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ON UNIT SOLUTIONS OF THE EQUATION $xyz = x + y + z$ IN NOT TOTALLY REAL CUBIC FIELDS

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ABSTRACT. It is shown that the equation $xyz = x+y+z$ has unit solutions in only four not totally real cubic fields: two fields which are real and two fields which are imaginary. These fields are then listed.

1. **Introduction.** The equation

$$xyz = x + y + z = 1$$

has been studied and shown to have no solution in the rational number field $\mathbb{Q}$ ([1], [3], [4]). This leads to the study of the equation

$$u_1u_2u_3 = u_1 + u_2 + u_3, \tag{1.1}$$

where u_i, $i = 1, 2, 3$, is a unit in the ring of integers of an algebraic number field K. If K is a quadratic extension of the rationals, this problem has been completely solved by Mollin, Small, Varadarajan and Walsh [2]. We have shown that in a real number field, with unit group of rank 1 and a fundamental unit $\eta > 3$, (1.1) has no solution, and consequently (1.1) has no solution in any pure cubic field [5]. In this paper, we shall solve this problem for not totally real cubic extensions of the rationals.

2. **Results.**

THEOREM. *Let K be a real but not totally real cubic field. Then (1.1) has no solution except for two fields:*

$$K_1 = \mathbb{Q}(\eta_1),$$

where

$$\mathrm{Irr}(\eta_1, \mathbb{Q}, x) = x^3 - x^2 - x - 1$$

and

$$K_2 = \mathbb{Q}(\eta_2),$$

where

$$\mathrm{Irr}(\eta_2, \mathbb{Q}, x) = x^3 - x - 1, \textit{and } \eta_i \in \mathbb{R}.$$

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PROOF. Let U_K be the group of units of the ring of integers of K. Then from the Dirichlet unit theorem, the rank of U_K is 1. Let η be the fundamental unit of K which is greater than 1, then $K = \mathbb{Q}(\eta)$.

Let $f(x) = \mathrm{Irr}(\eta, \mathbb{Q}, x)$. Since η is a unit,

$$f(x) = x^3 + ax^2 + bx - 1, \quad a, b \in \mathbb{Z}$$

If U_K contains a solution of (1.1) then η satisfies

$$\eta^{\ell_1+\ell_2+\ell_3} = \eta^{\ell_1} + \eta^{\ell_2} + \eta^{\ell_3}$$

or

$$\eta^{\ell_1+\ell_2+\ell_3} = \eta^{\ell_1} - \eta^{\ell_2} - \eta^{\ell_3} \text{ for some } \ell_i \in \mathbb{Z}.$$

Here we may assume $\ell_1 \geq \ell_2 \geq \ell_3$. So η satisfies some polynomial equation $h(x) = 0$, where

$$h(x) = x^{\ell_1+\ell_2} - x^{\ell_1-\ell_3} - x^{\ell_2-\ell_3} - 1$$

or

$$h(x) = x^{\ell_1+\ell_2} - x^{\ell_1-\ell_3} + x^{\ell_2-\ell_3} + 1$$

Let

$$h_1(x) = x^{\ell_1+\ell_2} - x^{\ell_1-\ell_3} - x^{\ell_2-\ell_3} - 1$$

and

$$h_2(x) = x^{\ell_1+\ell_2} - x^{\ell_1-\ell_3} + x^{\ell_2-\ell_3} + 1,$$

so $h_1(1) = -2$ and $h_2(1) = 2$.

If $\ell_1 + \ell_2$ is even, then $\ell_1 - \ell_3$ and $\ell_2 - \ell_3$ have the same parity. Consequently,

$$h_1(-1) = 1 - 1 - 1 - 1 = -2$$

or

$$h_1(-1) = 1 + 1 + 1 - 1 = 2.$$

On the other hand, if $\ell_1 + \ell_2$ is odd, then $\ell_1 - \ell_3$ and $\ell_2 - \ell_3$ have opposite parity. Consequently,

$$h_1(-1) = -1 + 1 - 1 - 1 = -2$$

or

$$h_1(-1) = -1 - 1 + 1 - 1 = -2.$$

Thus $h_1(-1)$ is always ± 2. Similarly, $h_2(-1)$ is always ± 2. Therefore $h(\pm 1) = \pm 2$.

Since $f(x) = \mathrm{Irr}(\eta, \mathbb{Q}, x)$, $f(x)$ divides $h(x)$, and f(± 1) divides $h(\pm 1)$. Thus $f(\pm 1) = \pm 1$ or ± 2. And since K is not totally real, $f(x)$ has only one real root, $\eta > 1$, and thus

$f(x) < 0$ if $x < \eta$. So since $\pm 1 < \eta$, $f(\pm 1) < 0$, which implies that $f(\pm 1) = -1$ or -2.

