least element of $\Gamma = \Gamma(G, S)$. Then (G, S) has a compatible tight Riesz order.*

Proof. Assume the hypotheses of the theorem. Let $\mathscr{F}$ be the filter on $\{s\mathscr{S}^\mu : s \in S\}$ defined by $\{s_i\mathscr{S}^\mu : i \in I\} \in \mathscr{F}$ if and only if $\{s\mathscr{S}^\mu : s\mathscr{S}^\mu \neq s_i\mathscr{S}^\mu$ for all $i \in I\}$ is finite. Define T_1 by: $g \in T_1$ if and only if $g \in G^+$, $(s\mathscr{S}^\mu)g = s\mathscr{S}^\mu$ for all $s \in S$ and $\{s\mathscr{S}^\mu : g|s\mathscr{S}^\mu \in T_0\} \in \mathscr{F}$. Clearly $T_1 \lhd G$, $e \notin T_1$, $T_1 \cdot T_1 = T_1$ and if $g, h \in T_1$, then $g \wedge h \in T_1$. Moreover, since G is weakly depressible and weakly laterally complete, $T_1 \neq \emptyset$ and $\inf\ (T_1) = e$. Thus $T = \{g \in G : g \geqq f$ for some $f \in T_1\}$ is a compatible tight Riesz order for G.

COROLLARY 2.3. *Let (G, S) be an l-permutation group that is weakly laterally complete, weakly depressible and locally o-primitive. If the local component has a compatible tight Riesz order, then G has a compatible tight Riesz order.*

COROLLARY 2.4. *Let $(A(S), S)$ be transitive and locally o-primitive. If the local component has a compatible tight Riesz order, then so does $A(S)$.*

We now turn our attention to the case when (G, S) is a transitive l-permutation group that is not locally o-primitive. So $\Gamma = \Gamma(G, S)$ has no least element.

THEOREM 2.5. *Let (G, S) be a transitive l-permutation group that is weakly laterally complete, weakly depressible and is not locally o-primitive. Suppose that $\Gamma' \subseteq \Gamma$ is coinitial in Γ and that G_γ has a pseudo-compatible tight Riesz order for each $\gamma \in \Gamma'$. Then (G, S) has a compatible tight Riesz order* (cf. Theorem 4.11).

Proof. For each $\gamma \in \Gamma'$, let $\mathscr{F}_\gamma$ be the filter on $\{s\mathscr{S}^\gamma : s \in S\}$ defined by: $\{s_i\mathscr{S}^\gamma : i \in I\} \in \mathscr{F}_\gamma$ if and only if $\{s\mathscr{S}^\gamma : s\mathscr{S}^\gamma \neq s_i\mathscr{S}^\gamma$ all $i \in I\}$ is bounded in $\{s\mathscr{S}^\gamma : s \in S\}$. For each $g \in G$, $\gamma \in \Gamma$ and $s \in S$, if $(s\mathscr{S}^\gamma)g = s\mathscr{S}^\gamma$, let $g_{\gamma,s}$ be the image of g in $(G_\gamma, s\mathscr{S}^\gamma/\mathscr{S}_\gamma) \cong (G_\gamma, S_\gamma)$. For each $\gamma \in \Gamma'$, let $T_\gamma' = \{g \in G^+ : (s\mathscr{S}^\gamma)g = s\mathscr{S}^\gamma$ and $\{s\mathscr{S}^\gamma : g_{\gamma,s} \in T_\gamma\} \in \mathscr{F}_\gamma\}$ where T_γ is the pseudo-compatible tight Riesz order on (G_γ, S_γ). T_γ' is clearly a normal subset of G^+, $T_\gamma' \cdot T_\gamma' = T_\gamma'$, $e \notin T_\gamma'$ and if $g_1, g_2 \in T_\gamma'$, then $g_1 \wedge g_2 \in T_\gamma'$. Let T be the filter on G^+ generated by $\{T_\gamma' : \gamma \in \Gamma'\}$. Clearly, T is a normal subset of G^+, $T \cdot T = T$ and since Γ' is coinitial in Γ, (G, S) is transitive, weakly laterally complete and weakly depressible, $T \neq \emptyset$ and $\inf T = e$. Finally, if $g^{(i)} \in T_{\gamma_i}'$ $(i = 1, 2)$ and $\gamma_1 < \gamma_2$, then there exist $u, v \in S$, $u < v$ such that $g_{\gamma,s}{}^{(i)} \in T_{\gamma_i}$ if $s\mathscr{S}^{\gamma_2} \supseteq u\mathscr{S}^{\gamma_2}$ or $s\mathscr{S}^{\gamma_2} \subseteq v\mathscr{S}^{\gamma_2}$ $(i = 1, 2)$. Now for each such s, by Theorem 2.1, there exist $x, y \in S$, $x < y$ such that for all $z \in s\mathscr{S}^{\gamma_2}$, $z \leqq x$ or $z \geqq y$ implies $(z\mathscr{S}_{\gamma_2})g^{(2)} > z\mathscr{S}_{\gamma_2}$. Hence for any such z, $(z\mathscr{S}_{\gamma_1})(g^{(1)} \wedge g^{(2)}) = z\mathscr{S}_{\gamma_1}g^{(1)}$ and this exceeds $z\mathscr{S}_{\gamma_1}$ for sufficiently large z in $z\mathscr{S}^{\gamma_1}$. Hence $g^{(1)} \wedge g^{(2)} \neq e$. Similarly, if $g_1, \ldots, g_n \in T$, then $g_1 \wedge \ldots \wedge g_n \neq e$. Thus T is a compatible tight Riesz order for G.

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COROLLARY 2.6. *If $(A(S), S)$ is transitive, not locally o-primitive and has a coinitial set of components each of which has a compatible tight Riesz order, then $A(S)$ has a compatible tight Riesz order.*

COROLLARY 2.7. *If $(A(S), S)$ is transitive and every o-primitive component supports a compatible tight Riesz order, then $A(S)$ has a compatible tight Riesz order.*

We now consider the relation between certain filters on (the set of all subsets of) S and compatible tight Riesz orders on G. Let $\mathscr{G}$ be a filter on $S(\bar{S})$. $\mathscr{G}$ is said to be *G-invariant* if $\mathscr{G}$ is a proper filter on S $(\bar{S})$ and for each $X \subseteq S$ $(X \subseteq \bar{S})$, if $X \in \mathscr{G}$ then $Xg \in \mathscr{G}$ for all $g \in G$.

