BULL. AUSTRAL. MATH. SOC. 18A30, 18C10, 18D35
VOL. 24 (1981), 277-288.

FREE FINITARY ALGEBRAS ON COMPACTLY GENERATED SPACES

BRIAN J. DAY

An explicit colimit formula is used to describe the free k-space algebra on a given k-space for any k-enriched finitary theory. A question, raised and solved affirmatively by several authors, has been that of whether the free k-space group on a weakly hausdorff k-space is again weakly hausdorff and admits a closed embedding of the generators. In the present article both these features of finitary k-space algebra are combined to answer analogous questions regarding the free finitary k-space algebras in general, and the weakly hausdorff separation axiom. Relationships with other problems in k-space theory are described.

Introduction

The main motivation for the present article is LaMartin's definitive account [6] on the foundations of k-space group theory. There exist many accounts of the theory of *k-spaces* in the literature (both published and otherwise) and, as there is no real point in enumerating them, we shall simply assume some familiarity with LaMartin's exposition [6, Part 1] and proceed forthwith. As we proceed, several additional needed properties of k-spaces shall be recorded.

Now we turn to k-space algebra. As regards finitary k-space universal algebra, for a given k-enriched theory, there exists a variety of ways of *presenting* the free k-space algebra as a colimit (or direct

Received 22 April 1981.

limit) built on the given k-space. The presentation chosen by LaMartin, for the case of free k-space groups, is derived directly from the coproduct presentation of the free k-space monoid on a given k-space (see also Ordman [7]). However, for the general case considered in this article, a sometimes-better alternative seems to be the *coend* presentation of the free algebra (see, for example, Borceux and Day [2]). The computational advantages of this alternative will become apparent. Thus we shall assume some familiarity with the first section of [2] in order not to make the present text too voluminous.

There will be several categories of importance in the present study. The first is the cartesian closed category K of k-spaces (equals *all* topological quotients of locally compact hausdorff spaces) and continuous maps. The second is the full reflective cartesian closed subcategory WK of K comprising the weakly hausdorff, or t_2, k-spaces. For notational convenience, we shall term such a space a *wk-space*. Thirdly, we shall make use of the quasi-topos $\mathcal{Q}$ (see Penon [8]) of all Spanier's [9] quasi-topological spaces (here called q-spaces) and "continuous" maps (called q-maps). The particular property we use is that $\mathcal{Q}/Q$ is cartesian closed for all objects $Q \in \mathcal{Q}$ (originally due to Booth [1]). The category $\mathcal{Q}$ contains K as a full reflective subcategory with the reflector preserving finite products and certain other pullbacks (as discussed in Day [3]). Finally, we refer (but briefly) to the cartesian closed category $W\mathcal{Q}$ of "separated" q-spaces (called wq-spaces).

In addition to the main aim of this article, namely to generalise *some* of LaMartin's results, we can naturally make some deductions about free topological k_ω-algebras over a k_ω-theory. These results are just corollaries to the separation properties of free k_ω-algebras, and they will follow the results of [6, p. 21] analogously. In conclusion, some questions concerning colimits are raised.

1. Separation properties

The idea of a wk-space has an analogue in $\mathcal{Q}$. Let Q denote the k-space $\{0, 1\}$ where 0 is open and 1 is not open (this is a topological quotient of the unit interval). Also, let $t : 1 \to Q$ denoted insertion of 1. A q-map $m : X \to Y$ is called *closed* if there exists a

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

pullback diagram in $\mathcal{Q}$ of the following form:

$$\begin{array}{ccc} X & \xrightarrow{m} & Y \\ \downarrow & & \downarrow \chi(m) \\ 1 & \xrightarrow{t} & Q \end{array}$$

For example, let Y be a k-space regarded as a q-space and let X be a topologically closed subspace of Y. Then $X \subset Y$ is a closed q-map since the natural embedding of $\mathcal{K}$ into $\mathcal{Q}$ preserves limits, hence pullbacks. In fact the inclusion of $\mathcal{K}$ in $\mathcal{Q}$ has a left adjoint called "realisation" and here denoted by R.

PROPOSITION 1.1. *If* $m : X \to Y$ *is a closed* q*-map then* $Rm : RX \to RY$ *is closed in* $\mathcal{K}$*, hence in* $\mathcal{Q}$.

