THE CATEGORIES OF BOOLEAN LATTICES, BOOLEAN RINGS AND BOOLEAN SPACES

Hoshang P. Doctor

(received October 8, 1963)

Introduction. In Theorem 4 of [5] Stone stated that the theory of Boolean rings was "mathematically equivalent" to the theory of Boolean spaces without, however, properly defining the phrase "mathematically equivalent". It is the main purpose of this note to establish a precise reformulation of Theorem 4 in [5]. This is accomplished by introducing special classes of maps between Boolean lattices, Boolean rings and Boolean spaces respectively, and showing the categories arising in conjunction with these maps to be equivalent in the sense of Grothendieck [2]. Thus the notion of equivalence of categories will replace the phrase "mathematically equivalent" in [5]. In addition the well-known axiomatic characterization of meet and complementation of Boolean lattices with unit is discussed in analogous terms.

The author would like to thank Professor B. Banaschewski for his help during the preparation of this paper. The content of this paper is a summary of the Master's thesis of the author written at McMaster University under the supervision of Professor B. Banaschewski.

1. Definitions and notation. We recall that a Boolean lattice is a relatively complemented, distributive lattice with a zero element; a Boolean space is a zero-dimensional locally compact Hausdorff space, and a Boolean ring is a ring in which every element is idempotent.

Let B, C be Boolean lattices. A mapping f from B into C is called a Boolean lattice homomorphism if f preserves meets, joins and relative complements. If B, C have units

Canad. Math. Bull. vol. 7, no. 2, April 1964

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press

e, e' respectively, a Boolean lattice homomorphism f from B into C is called unitary if $f(e) = e'$. We call a Boolean lattice homomorphism f from B into C proper if for any c in C there exists a b in B with $f(b) \geq c$. It is clear that any unitary homomorphism between Boolean lattices with unit is proper. On the other hand any proper homomorphism between Boolean lattices with unit is unitary.

Let E, F be Boolean spaces. A continuous map f from E into F is called proper if the inverse image of every compact set in F under f is compact in E. It is easily seen that a map f from E into F is proper continuous if and only if the inverse image under f of every compact open set in F is compact open in E.

For any pair of elements x, y in a Boolean ring R, put $x \leq y$ if and only if there exists a z in R with $x = yz$. The relation so defined in R is called the divisibility relation, and it is easily seen that it partially orders R. Let R, S be Boolean rings and let $\leq_R$, $\leq_S$ denote the divisibility relations in R, S respectively. We call a ring homomorphism f from R into S proper if for any s in S there exists a r in R with $f(r) \geq_S s$. Again it is clear that the classes of proper ring homomorphisms and unitary ring homomorphisms between Boolean rings with unit coincide.

It is easily seen that the above definitions give rise to categories. Throughout this paper K will denote the category of Boolean lattices and proper Boolean lattice homomorphisms, K_1 the category of Boolean lattices and Boolean lattice homomorphisms, K_2 the category of Boolean lattices with unit and unitary Boolean lattice homomorphisms, L the category of Boolean spaces and proper continuous maps, M the category of Boolean rings and proper ring homomorphisms and M_1 the category of Boolean rings and ring homomorphisms. For any category N, I_N will denote the identity functor on N.

Finally we recall the definition given by Grothendieck [2] for the equivalence of categories based on the notion of natural equivalence between functors [1]. A category N is said to be equivalent (dually equivalent) to a category P if there exists a covariant (contravariant) functor S from N into P and a covariant (contravariant) functor T from P into N such that there is a natural equivalence $\Gamma = (\varphi_X)_{X \in N}$ between the composite functor TS and I_N and a natural equivalence $\Gamma' = (\psi_Y)_{Y \in P}$ between the composite functor ST and I_P such that

$$S(\varphi_X) = \psi_{S(X)}\left(\psi^{-1}_{S(X)}\right), \quad T(\psi_Y) = \varphi_{T(Y)}\left(\varphi^{-1}_{T(Y)}\right)$$

for any X in N and Y in P.