THEOREM 2.8. *Let (G, S) be an l-permutation group.*
1. *If T is a compatible tight Riesz order on G, then*

$$\mathscr{G}(T, S) = \{X \subseteq S : X \supseteq \text{supp}\,(g, S) \text{ for some } g \in T\} \text{ and}$$
$$\mathscr{G}(T, \bar{S}) = \{X \subseteq \bar{S} : X \supseteq \text{supp}\,(g, \bar{S}) \text{ for some } g \in T\}$$

are G-invariant filters on S and $\bar{S}$ respectively.
2. *Let $\mathscr{G}$ be a G-invariant filter on S (or $\bar{S}$). Then*

$$\mathscr{T}(\mathscr{G}) = \{g \in G^+ : \text{supp}\,(g, S) \supseteq F \text{ for some } F \in \mathscr{G}\}$$
$$(= \{g \in G^+ : \text{supp}\,(g, \bar{S}) \supseteq F \text{ for some } F \in \mathscr{G}\})$$

is a normal subset of G^+ that is either empty or a proper filter on G^+. If $\bigcap \mathscr{G}$ contains the support of no element of G other than the identity, then either $\mathscr{T}(\mathscr{G}) = \emptyset$ or $\inf\,(\mathscr{T}(\mathscr{G})) = e$.

Proof. Routine verification.

In any case, observe that if $T_1 \subseteq T_2$ are compatible tight Riesz orders for G, then $\mathscr{G}(T_1, S) \subseteq \mathscr{G}(T_2, S)$, $\mathscr{G}(T_1, \bar{S}) \subseteq \mathscr{G}(T_2, \bar{S})$ and $T_i \subseteq \mathscr{T}(\mathscr{G}(T_i, S)) \subseteq \mathscr{T}(\mathscr{G}(T_i, \bar{S}))$ $(i = 1, 2)$. If $T = \mathscr{T}(\mathscr{G}(T, S))$ or $T = \mathscr{T}(\mathscr{G}(T, \bar{S}))$, then we will say that T is a *full* compatible tight Riesz order (cf. [**1**]). Davis and Bolz [**3**] use the word "algebraic" instead of full. In view of Ball's work (op. cit.), we prefer, for technical reasons, the use of "full". Also if $\mathscr{G}$ is a G-invariant filter on S or $\bar{S}$, then $\mathscr{G}(\mathscr{T}(\mathscr{G}), S) \subseteq \mathscr{G}$ (respectively, $\mathscr{G}(\mathscr{T}(\mathscr{G}), \bar{S}) \subseteq \mathscr{G}$) provided $\mathscr{T}(\mathscr{G})$ is a compatible tight Riesz order on G. Moreover, if $\mathscr{G}_1 \subseteq \mathscr{G}_2$ are G-invariant filters on S or $\bar{S}$, then $\mathscr{T}(\mathscr{G}_1) \subseteq \mathscr{T}(\mathscr{G}_2)$.

3. *o*-primitive *l*-permutation groups. Let (G, S) be an l-permutation group that is o-primitive. If G is static, i.e., $G = \{e\}$ and $|S| > 1$, then vacuously G has a compatible tight Riesz order whereas if G is integrally derived, then it clearly does not. Any other o-primitive l-permutative group is derived from an o-primitive transitive l-permutation group (not *isomorphic* to $(A(Z), Z)$) in the manner described in [**11**] and it is enough to look at *transitive* o-primitive l-permutation groups to discover the compatible tight Riesz orders on all (non-static and not integrally derived) o-primitive l-permutation groups.

LEMMA 3.1. *Let (G, S) be an o-2 transitive l-permutation group ($|S| > 2$) and $\mathscr{G}$ a G-invariant filter on S or $\bar{S}$. Then $\mathscr{T}(\mathscr{G}) = \emptyset$ or $\inf \mathscr{T}(\mathscr{G}) = e$.*

Proof. Assume $\mathscr{T}(\mathscr{G}) \neq \emptyset$.

Case 1. For every $h \in \mathscr{T}(\mathscr{G})$, supp $(h, S) = S$. Suppose $e < g \leqq \inf \mathscr{T}(\mathscr{G})$. Let $y \in$ supp (g, S). Let $h \in \mathscr{T}(\mathscr{G})$. Then $yh > y$. There exists $f \in G$ such that $yf = y$ and $y < yhf < yg$. Now $f^{-1}hf \in \mathscr{T}(\mathscr{G})$ so $g \leqq f^{-1}hf$. Thus $yg \leqq yf^{-1}hf = yhf < yg$, a contradiction. Hence $\inf \mathscr{T}(\mathscr{G}) = e$.

Case 2. There exists $h \in \mathscr{T}(\mathscr{G})$ such that $zh = z$ for some $z \in S$. Let $e < g \leqq \inf \mathscr{T}(\mathscr{G})$. Let $y \in$ supp (g, S). Since (G, S) is o-primitive, $zG \cap (y, yg) \neq \emptyset$. Hence let $f \in G$ by such that $zf \in (y, yg)$. Now $f^{-1}hf \in \mathscr{T}(\mathscr{G})$ so $g \leqq f^{-1}hf$. Thus $(zf)g \leqq (zf)f^{-1}hf = zhf = zf$. But $y < zf$ so $(zf)g \leqq zf < yg < (zf)g$, a contradiction. Thus $\inf \mathscr{T}(\mathscr{G}) = e$.

If (G, S) has only elements of bounded support, $\mathscr{T}(\mathscr{G}) = \emptyset$ (in compliance with Theorem 2.1).