Proof. Let $f : C \to Y$ be any "admissible" map defining Y. Form the double pullback:

$$\begin{array}{ccc} f^{-1}X & \longrightarrow & C \\ \downarrow & & \downarrow f \\ X & \xrightarrow{m} & Y \\ \downarrow & & \downarrow \chi \\ 1 & \xrightarrow{t} & Q \end{array} .$$

This shows that $f^{-1}X$ is closed in C, as required. //

Call a q-space X a *wq-space* if the diagonal map $X \to X \times X$ is a closed q-map. Proposition 1.1 thus asserts that if X is a wq-space then RX is a wk-space, since R preserves finite products (see Day [3]). Thus one obtains a full reflective embedding of $\mathcal{WK}$ in $\mathcal{WQ}$.

An indexing functor $X : \Phi \to \mathcal{Q}$ will be called *monofiltered* if Φ is filtered and each of the transition maps in the diagram X, is a monomorphism in $\mathcal{Q}$.

LEMMA 1.2. *In* $\mathcal{Q}$*, a monofiltered colimit of* wq*-spaces is a* wq*-space.*

Proof. Each diagonal $\delta = \delta(\phi) : X(\phi) \to X(\phi) \times X(\phi)$ gives a pullback

$$\begin{array}{ccc} X(\phi) & \xrightarrow{\ \delta\ } & X(\phi)\times X(\phi) \\ \downarrow & & \downarrow{\scriptstyle \chi(\delta)} \\ 1 & \longrightarrow & Q \end{array}$$

in $\mathcal{Q}$. Since each transition map h is a monomorphism in $\mathcal{Q}$, one has that

$$\begin{array}{ccc} X(\phi)\times X(\phi) & \xrightarrow{\ h\times h\ } & X(\phi')\times X(\phi') \\ & {\scriptstyle \chi(\delta)}\searrow \quad \swarrow{\scriptstyle \chi(\delta)} & \\ & Q & \end{array}$$

commutes. Thus one obtains a colimit

$$\operatorname{colim}_{\phi}\big(X(\phi) \times X(\phi)\big) \to Q$$

in $\mathcal{Q}/Q$. But $\mathcal{Q}/Q$ is cartesian closed, so filtered colimits commute with finite products. This means that

$$\begin{array}{ccc} \operatorname{colim} X(\phi) & \xrightarrow{\ \operatorname{colim}\,\delta\ } & \operatorname{colim}\big(X(\phi)\times X(\phi)\big) \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\qquad t \qquad} & Q \end{array}$$

is a pullback diagram in $\mathcal{Q}$. Since $\mathcal{Q}$ is cartesian closed, one has $\operatorname{colim}\big(X(\phi) \times X(\phi)\big) \cong \operatorname{colim} X(\phi) \times \operatorname{colim} X(\phi)$ in $\mathcal{Q}$, whence $\operatorname{colim} X(\phi)$ is a *wq*-space. //

LEMMA 1.3. *In* K *, a monofiltered colimit of wk-spaces is a wk-space.*

Proof. Form the colimit in $\mathcal{Q}$ then apply R to the result, using Proposition 1.1 to show that $\delta : \operatorname{colim} X(\phi) \to \operatorname{colim} X(\phi) \times \operatorname{colim} X(\phi)$ is closed in K . //

THEOREM 1.4. *A k-space* X *is a wk-space if and only if it is a monofiltered colimit (in* K *) of compact hausdorff spaces.*

Proof. Any *wk*-space is the (monofiltered) colimit in K of its compact hausdorff subspaces; this observation appears in Hofmann [5]. The converse is by Lemma 1.3. //

Similarly, one obtains:

PROPOSITION 1.5. *If a monofiltered system* $u(\phi) : X(\phi) \to Y(\phi)$,

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

$\phi \in \Phi$, of closed maps in K has the property that

$$\begin{array}{ccc} Y(\phi) & \xrightarrow{\quad h \quad} & Y(\phi') \\ {\scriptstyle \chi(\phi)} \searrow & & \swarrow {\scriptstyle \chi(\phi')} \\ & Q & \end{array}$$

commutes for all transition maps h, then colim $u(\phi)$ *is a closed map in K.* //

2. Separation theorem

Throughout this section all categories, functors, Kan extensions, and so forth, are assumed to be *k-enriched*. The internal-hom of K will be denoted by $[-, -]$.

To each k-space X one can assign the *free k-space group* GX on X. Then we have

$$\int^{n} [n, X] \times Gn \xrightarrow{\cong} GX$$

in K, where $n \in \mathit{Fin}$ (the category of discrete finite sets). By use of this coend formula, let us first reprove a part of [6, Theorem 2.12] in a manner which will lend itself to generalisation.