For any Boolean lattice B, $\Omega(B)$ will denote the Boolean space associated with B, that is, the ultrafilter space of B, and for any a in B, $\Omega(a)$ will denote the set of ultrafilters of B containing the element a. For any Boolean space E, $\mathscr{L}(E)$ will denote the Boolean lattice of compact open sets of E and for any x in E, $W(x)$ will denote the ultrafilter in $\mathscr{L}(E)$ consisting of all those compact open sets of E containing x. For any Boolean ring R, $V(R)$ will denote the Boolean lattice associated with R whose partial order is the divisibility relation of R, and for any Boolean lattice B, $A(B)$ will denote the Boolean ring associated with B whose addition and multiplication are given by $x + y = (x \sim (x \wedge y)) \vee (y \sim (x \wedge y))$ and $xy = x \wedge y$, respectively, where the signs $\wedge$, $\vee$ and $\sim$ have the usual meaning.

2. Boolean lattices and Boolean spaces. Let f be a proper homomorphism from a Boolean lattice B into a Boolean lattice C. f then gives rise to a proper continuous map f_Ω from $\Omega(C)$ into $\Omega(B)$ defined by $f_\Omega(U) = f^{-1}(U)$ where U is any ultrafilter in C. On the other hand, let E, F be Boolean spaces and let f from E into F be proper continuous. f then gives rise to a proper Boolean lattice homomorphism $f_{\mathscr{L}}$ from $\mathscr{L}(F)$ into $\mathscr{L}(E)$ defined by $f_{\mathscr{L}}(X) = f^{-1}(X)$ where X is any compact open set of F.

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press

THEOREM 1. The correspondences $B \to \Omega(B)$, $f \to f_\Omega$ and $E \to \mathcal{L}(E)$, $f \to f_{\mathcal{L}}$ are contravariant functors S from K into L and T from L into K respectively which establish the dual equivalence of the categories K and L.

Proof. It is easy to see that S, T as defined are contravariant functors. For any Boolean lattice B, let i_B denote the isomorphism from B onto $\mathcal{L}(\Omega(B))$ given by $i_B(a) = \Omega(a)$. Define a map Γ_1 which assigns to each B in K the map i_B from B into TS(B). Let B, C be in K and let f from B into C be a proper homomorphism. We show that $i_C \cdot f = (f_\Omega)_{\mathcal{L}} \cdot i_B$. For any a in B we have $(i_C \cdot f)(a)$ $= i_C(f(a)) = \Omega(f(a))$, the compact open set in $\Omega(C)$ determined by f(a). On the other hand we have $((f_\Omega)_{\mathcal{L}} \cdot i_B)(a) =$ $= (f_\Omega)_{\mathcal{L}}(\Omega(a)) = f_\Omega^{-1}(\Omega(a)) = \Omega(f(a))$. Also for each B in K the map i_B being an isomorphism has an inverse. Hence Γ_1 is a natural equivalence between the functors TS and I_K. Similarly for each E in L let i_E denote the homeomorphism from E into (ST)(E) given by $i_E(x) = W(x)$. Consider the map Γ_2 which assigns to each E in L the map i_E. Let E, F be in L and let f from E into F be a proper continuous map. We show that $(i_F \cdot f) = (f_{\mathcal{L}})_\Omega \cdot i_E$. For any x in E $(i_F \cdot f)(x) = W(f(x))$. On the other hand $(f_{\mathcal{L}})_\Omega \, i_E(x)$ $= (f_{\mathcal{L}})_\Omega(W(x)) = f_{\mathcal{L}}^{-1}\{Y \,/\, x \in Y \in \mathcal{L}(E)\} = W(f(x))$. Since for each E in L i_E has an inverse, it follows that Γ_2 is a natural equivalence between the functors ST and I_L. It remains to show (i) $S(i_B) = i^{-1}_{S(B)}$ for any B in K and (ii) $T(i_E) = i^{-1}_{T(E)}$ for any E in L. Now $S(i_B) = (i_B)_\Omega$ is a map from (STS)(B) into S(B) given by $(i_B)_\Omega(W(U)) = i_B^{-1}(W(U))$ for any ultrafilter U in B, where in accordance with our notation $W(U) = \{\Omega(a) \,/\, U \in \Omega(a)\}$. Thus $i_B^{-1}(W(U)) = U$.