LEMMA 3.2. *Let (G, S) be a periodic o-primitive l-permutation group and $\mathscr{G}$ a G-invariant filter on S or $\bar{S}$. Then $\mathscr{T}(\mathscr{G}) = \emptyset$ or $\inf \mathscr{T}(\mathscr{G}) = e$.*

Proof. Assume $\mathscr{T}(\mathscr{G}) \neq \emptyset$. If $\mathscr{T}(\mathscr{G})$ contains an element h such that h fixes some point of S, then $\inf \mathscr{T}(\mathscr{G}) = e$ as in Case 2 of Lemma 3.1. So assume every $h \in \mathscr{T}(\mathscr{G})$ fixes no point of S. Suppose $e < g \leqq \inf \mathscr{T}(\mathscr{G})$. Let $y \in$ supp (g, S). We first show that we may assume that $yg \leqq yf_0$ where f_0 is the period of (G, S). If $yg > yf_0$, let $z \in (yf_0, yg) \cap S$. There exists $h \in G$ such that $yh = z$. Then h fixes no point of (y, yf_0). Hence supp $(h, S) = S$. Therefore $h \in \mathscr{T}(\mathscr{G})$. Hence $h \geqq g$. But $z = yh \geqq yg > z$, a contradiction, as claimed. Now assume $yg \leqq yf_0$. Since $f_0 g = gf_0$, it follows that $yf_0 g = ygf_0 > yf_0$ so there exists $w \in (y, yf_0) \cap S$ such that if $x \in S \cap (y, yf_0)$ and $x \geqq w$, then $xg > x$. There exists $e < f \in G_y$ such that $zf = w$ where $z \in (y, yg) \cap S$. Let $h = f \vee g$. Then h fixes no point of (y, yf_0) and hence $h \in \mathscr{T}(\mathscr{G})$. But $yh = \max\{yf, yg\} \leqq \max\{zf, yg\} = \max\{w, yg\}$. Thus $yh < yf_0$ if $g \neq f_0$. There exists $k \in G_y$ such that $yhk < yg$. Hence $yk^{-1}hk = yhk < yg$. Hence $k^{-1}hk \ngeqq g$ but $k^{-1}hk \in \mathscr{T}(\mathscr{G})$, a contradiction. Therefore, $g = f_0$. Let $z \in (y, yf_0) \cap S$. Let $z < u < yf_0$ with $u \in S$. So $uf_0^{-1} < y < z < u$. Consequently, there exists $e < g' \in G_u$ such that $yg' = z$. Let $x \in S$. There exists $n \in \mathbf{Z}$ such that $x \in [uf_0^n, uf_0^{n+1})$. Now $xg' < uf_0^{n+1}g' = ug'f_0^{n+1} = uf_0^n f_0 \leqq xf_0$. Thus $e < g' < f_0 \leqq \inf \mathscr{T}(\mathscr{G})$ and we have the previous case. This yields the required contradiction and so $\inf \mathscr{T}(\mathscr{G}) = e$.

LEMMA 3.3. *If (G, S) is an o-primitive l-permutation group and G is an o-group not o-isomorphic to $\mathbf{Z}$, then $\inf \mathscr{T}(\mathscr{G}) = e$ for any G-invaraint filter $\mathscr{G}$ on S or $\bar{S}$.*

Proof. $\mathscr{T}(\mathscr{G}) = \{g \in G^+ : g > e\} \neq \emptyset$ and since G is dense, $\inf \mathscr{T}(\mathscr{G}) = e$.

Let X be a subset of G^+. X is said to be *factorable* if for each $g \in X$, there exist $g_1, g_2 \in X$ such that $g_1 g_2 = g$ and supp $(g_i, \bar{S}) =$ supp $(g, \bar{S})$ $(i = 1, 2)$. In the case that G^+ is factorable, we will simply say that G is factorable. Clearly, if G is divisible, then it is factorable. However, the converse is false. For example, let G_1 be the l-subgroup of $A(\mathbf{R})$ of eventually (initially and finally) constant functions; i.e., there exist $y, z \in \mathbf{R}$ such that for all $x \in \mathbf{R}$, if $x \geqq y$ or $x \leqq z$, then $xg - x = yg - y = zg - z$. Let G be the l-subgroup of G_1 containing translations by $n/3^m$ ($n \in \mathbf{Z}$, $m \in \mathbf{Z}^+$) and maximal with respect to not containing any square roots of translation by $+1$. Thus G is not divisible. But G is o-2 transitive (since it contains all elements of bounded support and is clearly factorable since inf $\{n/3^m : n, m \in \mathbf{Z}^+\} = 0$.

It should be observed that if G is factorable, so is every full subset of G^+ ($X \subseteq G^+$ is *full* if supp $(g, \bar{S})$ = supp $(h, \bar{S})$ and $h \in X$ imply $g \in X$). Moreover, if $A(S)$ is transitive and o-primitive, then it is factorable if S is not order-morphic to $\mathbf{Z}$. Consequently:

THEOREM 3.4. *Let (G, S) be an o-primitive l-permutation group which is not static or integrally derived. If $\mathscr{G}$ is a G-invariant filter on S or $\bar{S}$ such that $\mathscr{T}(\mathscr{G})$ is factorable and non-empty, then $\mathscr{T}(\mathscr{G})$ is a compatible tight Riesz order for G.*

COROLLARY 3.5. *Let (G, S) be an o-primitive l-permutation group which is not static or integrally derived. If G is factorable and $\mathscr{G}$ is a G-invariant filter on S or $\bar{S}$ with $\mathscr{T}(\mathscr{G}) \neq \emptyset$, then $\mathscr{T}(\mathscr{G})$ is a compatible tight Riesz order for G.*

COROLLARY 3.6. *If $(A(S), S)$ is transitive, o-primitive and not isomorphic to $(A(Z), Z)$, and $\mathscr{G}$ is an $A(S)$-invariant filter on S or $\bar{S}$, then $\mathscr{T}(\mathscr{G}) = \emptyset$ or $\mathscr{T}(\mathscr{G})$ is a compatible tight Riesz order for $A(S)$.*

We have been unable to decide whether G is factorable if (G, S) is a periodic o-primitive l-permutation group. However, we believe this to not be the case in general. More importantly, we have been unable to show that if $(A(S), S)$ is o-2 transitive, it has a compatible tight Riesz order (If $(A(S), S)$ is transitive, o-primitive and $A(S)$ is not an o-group, $(A(S), S)$ is o-2 transitive [**6**].). For example, if S is the long line, does $(A(S), S)$ have a compatible tight Riesz order? (*Added in proof*: Richard N. Ball (and independently Gary Davis and Colin D. Fox) have shown that if $(A(S), S)$ is o-2 transitive, it has a compatible tight Riesz order. The not too dense reader should be able to come up with a solution!)

