A SECOND LOOK AT A QUEUEING SYSTEM WITH MOVING AVERAGE INPUT PROCESS

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1. Introduction

We consider a single-server queueing system with first-come first-served queue discipline in which

(i) customers arrive at the instants $0 = A_0 < A_1 < A_2 < \cdots$, with time interval between the mth and $(m+1)$th arrivals

$$A_m - A_{m-1} = U_m + \beta U_{m-1}, \qquad m \geqq 1,\ \beta \geqq 0, \tag{1.1}$$

where $\{U_m\}$ is a sequence of identically and independently distributed non-negative random variables with common distribution function

$$A(x) = P(U_m \leqq x), \qquad m \geqq 0,\ x \geqq 0,$$

such that

$$\int_0^\infty x dA(x) < \infty,$$

and

(ii) the service time of the arrival at A_m is S_m, where $\{S_m\}$ is a sequence of identically and independently distributed non-negative random variables, distributed independently of the sequence $\{U_m\}$, with common distribution function

$$P(S_m \leqq x) = 1 - \exp(-\mu x), \qquad x \geqq 0,\ \mu > 0.$$

If P_j^m, $j \geqq 0$, $m \geqq 1$, is the probability that the arrival at A_m finds exactly j customers in the system, then it follows from the results of Finch [1] that

$$P_j = \lim_{m\to\infty} P_j^m, \qquad j \geqq 0,$$

exists. In fact, the general formulae of [1] enable one to write down explicit expressions for the probabilities P_j^m and P_j; however, these general formulae are applicable whatever the input process, and such general applicability means that for any particular input process the relevant expressions are neither the simplest nor the most elegant that can be obtained.

In an attempt to simplify these general formulae when the input

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process is that considered here, Finch used a heuristic symbolic argument which yielded simple and elegant expressions for the probabilities P_j. These expressions are the substance of Finch's conjecture. In this paper we examine the problem by a different method and show that the conjecture is false. We cannot, however, point to any error in the symbolic argument, and, since the form of the solution obtained is very close to that conjectured, we feel that a re-examination of that argument would be worthwhile, for if it could be made non-heuristic and error free, it would be simpler to apply to more general input processes than the methods of this paper.

2. Definitions and notation

We use capital letters to denote random variables and corresponding lower case letters to denote particular values taken by random variables, thus, for example, u_m denotes a numerical value taken by the random variable U_m. We denote the $(n+1)$-tuple $(u_0, u_1, \cdots, u_n)$ by $u^{(n)}$ and the corresponding vector random variable $(U_0, U_1, \cdots, U_n)$ by $U^{(n)}$.

$P_j(u^{(n)})$, $j \geqq 0$, denotes the conditional probability, given $U^{(n)} = u^{(n)}$, that the arrival at A_n finds exactly j customers in the system. Thus $E \cdot P_j$ $(U^{(n)})$, where $E\cdot$ denotes expectation, is the (unconditional) probability that the $(n+1)$th arrival finds exactly j customers in the system.

Throughout the paper we use, without further comment, the following notation.

$$\psi(\alpha) = E \cdot \exp(-\mu\alpha U_m) = \int_0^\infty \exp(-\mu\alpha u)dA(u),$$

$$k_j(x, y) = [\{\mu(y+\beta x)\}^j/j!] \exp\{-\mu(y+\beta x)\}, \qquad j \geqq 0,$$

$$K_j(x, y) = \sum_{i=j}^{\infty} k_i(x, y),$$

$$P(u^{(n)}; z) = \sum_{i=0}^{\infty} P_i(u^{(n)})z^i, \qquad |z| \leqq 1,$$

$$k(x, y; z) = \sum_{i=0}^{\infty} k_i(x, y)z^i = \exp\{-(1-z)\mu(y+\beta x)\},$$

$$c_i(u^{(n+1)}) = \sum_{j=0}^{\infty} P_j(u^{(n)})k_{j+i+1}(u_n, u_{n+1}), \qquad i \geqq 0,$$

$$P^*(s; z; n) = E \cdot [P(U^{(n)}; z) \exp\{-sU_n\}]$$

$$= \sum_{i=0}^{\infty} P_i^*(s, , n)z^i, \qquad |z| \leqq 1, \ \mathrm{Re}\cdot s \geqq 0,$$

$$c_i^*(s; n) = E \cdot [c_i(U^{(n)}) \exp\{-sU_n\}], \qquad \mathrm{Re}\cdot s \geqq 0.$$