PROPOSITION 2.1. *If X is a wk-space then so is GX.*

Proof. As we are dealing with a k-space group it suffices to show that the identity element $e \in GX$ is a closed singleton. In order to do this, first observe that the following diagram commutes by the definition of a coend:

$$\begin{array}{ccc} \sum_m \mathit{Fin}(n,m)\times[m,X]\times Gm & \longrightarrow\!\!\!\!\rightarrow & [n,X]\times Gn \\ \Big\downarrow & (*) & \Big\downarrow {\scriptstyle q_n} \\ \sum_m [m,X]\times Gm & \xrightarrow{\quad q \quad}\!\!\!\!\rightarrow & \int^n [n,X]\times Gn \end{array}$$

where the unlabelled arrows are canonical and q is the coend quotient map with nth component q_n. Let $\mathit{Sur}(n, m) \subset \mathit{Fin}(n, m)$ denote the surjections n to m, and consider the following diagram derived from $(*)$:

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

$$\begin{array}{ccc}
\sum_m [m,X]\times K(m,n) & \longrightarrow\!\!\!\!\rightarrow & q_n^{-1}(e) \\
\cap & & \cap \\
\sum_m [m,X]\times \mathcal{S}ur(n,m)\times Gn & \xrightarrow{\;r_n\times 1\;}\!\!\!\!\rightarrow & [n,X]\times Gn \\
\Big\downarrow \sum (1\times p_{mn}) & & \Big\downarrow q_n \\
\sum_m [m,X]\times Gm & \xrightarrow[\;q\;]{}\!\!\!\!\rightarrow & GX
\end{array}$$

Now $r_n \times 1$ is a closed retraction since X is a wk-space and Gn is discrete for all $n \in \mathcal{F}in$. Also

$$[m,\ X] \times K(m,\ n) \subset [m,\ X] \times \mathcal{S}ur(n,\ m) \times Gn$$

is a closed subset, for all $m,\ n \in \mathcal{F}in$, where $K(m,\ n) = p_{mn}^{-1}(e)$ for e the identity of Gm , and

$$p_{mn} : \mathcal{S}ur(n,\ m) \times Gn \to Gm$$

is the canonical map. But, by factoring each map $n \to X$ into a surjection $n \to m$ followed by an injection $m \to X$, it is seen that

$$(r_n \times 1)\left(\sum_m [m,\ X] \times K(m,\ n)\right) = q_n^{-1}(e)$$

for all $n \in \mathcal{F}in$. This implies that $q^{-1}(e)$ is closed in $\sum_n [n,\ X] \times Gn$, as required. //

Given a k-space X , we consider the left Kan extension of $[-,\ X] : \mathcal{F}in^{\mathrm{op}} \to K$ along the Yoneda embedding:

$$Y : \mathcal{F}in^{\mathrm{op}} \to [\mathcal{F}in,\ WK] .$$

The value $\mathrm{Lan}(F)$ of this extension at $F \in [\mathcal{F}in,\ WK]$ is given by the coend formula (see [4]):

$$\mathrm{Lan}(F)(X) = \int^n [n,\ X] \times Fn$$

computed in K . By the k-Yoneda-lemma, we have a natural isomorphism:

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

$$\mathrm{Lan}(F)(m) = \int^n [n,\ m] \times Fn \cong Fm\ .$$

Thus one essentially considers those endofunctors on K for which:

(i) the canonical transformation $\int^n [n,\ X] \times Fn \to FX$ in an isomorphism; and

(ii) for each $n \in Fin$, Fn is a wk-space.

Such an endofunctor on K will be called *wk-finitary*.

THEOREM 2.2 (Separation theorem). *Given a wk-finitary endofunctor F on K, its value FX at a wk-space X is again a wk-space.*

Proof. It is easily seen that the canonical map $FX \to GFX$ is an injection (see Lemma 3.1), so it suffices to show that the identity element in GFX is closed. However, on combining the properties of G and coends with the k-enriched Yoneda lemma, we have the following isomorphism:

$$\begin{aligned}
GFX &\cong G\left(\int^n [n,\ X] \times Fn\right) \\
&\cong \int^m \left[m,\ \int^n [n,\ X] \times Fn\right] \times Gm \\
&\cong \int^m Gm \times \left(\int^{m_1} [n_1,\ X] \times Fn_1\right) \times \ldots \times \left(\int^{n_m} [n_m,\ X] \times Fn_m\right) \\
&\cong \int^m Gm \times \int^{n_1 \ldots n_m} \left([n_1 + \ldots\ n_m,\ X] \times Fn_1 \times \ldots \times Fn_m\right) \\
&\cong \int^m Gm \times \int^n \left([n,\ X] \times Fn \times \ldots \times Fn\right)
\end{aligned}$$

by the k-enriched Yoneda lemma applied twice,

$$\begin{aligned}
&\cong \int^m Gm \times \int^n [n,\ X] \times [m,\ Fn] \\
&\cong \int^n [n,\ X] \times GFn\ .
\end{aligned}$$