The following are examples of G-invariant filters $\mathscr{G}$ on $\bar{S}$ that *may* give rise to $\mathscr{T}(\mathscr{G}) \neq \emptyset$:

$\mathscr{G}_1 = \{\bar{S}\}$
$\mathscr{G}_2 = \{X \subseteq \bar{S} : X \supseteq S\}$
$\mathscr{G}_3 = \{X \subseteq \bar{S} : X \supseteq \text{initial segment of } \bar{S}\}$
$\mathscr{G}_4 = \{X \subseteq \bar{S} : X \supseteq \text{final segment of } \bar{S}\}$
$\mathscr{G}_5 = \{X \subseteq \bar{S} : X \supseteq \text{initial segment of } S\}$
$\mathscr{G}_6 = \{X \subseteq \bar{S} : X \supseteq \text{final segment of } S\}$
$\mathscr{G}_7 = \{X \subseteq \bar{S} : \text{the complement of } X \text{ in } \bar{S} \text{ is nowhere dense in } \bar{S}\} =$

$\{X \subseteq \bar{S}$: the complement of $X \cap S$ in S is nowhere dense in $S\}$

$\mathscr{G}_8 = \{X \subseteq \bar{S}$: complement of X in $\bar{S}$ is bounded$\}$

$\mathscr{G}_9 = \{X \subseteq \bar{S}$: complement of $X \cap S$ in S is bounded$\}$

$\mathscr{G}_{10,\alpha} = \{X \subseteq \bar{S}$: complement of X in $\bar{S}$ has cardinality $< \aleph_\alpha\}$

$\mathscr{G}_{11,\alpha} = \{X \subseteq \bar{S}$: complement of X in S has cardinality $< \aleph_\alpha\}$.

Two good candidates for $\aleph_\alpha$ in $\mathscr{G}_{10,\alpha}$ and $\mathscr{G}_{11,\alpha}$ are $\aleph_0$ and $|\bar{S}|$ in $\mathscr{G}_{10,\alpha}$ ($|S|$ in $\mathscr{G}_{11,\alpha}$).

$\mathscr{G}_{12} = \{X \subseteq \bar{S} : X$ is cofinal in $\bar{S}\}$

$\mathscr{G}_{13} = \{X \subseteq \bar{S} : X$ is coinitial in $\bar{S}\}$

$\mathscr{G}_{14} = \{X \subseteq \bar{S} : X \cap S$ is cofinal in $S\}$

$\mathscr{G}_{15} = \{X \subseteq \bar{S} : X \cap S$ is coinitial in $S\}$

$\mathscr{G}_{16} = \{X \subseteq \bar{S} : X$ is not well-ordered$\}$

$\mathscr{G}_{17} = \{X \subseteq \bar{S} : X$ is not inversely well-ordered$\}$

$\mathscr{G}_{18} = \{X \subseteq \bar{S} : X \cap S$ is not well-ordered$\}$

$\mathscr{G}_{19} = \{X \subseteq \bar{S} : X \cap S$ is not inversely well-ordered$\}$.

Note that $\mathscr{G}_1$ is contained in all G-invariant filters on $\bar{S}$ so if $\mathscr{T}(\mathscr{G}_1) \neq \emptyset$, $\mathscr{T}(\mathscr{G}) \neq \emptyset$ for each G-invariant filter on $\bar{S}$. If $\bar{S}$ is a totally ordered field, $\mathscr{T}(\mathscr{G}_1) \neq \emptyset$. If S is a totally ordered field, $\mathscr{T}(\mathscr{G}_2) \neq \emptyset$ whence $\mathscr{T}(\mathscr{G}_i) \neq \emptyset$ where $i = 5, 6, 7, 9, 11, \alpha, 14, 15, 18$ or 19. This recovers all of Davis and Bolz's results in [**3**], since Theorem 3.4 obtains (If S is a totally ordered field, $(A(S), S)$ is o-2 transitive). Moreover, the list yields more compatible tight Riesz orders than were found in [**3**].

It should be observed that if T is a compatible tight Riesz order on G, then so are $\mathscr{T}(\mathscr{G}(T, S))$ and $\mathscr{T}(\mathscr{G}(T, \bar{S}))$, provided they are factorable.

Added in proof:

THEOREM 3.7. *If $(A(S), S)$ is transitive, then $A(S)$ has a compatible tight Riesz order if and only if $(A(S), S)$ is not locally the integers.*

Proof. Every o-primitive component of $(A(S), S)$ has the form $(A(T), T)$ where $(A(T), T)$ is o-2 transitive or regular. In the former case, $A(T)$ has a compatible tight Riesz order by Ball's result; in the latter case, $A(T)$ has a pseudo-compatible tight Riesz order which is a compatible tight Riesz order if $A(T)$ is not o-isomorphic to $\mathbf{Z}$. Hence, if $(A(S), S)$ is not locally the integers, $A(S)$ has a compatible tight Riesz order (by Theorems 2.2 and 2.5). The converse direction is obvious.

We now turn our attention to breaking down compatible tight Riesz orders on (G, S) to see what happens on the o-primitive components.

4. o-primitive components of compatible tight Riesz orders on l-permutation groups. We first observe that it is an easy exercise to see that if G is an l-group with H a convex l-subgroup and T is a compatible tight Riesz order for G, then $T \cap H = \emptyset$ or $T \cap H$ is a compatible tight Riesz order for H. If $G\phi$ is an l-homomorphic image of G under ϕ, then $T\phi$ is a normal subset of

$G\phi$, $T\phi \cdot T\phi = T\phi$ and $T\phi$ is a filter on $(G\phi)^+$. If inf $T\phi = e$ and $T\phi$ is a proper filter on $(G\phi)^+$, then $T\phi$ is a compatible tight Riesz order on $G\phi$. Note that $e \notin T\phi$ if and only if ker $(\phi) \cap T = \emptyset$. In order to examine the o-primitive components of an l-permutation group (G, S), we must first consider $(G_{(B)}, S)$ where $G_{(B)} = \{g \in G : Bg = B\}$ and B is a natural o-block. Since $G_{(B)}$ is a convex l-subgroup of G, $T \cap G_{(B)}$ is a compatible tight Riesz order for $G_{(B)}$ or $T \cap G_{(B)} = \emptyset$.

An ordered-permutation group (G, S) satisfies the *mild support property* if whenever B is a non-dead natural o-block containing at least two distinct points of S, there exists $g \in G$ such that $e < g$ and supp $(g, S) \subseteq B$.