We assume throughout that

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$$\mu(1+\beta)\int_0^\infty x dA(x) > 1, \tag{2.1}$$

so that

$$P_j = \lim_{n\to\infty} E \cdot P_j(U^{(n)}), \qquad j \geqq 0,$$

is a genuine probability distribution.

Under this assumption, it follows from Rouché's theorem that the equation

$$z = \psi\{(1+\beta)(1-z)\} \tag{2.2}$$

has a unique solution within the unit circle. This solution will be denoted by T.

3. Fundamental equations

It follows from the exponential form of the service time distribution that, conditional on $U^{(n+1)} = u^{(n+1)}$, the queue lengths at the instants $A_m - 0$, $m = 0, 1, 2, \cdots, n+1$ form a Markov chain, and that

$$P_j(u^{(n+1)}) = \sum_{i=0}^{\infty} P_{j+i-1}(u^{(n)})k_i(u_n, u_{n+1}), \qquad n \geqq 0, j \geqq 1. \tag{3.1}$$

Since $P(u^{(m)}; 1) = 1 = k(x, y; 1)$, it follows at once that

$$P_0(u^{(n+1)}) = \sum_{i=0}^{\infty} P_i(u^{(n)})K_{i+1}(u_n, u_{n+1}), \qquad n \geqq 0.$$

Note also that

$$\sum_{i=0}^{\infty} c_i(u^{(n+1)}) = P_0(u^{(n+1)}). \tag{3.2}$$

Forming the product of the power series $k(u_n, u_{n+1}; z)$, $P(u^{(n)}; z)$ and using the equations above, we obtain

$$P(u^{(n+1)}; z) = \sum_{i=0}^{\infty} (1-z^{-i})c_i(u^{(n+1)}) + zP(u^{(n)}; z) \cdot [\exp\{-(1-z^{-1})\mu(u_{n+1}+\beta u_n)\}]$$

for $|z| \leqq 1$, $z \neq 0$. Hence

$$\begin{aligned} P^*(s; z; n+1) = {} & \sum_{i=0}^{\infty} (1-z^{-i})c_i^*(s; n+1) \\ & + zP^*\{(1-z^{-1})\mu\beta; z; n\}\psi\{1-z^{-1}+s/\mu\} \end{aligned} \tag{3.3}$$

for $|z| \leqq 1$, $z \neq 0$, $Re \cdot s \geqq 0$, $Re \cdot [(1-z^{-1})\mu\beta] \geqq 0$. The restrictions on z require z to lie within or on the unit circle but to be without or on the circle

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with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$, with the point $z = 0$ deleted. This domain of the z-plane we denote by R.

By the methods of [1], it can be shown that assuming (2.1)

$$P(u; z) = \lim_{n\to\infty} E \cdot P(U_0, U_1, \cdots, U_{n-1}, u_n; z), \qquad u_n = u, \tag{3.4}$$

exists for $|z| \leqq 1$, $Re \cdot s \geqq 0$, and is the generating function of a probability distribution. In equation (3.4) we have departed momentarily from our usual notation for n-tuples and the expectation is with respect to the random variables $U_0, U_1, \cdots, U_{n-1}$.