Recall $f(x) = x^3 + ax^2 + bx - 1$, so $f(1) = a + b$ and $f(-1) = a - b - 2$. Then solving the system of simple linear equations in a and b:

$$a + b = -1 \text{ or } -2,$$
$$a - b - 2 = -1 \text{ or } -2,$$

and considering that a and b are integers, we find that $(a, b) = (-1, -1)$ or $(0, -1)$. Therefore

$$\mathrm{Irr}(\eta_1, \mathbb{Q}, x) = x^3 - x^2 - x - 1$$

or

$$\mathrm{Irr}(\eta_2, \mathbb{Q}, x) = x^3 - x - 1.$$

In $K_1 = \mathbb{Q}(\eta_1)$, since $\eta_1^3 - \eta_1^2 - \eta_1 - 1 = 0$,

$$u_1 = \eta_1^2,\ u_2 = \eta_1,\ u_3 = 1 \text{ is a solution of } (1.1).$$

In fact,

$$\eta_1 = \frac{\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1}{3}.$$

In $K_2 = \mathbb{Q}(\eta_2)$, since $\eta_2^3 - \eta_2 - 1 = 0$, and

$$(\eta_2^3 + \eta_2)(\eta_2^3 - \eta_2 - 1) = \eta_2^6 - \eta_2^3 - \eta_2^2 - \eta_2 = 0,$$
$$u_1 = \eta_2^3, u_2 = \eta_2^2, u_3 = \eta_2 \text{ is a solution of } (1.1).$$

In fact,

$$\eta_2 = \frac{\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}}{6}.$$

We have proved the theorem.

COROLLARY. *Let K' be an imaginary cubic field, then (1.1) has a solution if and only if*

$$K' = K_1' = \mathbb{Q}(\eta_1')$$

where

$$\mathrm{Irr}(\eta_1', \mathbb{Q}, x) = x^3 - x^2 - x - 1$$

or

$$K' = K_2' = \mathbb{Q}(\eta_2')$$

where

$$\mathrm{Irr}(\eta_2', \mathbb{Q}, x) = x^3 - x - 1,\ \eta_i' \in C - \mathbb{R}.$$

PROOF. Let K' be an imaginary cubic field and let η' be a fundamental unit with $|\eta'| < 1$. Then $K' = \mathbb{Q}(\eta')$.

Let $f(x) = \mathrm{Irr}(\eta', \mathbb{Q}, x)$, η be the real conjugate root of η' and $K = \mathbb{Q}(\eta)$. Then it is obvious that there is an isomorphism $\sigma: K' \longrightarrow K$ with $\sigma(\eta') = \eta$. Thus

$$\eta'^{\ell_2+\ell_2+\ell_3} = \eta'^{\ell_1} \pm \eta'^{\ell_2} \pm \eta'^{\ell_3}$$

if and only if

$$\eta^{\ell_2+\ell_2+\ell_3} = \eta^{\ell_1} \pm \eta^{\ell_2} \pm \eta^{\ell_3}$$

Then the result follows from the theorem.

REMARK. It is not difficult to calculate the values of η'. In fact,

$$\eta_1' = \frac{-\sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}} + 2}{6} + \frac{\sqrt{3}\left(\sqrt[3]{19-3\sqrt{33}} - \sqrt[3]{19+3\sqrt{33}}\right)\sqrt{-1}}{6},$$

or

$$\eta_1' = \frac{-\sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}} + 2}{6} - \frac{\sqrt{3}\left(\sqrt[3]{19-3\sqrt{33}} - \sqrt[3]{19+3\sqrt{33}}\right)\sqrt{-1}}{6};$$

$$\eta_2' = \frac{-\sqrt[3]{108+12\sqrt{69}} - \sqrt[3]{108-12\sqrt{69}}}{12} + \frac{\sqrt{3}\left(\sqrt[3]{108+12\sqrt{69}} - \sqrt[3]{108-12\sqrt{69}}\right)\sqrt{-1}}{12},$$

or

$$\eta_2' = \frac{-\sqrt[3]{108+12\sqrt{69}} - \sqrt[3]{108-12\sqrt{69}}}{12} - \frac{\sqrt{3}\left(\sqrt[3]{108+12\sqrt{69}} - \sqrt[3]{108-12\sqrt{69}}\right)\sqrt{-1}}{12}.$$

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