LEMMA 4.1. *Let B be a non-dead natural o-block of the ordered-permutation group (G, S) such that B has at least two elements. If (G, S) satisfies the mild support property and T is a compatible tight Riesz order for G, then $T \cap G_{(B)} \neq \emptyset$ and hence is a compatible tight Riesz order for $G_{(B)}$.*

Proof. If $T \cap G_{(B)} = \emptyset$, then $Bh > B$ for every $h \in T$. Let $e < g \in G$ have supp $(g, S) \subseteq B$. Then $e < g \leqq h$ for all $h \in T$. Consequently, $e < g \leqq \inf T$, a contradiction.

LEMMA 4.2. *If $B \neq S$ is a non-dead natural o-block of the ordered-permutation group (G, S), B has at least two elements, T is a compatible tight Riesz order for G and (G, S) satisfies the mild support property, then $g \notin T$ provided* supp $(g, S) \subseteq B$.

Proof. Let supp $(g, S) \subseteq B$ and $h \in G$ be chosen so that $Bh \neq B$. Now if $g \in T$, then $h^{-1}gh \in T$ and supp $(h^{-1}gh, S) \subseteq Bh$. Hence $g \wedge h^{-1}gh = e$, a contradiction. Therefore, $g \notin T$.

Indeed, by the same method, we see:

COROLLARY 4.3. *Let B, T and (G, S) satisfy the hypotheses of Lemma 4.2. If* supp $(g, S) \subseteq Bh_1 \cup \ldots \cup Bh_n$ *for some $h_1, \ldots, h_n \in G$, then $g \notin T$ provided $\{Bh : h \in G\}$ is infinite.*

$G_{(B)}$ acts on S. If $g \in G_{(B)}$, then $\hat{g}$ acts on B where: $b\hat{g} = bg$ for all $b \in B$. The map $g \mapsto \hat{g}$ $(g \in G_{(B)})$ is an l-homomorphism with kernel K, say. Let $\hat{G}_{(B)} = G_{(B)}/K$. $(\hat{G}_{(B)}, B)$ is an ordered permutation group in the natural way. Moreover, if G is an l-group, so is $\hat{G}_{(B)}$.

THEOREM 4.4. *Let B, T and (G, S) satisfy the hypotheses of Lemma 4.2. Suppose further that (G, S) is transitive. Either there exists $h \in T$ such that $bh = b$ for all $b \in B$, or $T_B = (T \cap G_{(B)})\hat{}$ is a compatible tight Riesz order on $\hat{G}_{(B)}$—and not both.*

Proof. Clearly, if such an h exists, $e = \hat{h} \in T_B$ and T_B is not a compatible tight Riesz order on $\hat{G}_{(B)}$. If no such h exists, then for all $h \in T$, $\hat{h} \neq e$ so $e \notin T_B$. If $e < \hat{g} \leqq \inf T_B$, then let $e < g_1 \in G$ with supp $(g_1, S) \subseteq B$. Now $(g \wedge g_1)\hat{} = \hat{g} \wedge \hat{g}_1$ and supp $(g_1 \wedge g, S) \subseteq B$. So $e \leqq g_1 \wedge g \leqq \inf T$. Hence

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$g_1 \wedge g = e$ for all $e < g_1 \in G$ with supp $(g_1, S) \subseteq B$. Also, by transitivity, such g_1 exist. Let $b_1 \in$ supp (g_1, S) and $b \in$ supp $(g, S) \cap B$ $(\hat{g} > e)$. There exists $h \in G$ such that $b = b_1h$. Now $Bh = B$ since B is an o-block of (G, S). Thus supp $(h^{-1}g_1h, S) =$ supp $(g_1, S)h \subseteq Bh = B$. Also $b(g \wedge h^{-1}g_1h) = \min\{(bg, b_1g_1h), bg\} > b$ and $b_1g_1h > b_1h = b$. Therefore $g \wedge h^{-1}g_1h \neq e$, a contradiction. Consequently, inf $T_B = e$ and by the remarks at the beginning of this section, T_B is a compatible tight Riesz orderfor $(\hat{G}_{(B)}, B)$.

Note that if there exists $h \in T$ such that $bh = b$ for all $b \in B$, then $T_B = (\hat{G}_{(B)})^+$ and $T_{Bg} = (\hat{G}_{(Bg)})^+$, for all $g \in G$. Moreover, $T_B \cong T_{Bg}$ for all $g \in G$. This can arise as can be seen by letting $(G, S) = (A(\mathbf{R})\text{Wr}A(\mathbf{Q}), R \overleftarrow{\times} \mathbf{Q}) = (A(\mathbf{R} \overleftarrow{\times} \mathbf{Q}), \mathbf{R} \overleftarrow{\times} \mathbf{Q})$. Let $T = \{(\{g_q\}, \bar{g}) \in G^+ : \bar{x}\bar{g} > \bar{x}$ for all but a finite number of $\bar{x} \in \mathbf{Q}\}$. T is a compatible tight Riesz order for G, but if $B = \{(r, \bar{0}) : r \in R\}$, then $T_B = (\hat{G}_{(B)})^+$ since such an h is furnished by $(\{h_q\}, \bar{h})$ where $\bar{h} > e$ and supp $(\bar{h}, \mathbf{Q}) = \mathbf{Q}\backslash\{\bar{0}\}$, $h_q = e$ if $q = 0$ and $rh_q = r + 1 (r \in \mathbf{R})$ if $q \neq 0$.

THEOREM 4.5. *Assume that* (G, S) *is a transitive l-permutation group with* $(\mathscr{S}_\gamma, \mathscr{S}^\gamma)$ *a covering pair of G-congruences on S. Let T be a compatible tight Riesz order on G and B an* $\mathscr{S}^\gamma$*-class. If* $T_B \neq (\hat{G}_{(B)})^+$, *then* $T_\gamma = \{\tilde{g} \in G_\gamma : \hat{g} \in T_B\} = G_\gamma^+$ *or is a compatible tight Riesz order on G provided* (G_γ, S_γ) *is not pathological or isomorphic to* $(A(\mathbf{Z}), \mathbf{Z})$. *If* $T_B = (\hat{G}_{(B)})^+$, *then* $T_\gamma = G_\gamma^+$.