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press

On the other hand $i^{-1}_{S(B)}$ is also a mapping from $(STS)(B)$ into $S(B)$ given by $i^{-1}_{S(B)}(W(U)) = U$, for the sets $W(U)$ give precisely the ultrafilters in $(TS)(B)$. Hence (i) is established; one proves (ii) similarly. Thus the categories K and L are dually equivalent and this completes the proof of the theorem.

3. Boolean lattices and Boolean rings. In a similar manner as in Theorem 1 we establish in this section that the category K_1 is equivalent to the category M_1. Let B, C be Boolean lattices and let f from B into C be a Boolean lattice homomorphism. f then gives rise to a ring homomorphism f_A from $A(B)$ into $A(C)$ defined by $f_A(x) = f(x)$. On the other hand, let R, S be Boolean rings and let f from R into S be a ring homomorphism. Then f gives rise to a Boolean lattice homomorphism f_B from $V(R)$ into $V(S)$ defined by $f_V(x) = f(x)$.

THEOREM 2. The correspondences $B \to A(B)$, $f \to f_A$ and $R \to V(R)$, $f \to f_V$ are covariant functors S from K_1 into M_1 and T from M_1 into K_1 respectively; this establishes the equivalence of the categories K_1 and M_1. Moreover, the restrictions of S and T to the subcategories K of K_1 and M of M_1 respectively establish that K is equivalent to M.

Proof. One sees easily that $(ST)(R) = A(V(R)) = R$ and $(TS)(B) = V(A(B)) = B$ for any Boolean ring R and Boolean lattice B. Moreover, it is clear that S, T as defined are covariant functors. In view of the observation just made it follows easily that $ST = I_{M_1}$ and $TS = I_{K_1}$. This gives immediately that the categories K_1 and M_1 are equivalent. The remainder of the theorem follows from the fact that the divisibility relation of $A(B)$ is the same as the partial order of B and the partial order of $V(R)$ is the same as the divisibility relation of R.

Remark. Under the correspondence of Theorem 2 one observes that the functor S carries free Boolean lattices B with a set X of free generators into free Boolean rings $R = S(B)$ with a set X of free generators, and the functor T carries free Boolean rings R with a set X of free generators into free Boolean lattices $B = T(R)$ with a set X of free generators. This shows that free Boolean lattices can be constructed by algebraic methods.

4. Boolean semi-groups. We call a commutative semi-group G with zero element o a Boolean semi-group if the following holds: for any a in G there exists a b in G with 1) $ba = o$, 2) $bc = o$ for any c in G implies $ac = c$. The element b whose existence is indicated in the definition will be called a converse of a.

LEMMA. Every element in a Boolean semi-group G is idempotent and possesses exactly one converse.

Proof. Take any a in G and let b be a converse of a. Then by 1) $ba = o$ and hence by 2) $aa = a$ establishing that every element in G is idempotent. We now establish that a is a converse of b. Suppose there exists a c in G with $ac = o$ but $bc \neq c$. Let z be a converse of bc. Consider the element $zc = y$ say. Now $by = o$ and this implies (since b is a converse of a) that $ay = y$. Hence $acy = cy$, that is, $cy = o$. Thus $zcc = zc = y = o$. But $zc = o$ implies $bcc = bc = c$, since z is a converse of bc. This contradicts our assumption that $bc \neq c$. Hence our supposition is false and a is a converse of b. Next, suppose that u, v are converses of a. Then a is a converse of u, v respectively and $au = o$ implies $vu = u$. Similarly $av = o$ implies $uv = v$. Since G is commutative we have $u = uv = vu = v$. Hence each element possesses precisely one converse and this establishes the lemma.