Using a natural notation we write

$$P^*(s; z) = E \cdot [P(U; z) \exp \{-sU\}], \qquad |z| \leqq 1, Re \cdot s \geqq 0, \tag{3.5}$$

where U is a random variable with distribution function $A(x)$. Similarly we write

$$P^*(s; z) = \sum_{i=0}^{\infty} P_i^*(s) z^i$$

and

$$c_i^*(s) = \lim_{n\to\infty} c_i^*(s; n).$$

We have from (3.5)

$$P^*(s; 1) = \psi(s/\mu). \tag{3.6}$$

Letting $n \to \infty$ in (3.3) we obtain

$$P^*(s; z) = c(s; z) + zP^*\{(1-z^{-1})\mu\beta; z\}\psi\{1-z^{-1}+s/\mu\}, z \in R, Re \cdot s \geqq 0, \tag{3.7}$$

where

$$c(s; z) = \sum_{i=0}^{\infty} (1-z^{-i}) c_i^*(s), \ |z| \leqq 1, \ Re \cdot s \geqq 0. \tag{3.8}$$

4. Evaluation of $P^*(s; z)$

Putting $s = (1-z^{-1})\mu\beta$ in (3.7), we obtain, for $z \in R$, $Re \cdot s \geqq 0$,

$$P^*\{(1-z^{-1})\mu\beta; z) = [c\{(1-z^{-1})\mu\beta; z\}]/[1-z\psi\{(1-z^{-1})(1+\beta)\}],$$

so that for $z \in R$, $Re \cdot s \geqq 0$,

$$\begin{aligned} P^*(s; z) = c(s; z) + [c\{(1-z^{-1})\mu\beta; z\}] z\psi[1-z^{-1}+s/\mu] \\ \cdot (1-z\psi\{(1-z^{-1})(1+\beta)\})^{-1}. \end{aligned} \tag{4.1}$$

By a well known argument we simplify this expression, and prove that for $|z| \leqq 1$, $Re \cdot s \geqq 0$,

$$P^*(s; z) = (1-Tz)^{-1}[(1-T)\psi(s/\mu) - B(s) + zB(s)],$$

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where $B(s)$ is a function of s alone and T is the solution of (2.2) inside the unit circle.

Consider the function

$$F(s;z) = (1-Tz)P^*(s;z),\ |z| \leqq 1,\ Re \cdot s \geqq 0.$$

Since $P^*(s;z)$ is the generating function of a probability distribution, $P^*(s;z)$ and hence also $F(s;z)$ must be a regular function of z for $|z| \leqq 1$, $Re \cdot s \geqq 0$.

If we define

$$F(s;z) = (1-Tz)\{c(s;z)+[c\{(1-z^{-1})\mu\beta;z\}]z\psi\{1-z^{-1}+s/\mu\} \cdot (1-z\psi[(1-z^{-1})(1+\beta)])^{-1}\}$$

for $|z| \geqq 1$, $Re \cdot s \geqq 0$, then as the only zero of $1-z\psi[(1-z^{-1})(1+\beta)]$ outside the unit circle is the zero of $1-Tz$, $F(s;z)$ must be a regular function of z for $|z| \geqq 1$, $Re \cdot s \geqq 0$. Hence, by analytic continuation, $F(s;z)$ is a regular function of z for all finite z for $Re \cdot s \geqq 0$.

Also, using (3.2) and (3.8), it can be shown by Abel's theorem that $\lim_{z\to\infty} c(s;z)$ exists and equals $\sum_{i=1}^{\infty} c_i^*(s)$, and it can also be shown that $c\{(1-z^{-1})\mu\beta;z\}$, $c\{\mu\beta;z\}$ converge to the same limit $\sum_{i=1}^{\infty} c_i^*(\mu\beta)$ as $z \to \infty$.

Thus $\lim_{z\to\infty} F(s;z)/z$ exists, and, by (4.1), is given by

$$\text{(4.2)} \quad \lim_{z\to\infty} \frac{F(s;z)}{z} = T\{-\sum_{i=1}^{\infty} c_i^*(s)+\psi[1+s/\mu]\sum_{i=1}^{\infty} c_i^*(\mu\beta)(\psi[1+\beta])^{-1}\}.$$

Since a function $\theta(z)$ which is analytic for all finite values of z and $0(|z|)^k$, k a non-negative integer, as $z \to \infty$ is a polynomial of degree less than or equal to k, it follows that

$$F(s;z) = B_1(s)+B(s)z,\ Re \cdot s \geqq 0,$$

where $B_1(s)$, $B(s)$ are functions of s alone, and, from (4.2),

$$\text{(4.3)} \qquad B(\mu\beta) = 0.$$

(3.6) yields

$$P^*(s;z) = (1-Tz)^{-1}[(1-T)\psi(s/\mu)-B(s)+zB(s)],\ |z| \leqq 1,\ Re \cdot s \geqq 0.$$