Proof. If G_γ is not l-isomorphic to $\mathbf{Z}$, then S_γ is not discrete. Suppose that $e < \tilde{g} \leqq$ inf T_γ; then $C\tilde{g} > C$ for some $\mathscr{S}_\gamma$-class C. For all $\tilde{h} \in T_\gamma$, $C\tilde{h} \geqq C\tilde{g} > C$. Let C' be an $\mathscr{S}_\gamma$-class such that $C < C' < C\tilde{g}$. Let $x \in C$ and $y \in C'$. There exists $f \in G^+$ such that $xf = y$. Now $Bf = B$ since B is an o-block. Hence $\hat{f} \in G_{(B)}$ and $x\hat{f} > x$. So $\hat{f} > e$ and $\tilde{f} > e$.

Case 1. G_γ is an o-group. Then $C\tilde{h} \geqq C\tilde{g} > C' = C\tilde{f}$ for all $\tilde{h} \in T_\gamma$. Thus $C\tilde{h}\tilde{f}^{-1} > C$ so for every $\mathscr{S}_\gamma$-class $D \subseteq B$, $D\tilde{f} < D\tilde{h}$. Hence for all $z \in D$, $z\hat{f} \in D\tilde{f} < D\tilde{h}$ and $z\hat{h} \in D\tilde{h}$. So $z\hat{f} < z\hat{h}$ for all $z \in D$ and each $\mathscr{S}_\gamma$-class $D \subseteq B$. It follows that $\hat{h} > \hat{f}$ for all $\hat{h} \in T_B$. Consequently, $e < \hat{f} \leqq$ inf T_B, a contradiction.

Case 2. (G_γ, S_γ) is non-pathological o-2 transitive. There exists $\tilde{k} \in G_\gamma$ such that $e < \tilde{k}$ and supp $(\tilde{k}) \subseteq (C, C')$. Now if $z \in B$ and z belongs to some $\mathscr{S}_\gamma$-class strictly between C and C', then $z\hat{k} < x\hat{h} < z\hat{h}$ for all $\hat{h} \in T_B$. If $z \in B$ and is not in the above category, $z\hat{k} = z \leqq z\hat{h}$. Hence $e < \hat{k} \leqq$ inf T_B, a contradiction.

Case 3. (G_γ, S_γ) is periodic with period $\tilde{f}_0$. We may assume that $C < C' < C\tilde{f}_0$. Now there exists $e < \tilde{k} \in (G_\gamma)_C$ such that supp $(\tilde{k}, B/\mathscr{S}_\gamma) \cap (C, C\tilde{f}_0) \subseteq (C, C')$. If $z \in D$ and D is an $\mathscr{S}_\gamma$-class contained in (C, C'), then $z\hat{k} < x\hat{h} < z\hat{h}$ for all $\hat{h} \in T_B$ and if $z \in D$ an $\mathscr{S}_\gamma$-class contained in $[C', C\tilde{f}_0]$, then $z\hat{k} = z \leqq z\hat{h}$. By the periodicity, $z\hat{k} \leqq z\hat{h}$ for all $z \in B$ and $\hat{h} \in T_B$. Thus $e < \hat{k} \leqq$ inf T_B, a contradiction.

It follows that T_γ is a compatible tight Riesz order on G_γ if $e \notin T_\gamma$, and

$T_\gamma = G_\gamma^+$ if $e \in T_\gamma$. This latter occurs only if $Ch = C$ for all $\mathscr{S}_\gamma$-classes $C \subseteq B$ (for some $h \in T$).

Of course, if $(G_\gamma, S_\gamma) \cong (A(\mathbf{Z}), \mathbf{Z})$, then $T_\gamma = G_\gamma^+$ or $T_\gamma = \{\tilde{g} \in G_\gamma^+ : \tilde{g} \neq e\}$. We have been unable to decide what occurs in the case that (G_γ, S_γ) is pathological o-2 transitive.

We now prove a partial converse to Theorem 2.2.

THEOREM 4.6. *Let (G, S) be transitive and be weakly laterally complete and weakly depressible. Suppose T is a compatible tight Riesz order on G. If (G, S) is locally o-primitive with μ the least element of Γ, then $T_\mu = G_\mu^+$ or T_μ is a compatible tight Riesz order on G_μ.*

Proof. The proof is immediate from Theorem 4.5 unless $(G_\mu, S_\mu) \cong (A(\mathbf{Z}), \mathbf{Z})$ or (G_μ, S_μ) is pathological. If $(G_\mu, S_\mu) \cong (A(\mathbf{Z}), \mathbf{Z})$, let C be an $\mathscr{S}^\mu$-class and define $g \in G$ by:

$$xg = \begin{cases} x + 1 \text{ if } x \in C \\ x \text{ if } x \notin C. \end{cases}$$

If $T_\mu \neq G_\mu^+$, then $e < g \leqq \inf T$ as is easily seen. This is impossible so $T_\mu = G_\mu^+$. If (G_μ, S_μ) is pathological, suppose $e < \tilde{g} \leqq \inf T_\mu$. Then $\tilde{g} \leqq \tilde{h}$ for each $\hat{h} \in T_{(C)}$, where C is an $\mathscr{S}^\mu$-class. As before, we can construct $g \in G$ with

$$xg = \begin{cases} x\hat{g} \text{ if } x \in C \\ x \text{ if } x \notin C. \end{cases}$$

Now $e < g \leqq \inf T$, a contradiction. Thus $\inf T_\mu = e$ and the result follows.

As was seen in Example 4, this partial converse is the best possible.

THEOREM 4.7. *Let B, T and (G, S) satisfy the hypotheses of Lemma 4.2 and let (G, S) be transitive. Suppose further there exists $h \in T$ such that $bh = b$ for all $b \in B$.*

1. *If B contains an $\mathscr{S}^\gamma$-class, then $T_\delta = G_\delta^+$ for all $\delta \leqq \gamma$.*
2. *If B contains an $\mathscr{S}_\gamma$-class, then $T_\delta = G_\delta^+$ for all $\delta < \gamma$.*

Proof of 1. Let C be an $\mathscr{S}^\delta$-class contained in B, $\delta \leqq \gamma$. Let $h \in T$ satisfy the hypothesis. Then $ch = c$ for all $c \in C$ so $T_C = (\hat{G}_{(C)})^+$ by the remark following Theorem 4.4. Let D be any $\mathscr{S}_\delta$-class contained in C. Then $Dh = D$ so $e \in T_\delta$. Hence $T_\delta = G_\delta^+$. Part 2 has the same proof.