By Proposition 2.1, we have that GFn is a wk-space if Fn is a wk-space, for all finite n. Now the remainder of the proof that $\int^n [n,\ X] \times GFn$ is a wk-space if X is a wk-space is analogous to the

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

proof of Proposition 2.1. Simply replace Gn by GFn, Gm by GFm, and observe that the new $K(m, n)$ is closed in the space $Sur(n, m) \times GFn$ since the identity element is closed in the wk-space GFm. //

Now call a *monad* (T, μ, η) on K *wk-finitary* if this is so of its functor part T. For example, any monad on K generated by a finitary wk-theory is wk-finitary (see [2] for the notion of a finitary theory in a closed category).

COROLLARY 2.3. *Let* (T, μ, η) *be a wk-finitary monad on* K. *Then the free algebra* TX *on a wk-space* X *is a wk-space.*

3. Embedding theorem

A *pointed endofunctor* (T, η) on K (terminology of Kelly) is a natural transformation $\eta : 1 \to T$. We call it finitary if T is, and we call it *proper* if $\eta_n : n \to Tn$ is an injection for all $n \in Fin$.

LEMMA 3.1. *A finitary pointed endofunctor* (T, η) *on* K *is proper if and only if* η *is a monomorphism.*

Proof. The colimit $\int^n [n, X] \times Tn$, computed in K, is preserved by the faithful underlying-set functor $K \to Set$, so it suffices to consider $X \cong \operatorname{colim} n_\phi$ as the filted colimit of all its finite subsets. By well-known properties of filtered colimits in Set, we infer

$$\eta_X : X \rightarrowtail \int^n [n, X] \times Tn \cong TX$$

an injection from

$$\begin{array}{ccc} \operatorname{colim} n_\phi & \longrightarrow & \int^n [n, \operatorname{colim} n_\phi] \times Tn \\ \big\downarrow & & \big\uparrow \wr\wr \\ \operatorname{colim} Tn_\phi & \xleftarrow{\cong} & \operatorname{colim} \int^n [n, n_\phi] \times Tn \end{array}$$

where the lower isomorphism is by the Yoneda lemma. //

A proper monad (T, μ, η) on K is called χ-*proper* if, for all injections $u : m \to n$ in Fin, the diagram

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

$$\begin{array}{ccc} Tm & \xrightarrow{\;Tu\;} & Tn \\ & {\scriptstyle\chi(\eta)}\searrow \quad \swarrow{\scriptstyle\chi(\eta)} & \\ & Q & \end{array}$$

commutes in K. It is easily deduced that if (T, μ, η) is finitary and χ-proper on K then, for all injections $u : X \to Y$ in K, the diagram

$$\begin{array}{ccc} \operatorname{colim} Tm_\alpha & \xrightarrow{\;\operatorname{colim} Tu_\alpha\;} & \operatorname{colim} Tn_\alpha \\ \wr\Vert\downarrow & & \downarrow\Vert\wr \\ TX & \xrightarrow{\;Tu\;} & TY \\ & {\scriptstyle\chi(\eta)}\searrow \quad \swarrow{\scriptstyle\chi(\eta)} & \\ & Q & \end{array}$$

commutes in Set, hence in K.

THEOREM 3.2 (Embedding theorem). *Let* (T, μ, η) *be a* χ*-proper and wk-finitary monad on* K. *Then* $\eta_X : X \to TX$ *is a closed subspace embedding when* X *is a wk-space.*

Proof. Let $X = \operatorname{colim} C(\phi)$, $C(\phi) \subset X$ and $C(\phi)$ compact hausdorff for all $\phi \in \Phi$. Each unit

$$\eta(\phi) : C(\phi) \to \int^n [n, C(\phi)] \times Tn \cong TC(\phi)$$

is a closed morphism in K (Lemma 3.1 and Theorem 2.2). Moreover $\operatorname{colim} \eta(\phi)$ is closed (Proposition 1.5). But

$$\operatorname{colim} \int^n [n, C(\phi)] \times Tn \xrightarrow{\cong} \int^n [n, \operatorname{colim} C(\phi)] \times Tn$$

and the result follows. //

The inclusion $WK \hookrightarrow K$ has a finite-product-preserving left adjoint which is here denoted by H. On applying H to the expression $\int^n [n, X] \times Fn$ in K one obtains $\int^n [n, HX] \times HFn$ computed *in* K by Theorem 2.2. In this way, one can assign to each k-finitary pointed endofunctor (T, η) on K, a wk-finitary pointed endofunctor on K. If the result is proper (that is, $\eta_n : n \to HTn$ is an injection for all finite n) and X is a wk-space then, from commutativity of

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \int^n [n,X]\times Tn \\ \cong\big\downarrow & & \big\downarrow \\ HX & \xrightarrow{H\eta_X} & \int^n [n,X]\times HTn \end{array}$$

and Theorem 3.2, we conclude that η_X is a subspace embedding whenever $H\eta_X$ is such.