Hence

$$\text{(4.4)} \qquad P_j^*(s) = \begin{cases} (1-T)\psi(s/\mu)-B(s), & j > 0 \\ T^{j-1}(1-T)[B(s)+T\psi(s/\mu)], & j = 0. \end{cases}$$

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5. Determination of $B(s)$

From (3.1)

$$P_j(u^{(n+1)}) = \sum_{i=0}^{\infty} \sum_{l=0}^{i} P_{j+i-1}(u^{(n)}) \exp(-\mu\beta u_n) \frac{(\mu\beta u_n)^{i-l}}{(i-l)!} \exp(-\mu u_{n+1}) \frac{(\mu u_{n+1})^l}{l!}, \quad j \geqq 1,$$

whence

$$P_j^*(s, \ , n+1) = \sum_{i=0}^{\infty} \frac{\partial^i}{\partial \sigma^i} [P_{j+i-1}^*(\sigma\beta, \ , n)\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}, \quad j \geqq 1.$$

Letting $n \to \infty$ and using (4.4), we see that for $j \geqq 2$

$$T^{j-1}(1-T)[B(s)+T\psi(s/\mu)] = \sum_{i=0}^{\infty} \frac{(-\mu)^i}{i!} \frac{\partial^i}{\partial\sigma^i} [T^{j+i-2}(1-T)[\{B(\sigma\beta)+T\psi(\sigma\beta/\mu)\}\psi\{(\sigma+s)/\mu\}]_{\sigma=\mu}.$$

$$T[B(s)+T\psi(s/\mu)] = \{B[\mu\beta(1-T)]+T\psi[\beta(1-T)]\}\psi[1-T+s/\mu].$$

By (4.3), (4.4),

$$Q_0 = T\psi\{\beta\}\psi\{1-T\}/\psi\{1-T+\beta\},$$

where Q_0 is the limiting probability of a non-zero number of customers in the queue, so

$$Q_0 = \psi\{(1+\beta)(1-T)\}\psi\{\beta\}\psi\{1-T\}/\psi\{1-T+\beta\}, \tag{5.1}$$

and

$$P(q \leqq j) = 1-T^j Q_0, \quad j \geqq 1, \tag{5.2}$$

where q is the limiting queue length random variable.

The results obtained in [1] by the symbolic method were

$$Q_0 = \psi\{1-T\}\psi\{\beta(1-T)\}, \tag{5.1a}$$

$$P(q \leqq j) = 1-T^j Q_0, \quad j \geqq 1. \tag{5.2a}$$

(5.2), (5.2a) are of the same form, but (5.1), (5.1a) are different, as is shown by the following example.

Example. Taking

$$A(x) = 1-\exp(-\lambda x), \quad x \geqq 0, \quad \lambda > 0,$$

(2.2) yields $T = 1$ or $\lambda[\mu(1+\beta)]^{-1}$, of which only the latter root can lie within the unit circle. With $T = \lambda[\mu(1+\beta)]^{-1}$, we have

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$$\psi(\alpha) = \lambda[\lambda+\alpha\mu]^{-1}.$$

(5.1a), (5.1) then yield respectively

$$Q_0 = \lambda^2(1+\beta)^2[\lambda\beta+\mu(1+\beta)]^{-1}[\lambda+\mu\beta(1+\beta)]^{-1},$$
$$Q_0 = \lambda^2[\lambda\beta+(1+\beta)^2\mu][\mu(1+\beta)(\lambda+\beta\mu)\{\lambda\beta+\mu(1+\beta)\}]^{-1},$$

which are in general different, although they agree if $\beta = 0$, that is, when the time intervals between successive arrivals are identically and independently distributed.

Reference

[1] Finch, P. D., The single server queueing system with non-recurrent input process and Erlang service time, this Journal 3 (1963), 220—236.

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