THEOREM 4.8. *Let (G, S) be transitive, T a compatible tight Riesz order on G and let $\mathscr{B}$ be the largest convex G-congruence such that $\mathscr{S}^\delta \leqq \mathscr{B}$ implies $T_\delta = G_\delta^+$. If $\mathscr{B} \neq \{S\}$, then there exists a G-invariant filter $\mathscr{G}$ on $\bar{\mathscr{B}}$, the Dedekind completion of $\{B \subseteq S : B$ is a $\mathscr{B}$-class$\}$, such that if $g \in T$, then $\{B' \in \bar{\mathscr{B}} : B'g > B'\} \in \mathscr{G}$. The converse holds if T is full.*

Proof. Let $F(g) = \{B' \in \bar{\mathscr{B}} : B'g > B'\}, g \in T$. If $F(g) = \emptyset$ for some $g \in T$, then $Bg = B$ for all $B \in \mathscr{B}$. Hence if $\mathscr{S}_\gamma \geqq \mathscr{B}$, then g fixes each $\mathscr{S}_\gamma$-class so

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$T_\gamma = G_\gamma^+$. Thus $\mathscr{B} = \{S\}$, a contradiction. It follows that $F(g) \neq \emptyset$ for each $g \in T$. Now let $\mathscr{G} = \{X \subseteq \overline{\mathscr{B}} : X \supseteq F(g)$ for some $g \in T\}$. Clearly, $\mathscr{G}$ is a G-invariant filter and if $g \in T$, then $F(g) \in \mathscr{G}$.

How important is $\mathscr{B} \neq \{S\}$?

As in Section 2, we will form $\mathscr{T}(\mathscr{G})$ but this time in the following context. Let $\mathscr{B}$ be a convex G-congruence and $\mathscr{G}$ a G-invariant filter on $\overline{\mathscr{B}}$, the Dedekind completion of $\{B \subseteq S : B$ is a $\mathscr{B}$-class$\}$.

$$\mathscr{T}(\mathscr{G}) = \{g \in G^+ : \text{there exists } \mathscr{X} \in \mathscr{G} \text{ such that } B'g > B' \text{ for all } B' \in \mathscr{X}\},$$

a normal filter on G^+ with $e \notin \mathscr{T}(\mathscr{G})$.

THEOREM 4.9. *Let (G, S) be transitive and $\mathscr{B} \neq \{S\}$ a convex G-congruence. Let $\mathscr{G}$ be a G-invariant filter on $\overline{\mathscr{B}}$. Assume further that $\mathscr{T}(\mathscr{G})$ is factorable and that there exist $B_0 \in \mathscr{B}$ and $h \in \mathscr{T}(\mathscr{G})$ such that $B_0h = B_0$. Then $\mathscr{T}(\mathscr{G})$ is a compatible tight Riesz order on G if either:*

1. *Whenever $C \in \mathscr{S}^\delta \leqq \mathscr{B}$, there exists $g \in \mathscr{T}(\mathscr{G})$ such that D is an $\mathscr{S}_\delta$-class and $D \subseteq C$ imply $Dg = D$, or*
2. *(G, S) has the support property and for some $B_0 \in \mathscr{B}$ and $h \in \mathscr{T}(\mathscr{G})$, $bh = b$ for all $b \in B_0$.*

Proof. It is enough to show that $\inf \mathscr{T}(\mathscr{G}) = e$.

1. Let $e < g' \leqq \inf \mathscr{T}(\mathscr{G})$. Then $sg' > s$ for some $s \in S$. Let $\mathscr{C}$ be the intersection of all convex G-congruences $\mathscr{D}$ such that $sg'\mathscr{D}s$ and let $\mathscr{C}'$ be the union of all convex G-congruences $\mathscr{D}$ such that $sg'\mathscr{D}s$ is false. Then $(\mathscr{C}', \mathscr{C})$ is a covering pair so $(\mathscr{C}', \mathscr{C}) = (\mathscr{S}_\gamma, \mathscr{S}^\gamma)$ for some $\gamma \in \Gamma$. Let C be the $\mathscr{S}^\gamma$-class containing s and sg' and D the $\mathscr{S}_\gamma$-class containing s. If $\mathscr{S}^\gamma \leqq \mathscr{B}$, let $g \in \mathscr{T}(\mathscr{G})$ satisfy the hypotheses. Then $sg \in D < Dg'$ and $sg' \in Dg'$. Hence $g' \nleq g$, a contradiction. If $\mathscr{B} < \mathscr{S}^\gamma$, then if $s \in B$ a $\mathscr{B}$-class, $B < Bg'$. By the G-invariance and transitivity, there exists $g \in \mathscr{T}(\mathscr{G})$ such that $Bg = B$. Now $g' \nleq g$, a contradiction. Thus $\inf \mathscr{T}(\mathscr{G}) = e$.
2. If $e < g' \leqq \inf \mathscr{T}(\mathscr{G})$, then we may assume that supp $(g', S) \subseteq B$. Let h be as in the hypotheses. Clearly $g' \nleq h$, a contradiction.

We now wish to take a transitive l-permutation group (G, S) with compatible tight Riesz order T and embed it in the wreath product of its o-primitive components. We examine the image of T. Clearly, any compatible tight Riesz order on the wreath product which intersects the image of G gives rise—via the pull-back map—to a compatible tight Riesz order on G. So if the image of T is always a compatible tight Riesz order on the wreath product, then *all* compatible tight Riesz orders on G come from the wreath product.