Let (T, η) again denote a wk-finitary pointed endofunctor on K. For each k-space X and $\sigma \in Ts$, $s \in Fin$, let $W(\sigma) \subset TX$ denote the image of $i = i(\sigma)$, defined by commutativity of

$$\begin{array}{ccc} [s,X] & \xrightarrow{i} & \int^n [n,X]\times Tn \\ \cong\big\uparrow & & \big\uparrow q_s \\ [s,X]\times 1 & \xrightarrow{1\times\sigma} & [s,X]\times Ts \end{array} .$$

The following follows from Theorem 2.2.

PROPOSITION 3.3. *If C is a compact hausdorff space then $W(\sigma)$ is a closed subspace of TC.* //

4. Concluding remarks

REMARK 4.1. Let Top denote the category of all topological spaces and continuous maps, and let $t : Fin^{op} \to T$ be a finitary K-theory (in the sense of [2]). Let us call a finite-product-preserving functor from T to Top a topological t-algebra. Then we have

$$t^* \dashv [t, 1] : [T, Top] \to |Fin^{op}, Top| .$$

Thus, if $t^*(X) = \int^n X^n \times_c T(tn, t-) : T \to Top$ is a topological algebra then it is the free such on $X \in Top$. Here X^n denotes the nth power of X in Top and $X \times_c Y$ denotes the cartesian product in Top.

Now suppose each $T(tn, t1)$ is a k_ω-space (see [6, Proposition 2.2]), and let X be a k_ω-space. Then

$$\int^n X^n \times_c T(tn, t1) \cong \int^n [n, X] \times T(tn, t1)$$

is a k_ω-space. Also, for each $m \in Fin$, one has

$$\left[m, \int^n [n, X] \times T(tn, t1)\right] \cong \int^n [n, X] \times [m, T(tn, t1)]$$

by iterated use of the k-Yoneda lemma,

$$\cong \int^n X^n \times_c T(tn, t1)^m$$

$$\cong \int^n X^n \times_c T(tn, tm) .$$

Thus $\int^n [n, X] \times T(tn, t1)$ in K is the free topological t-algebra (k_ω-algebra) on the k_ω-space X. //

REMARK 4.2. One knows that coproducts exist in the K-category of (T, μ, η)-algebras if T is k-finitary. Also, the forgetful functor into K creates filtered colimits. Thus, by Theorem 1.4, a coproduct of wk-algebras is a wk-algebra if each finite summand is a wk-algebra which is canonically injected into the coproduct. In [6] this is shown to be true for k-space groups and it is obviously true if the K-category of (T, μ, η)-algebras is additive.

References

[1] P.I. Booth, "The exponential law of maps I", *Proc. London Math. Soc.* (3) 20 (1970), 179-192.

[2] Francis Borceux and Brian Day, "Universal algebra in a closed category", *J. Pure Appl. Algebra* 16 (1980), 133-147.

[3] Brian Day, "A reflection theorem for closed categories", *J. Pure Appl. Algebra* 2 (1972), 1-11.

[4] B.J. Day and G.M. Kelly, "Enriched functor categories", *Reports of the Midwest Category Seminar III*, 178-191 (Lecture Notes in Mathematics, 106. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[5] Karl Heinrich Hofmann, "Category theoretical methods in topological algebra", *Categorical topology*, 345-403 (Proc. Conf. Mannheim, 1975. Lecture Notes in Mathematics, **540**. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[6] W.F. LaMartin, "On the foundations of k-group theory", *Dissertationes Math. (Rozprawy Mat.)* **146** (1977).

[7] Edward T. Ordman, "Free k-groups and free topological groups", *General Topology Appl.* **5** (1975), 205-219.

[8] Jacques Penon, "Sur les quasi-topos", *Cahiers Topologie Géom. Différentielle* **18** (1977), 181-218.

[9] E. Spanier, "Quasi-topologies", *Duke Math. J.* **30** (1963), 1-14.

School of Mathematics and Physics,
Macquarie University,
North Ryde,
New South Wales 2113,
Australia.

https://doi.org/10.1017/S0004972700007644 Published online by Cambridge University Press