In view of the lemma, we will denote the converse of a by a' for each a in the Boolean semi-group G. For any x, y in G put $x \leq y$ if and only if there exists a z in G with $x = yz$. The relation $\leq$, again called the divisibility relation, partially orders G.

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press

PROPOSITION. Any Boolean semi-group G is a Boolean lattice with unit under the divisibility relation in which $x \wedge y = xy$, $x \vee y = (x'y')'$, and for any x the converse x' of x is the complement of x. The element o of G is the zero of the lattice and o' is its unit. Conversely any Boolean lattice B with unit is a Boolean semi-group under meet and complementation.

Proof. We give an outline of the steps of the proof, the details of which can be obtained from Rosenbloom [3]. One first establishes that $x'' = x$ for any x in G and uses this and the definition of the divisibility relation to show that $xy = \inf\{x, y\}$, $(x'y')' = \sup\{x, y\}$. This will give G, partially ordered by divisibility, to be a lattice with zero and unit. Also we have for any x in G, $x \wedge x' = xx' = o$ and $o' = (x \wedge x')' = (x'' \wedge x')' = x \vee x'$. Hence G is a complemented lattice. To establish that it is distributive, one first proves that $x \wedge (x' \vee y) = x \wedge y$ and uses this to obtain $x \wedge (y \vee z) \wedge (x \wedge y)' \wedge (x \wedge z)' = o$. Conversely let B be any Boolean lattice with unit e. Then trivially the set B under meet and complementation of the lattice B forms a Boolean semi-group. This completes the proof of the proposition.

Let G, H be Boolean semi-groups. A mapping f from G into H is called a Boolean semi-group homomorphism if 1) $f(xy) = f(x)f(y)$ for any x, y and 2) $f(x') = (f(x))'$ for any x in G. Let N_2 denote the category of Boolean semi-groups and Boolean semi-group homomorphisms. For any Boolean semi-group G let $V(G)$ denote the Boolean lattice with unit associated with it and for any Boolean lattice B with unit let $S(B)$ denote the associated Boolean semi-group as in the above proposition. Let f from G to H be a Boolean semi-group homomorphism. Then f gives rise to a unitary Boolean lattice homomorphism f_V from $V(G)$ into $V(H)$ defined by $f_V(x) = f(x)$. On the other hand let B, C be Boolean lattices with unit and let f from B into C be a unitary Boolean lattice homomorphism. Then f gives rise to a Boolean semi-group homomorphism f_S from $S(B)$ into $S(C)$ defined by $f_S(x) = f(x)$.

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press

THEOREM 3. The correspondences $B \to S(B)$, $f \to f_S$ and $G \to V(G)$, $f \to f_V$ are covariant functors X from K_2 into N_2 and Y from N_2 into K_2 which establish the equivalence of the categories K_2 and N_2.

Proof. It is clear that X, Y as defined are covariant functors. We note further that $V(S(B)) = B$ for any Boolean lattice with unit and $S(V(H)) = H$ for any Boolean semi-group H. It thus follows easily that $YX = I_{K_2}$ and $XY = I_{N_2}$.

This establishes the theorem.

REFERENCES

1. S. Eilenberg and S. MacLane, General theory of natural equivalences, Trans. Amer. Math. Soc. 58 (1945), 231-294.

2. A. Grothendieck, Sur quelques points d'algebre homologique, Tôhoku Math. J. 9 (1957), 119-121.

3. P. Rosenbloom, The elements of mathematical logic, New York, Dover Publications, Inc., 1950.

4. M.H. Stone, Theory of representations for Boolean algebras, Trans. Amer. Math. Soc. 40 (1936), 37-111.

5. M.H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41 (1937), 375-481.

McMaster University

https://doi.org/10.4153/CMB-1964-022-6 Published online by Cambridge University Press