THEOREM 4.10 *Let (G, S) be a transitive l-permutation group that is weakly depressible and weakly laterally complete. Let ϕ be the standard l-embedding of (G, S) into $(W, U) = \mathrm{Wr}\{(G_\gamma, S_\gamma) : \gamma \in \Gamma\}$. Let T be a compatible tight Riesz order on G and let $T' = \{w \in W : w \geqq g\phi$ for some $g \in T\}$. If T' is a normal subset of W, then T' is a compatible tight Riesz order on W.*

https://doi.org/10.4153/CJM-1976-024-4 Published online by Cambridge University Press

Proof. Clearly $e \notin T'$ and T' is a filter on G^+. If $w \in T'$, then $w \geqq g\phi$ for some $g \in T$. Now $g = g_1g_2$ for some $g_1, g_2 \in T$. Hence $w \geqq (g_1g_2)\phi = (g_1\phi)(g_2\phi)$. Therefore $w = (g_1\phi)(g_1\phi)^{-1}w$ and $(g_1\phi)^{-1}w \geqq g_2\phi$ so $g_1\phi, (g_1\phi)^{-1}w \in T'$. Thus $T' = T' \cdot T'$. It remains to prove that $\inf T' = e$. Suppose that $e < w \leqq \inf T'$. Then $e < w \leqq f\phi$ for all $f \in T$. If Γ has a minimal element μ, let C be an $\mathscr{S}^\mu$-class such that $\text{supp}\,(w, U) \cap C \neq \emptyset$. Without loss of generality, $\text{supp}\,(w, U) \subseteq C$. Now, by Theorem 4.6, $\inf T_\mu = e$; but there exists $g \in G$ such that $g|_C = w|_C \neq e$, a contradiction. If Γ has no least element, let $y \in \text{supp}\,(w, U) \cap S\phi$—since $w \leqq g\phi$. Let $\gamma \in \Gamma$ be such that $yw\mathscr{S}^\gamma y$ but $yw\mathscr{S}_\gamma y$ is false. Let $\gamma' < \gamma$. There exists $e \neq g \in C^+$ such that $\text{supp}\,(g, S) \subseteq C$ where C is an $\mathscr{S}^{\gamma'}$-class. Clearly, $e < g\phi \leqq w \leqq \inf T\phi$. Thus $e < g \leqq \inf T$, a contradiction.

We do not know if the conditions in Theorem 4.10 force (G, S) to be l-isomorphic to (W, U). If they do, the theorem is of no value.

So we now turn our attention to $(G, S) = \text{Wr}\{(G_\gamma, S_\gamma) : \gamma \in \Gamma\}$.

THEOREM 4.11. *Let* $(G, S) = \text{Wr}\{(G_\gamma, S_\gamma) : \gamma \in \Gamma\}$ *be a transitive l-permutation group. Suppose that* Γ *has no least element and that for each* $\gamma \in \Gamma$, *there exists* $\tilde{g} \in G_\gamma$ *such that* $\text{supp}\,(\tilde{g}, S_\gamma) = S_\gamma$. *Then* (G, S) *has a compatible tight Riesz order* (also see Theorem 2.5).

Proof. Let $T_1 = \{g \in G^+ : \text{there exists } \gamma \in \Gamma \text{ such that } (s\mathscr{S}_\gamma)g > s\mathscr{S}_\gamma \text{ for all } s\mathscr{S}_\gamma\}$. Obviously, $e \notin T_1$ and $T_1 \neq \emptyset$. Moreover, T_1 is a normal subset of G^+. If $g_1, g_2 \in T_1$, let $\gamma_1, \gamma_2 \in \Gamma$ be such $(s\mathscr{S}_{\gamma_i})g_i > s\mathscr{S}_{\gamma_i}$ for all $s\mathscr{S}_{\gamma_i}$ $(i = 1, 2)$. Let $\gamma_1 \leqq \gamma_2$ without loss of generality. Then if $\gamma_1 = \gamma_2$ $(s\mathscr{S}_{\gamma_1})(g_1 \wedge g_2) > s\mathscr{S}_{\gamma_1}$ for all $s\mathscr{S}_{\gamma_1}$ and if $\gamma_1 < \gamma_2$, since $(s\mathscr{S}_{\gamma_2})g_2 > s\mathscr{S}_{\gamma_2}$ for all $s\mathscr{S}_{\gamma_2}$, $(s\mathscr{S}_{\gamma_1})g_2 > s\mathscr{S}_{\gamma_1}$ for all $s\mathscr{S}_{\gamma_1}$. Thus $(s\mathscr{S}_{\gamma_1})(g_1 \wedge g_2) > s\mathscr{S}_{\gamma_1}$ for all $s\mathscr{S}_{\gamma_1}$ so $g_1 \wedge g_2 \in T_1$. Let $g \in T_1$ and $(s\mathscr{S}_\gamma)g > s\mathscr{S}_\gamma$ for all $s\mathscr{S}_\gamma$. Let $\delta < \gamma$. Then $(s\mathscr{S}_\delta)g > s\mathscr{S}_\delta$ so choose $h \in G^+$ so that $(s\mathscr{S}_\delta)h > s\mathscr{S}_\delta$ and $(s\mathscr{S}^\delta)h = s\mathscr{S}^\delta$ for all $s\mathscr{S}_\delta$ and $s\mathscr{S}^\delta$. Then $h \in T_1$ and $h < g$. It is easy to see that $(s\mathscr{S}_\gamma)h^{-1}g = s\mathscr{S}_\gamma g > s\mathscr{S}_\gamma$ for all $s\mathscr{S}_\gamma$ so $h^{-1}g \in T_1$. Now $g = h \cdot h^{-1}g$ so $T_1 = T_1 \cdot T_1$. If $e < g \leqq \inf T_1$, let $s \in S$ be such that $sg > s$. Let $\gamma \in \Gamma$ be chosen so that $sg\mathscr{S}^\gamma s$ but not $sg\mathscr{S}_\gamma s$ and let $\delta < \gamma$. There exists $h \in G^+$ such that $(s'\mathscr{S}^\delta)h = s'\mathscr{S}^\delta$ and $(s'\mathscr{S}_\delta)h > s'\mathscr{S}_\delta$ for all $s'\mathscr{S}^\delta$ and $s'\mathscr{S}_\delta$. Now $h \in T_1$ and $sh < sg$, a contradiction. Thus $\inf T_1 = e$ so $T = \{g \in G : g \geqq h \text{ for some } h \in T_1\}$ is a compatible tight Riesz order on G.

This completes the picture we have been able to obtain. The examples in Section **1** show the reasons for the very limited results and Section 2 and Theorem 4.11 show what can be managed. It would be less than truthful were we not to mention that we have obtained only a very muddied understanding to date of the compatible tight Riesz orders on l-permutation groups. However, even this is something of an advancement in the knowledge of the subject, albeit a less than pleasing one.

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