THE DIMENSION OF ATTRACTORS OF NONAUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS

T. CARABALLO[1], J. A. LANGA[2] and J. VALERO[3]

Abstract

The concept of nonautonomous (or cocycle) attractors has become a proper tool for the study of the asymptotic behaviour of general nonautonomous partial differential equations. This is a time-dependent family of compact sets, invariant for the associated process and attracting "from $-\infty$". In general, the concept is rather different to the classical global attractor for autonomous dynamical systems. We prove a general result on the finite fractal dimensionality of each compact set of this family. In this way, we generalise some previous results of Chepyzhov and Vishik. Our results are also applied to differential equations with a nonlinear term having polynomial growth at most.

1. Introduction

In this paper, we develop a general theory on the finite dimension of attractors for nonautonomous partial differential equations and we apply it, in particular, to estimate the fractal dimension of the attractor for the following nonautonomous equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(t, u) = h(t), \\ u \mid_{\partial\Omega} = 0, \\ u(\tau) = u_\tau, \end{cases}$$

where the function $h(t)$ is allowed to have polynomial growth in time (see condition (4.4) below). For these kinds of nonautonomous systems it is not possible in general

---

[1]Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain; e-mail: caraball@us.es.

[2]Dpto. Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain; e-mail: langa@numer.us.es.

[3]Universidad Cardenal Herrera CEU, Comisario 3, 03203 Elche, Alicante, Spain; e-mail: valer.el@ceu.es.

to obtain a uniform global attractor in the sense of [5], since the trajectories can be unbounded when time rises to infinity. A different approach was developed in [8, 9, 25] (see also [3, 16, 17, 24]), where the existence of attractors for some stochastic and nonautonomous equations was studied. The main definitions and theorems from the abstract theory of attractors for such systems are given in Section 2.

It is worth pointing out that in such systems the global attractor is not a compact set, but a parameterised family $\mathscr{A}(t)$ of compact sets. We are interested in proving the finite dimensionality of each of the sets $\mathscr{A}(t)$. We note that the union of all the attractors, that is, $\bigcup_{t \in \mathbb{R}} \mathscr{A}(t)$, can be infinite dimensional.

In the case of stochastic equations of parabolic and hyperbolic types such results were obtained in [10, 12] and [13]. There are some technical tools in the proofs of these papers that do not seem to be applicable to the nonautonomous case. As far as we know, the only result for the nonautonomous case was proved in [4] under the assumption that the function $h(t)$ was uniformly bounded in the variable t. In such a case, the union of the whole family of attractors $\bigcup_{t \in \mathbb{R}} \mathscr{A}(t)$ is bounded, and the well-known technique of Lyapunov exponents, developed in [7], can be adapted with slight modifications. However, when the function $h(t)$ is allowed to have polynomial growth, the supremum of the norm of the global attractor $\mathscr{A}(t)$ can also have polynomial growth, so that we cannot expect the union of attractors to be bounded.

In this paper we extend the general theory on the finite-dimensionality of compact invariant sets in Hilbert spaces (see [1, 14, 20, 23, 27]) to the case of a parameterised family of global attractors with polynomial growth at most. The invariance property for nonautonomous attractors is now stated for a time-dependent family of compact sets $\{\mathscr{A}(t)\}_{t \in \mathbb{R}}$ and the attraction is defined for trajectories with initial time going to $-\infty$. Thus the idea is to construct a sequence of coverings of $\mathscr{A}(t)$ by iterating n times an initial covering of $\mathscr{A}(t - nT^*)$, as $n \to \infty$.

Further we apply this abstract theorem to the attractor of the equation given above. We note that we are able to obtain the estimation of dimension in the case where the function $f(t, u)$ is globally Lipschitz on the second variable u. In the autonomous case it is possible to change the global Lipschitz condition to a local one by proving that the global attractor is bounded in $L_\infty(\Omega)$ (see [15, 21, 28]). In our case, in order to use a similar idea we would need to obtain an estimation of the norm in $L_\infty(\Omega)$ of the union $\bigcup_{\tau \le t} \mathscr{A}(\tau)$, $\forall t$, which is not possible in general as we have already remarked.

2. Attractors of nonautonomous equations

In this section, we introduce the general framework in which the theory of attractors for nonautonomous systems is going to be studied (see Crauel *et al.* [8] and Schmalfuss [26]). As a first step, we define semiprocesses as two-time dependent operators related

with the solutions of nonautonomous differential equations. In this way, we are able to treat these equations as dynamical systems. Secondly, we give the general definitions of invariance, absorption and attraction and we finish with a general theorem on the existence of global attractors for these equations.

Let (H, d) be a complete metric space (with the metric d) and $\{S(t, s)\}_{t \geq s}$, $t, s \in \mathbb{R}$, be a family of mappings satisfying:

(i) $S(t, t, \cdot) = \mathrm{Id}$,

(ii) $S(t, s, S(s, \tau, u)) = S(t, \tau, u)$, for all $\tau \leq s \leq t$, $u \in H$,

(iii) $u \mapsto S(t, \tau, u)$ is continuous in H.

This map is called a process (this term was introduced by Dafermos [11]). In general, we have to consider $S(t, \tau, u)$ as the solution of a nonautonomous equation at time t with initial condition u at time τ.

Let $\mathscr{D}$ be a non-empty set of parameterised families of non-empty bounded sets $\widehat{D} = \{D(t)\}_{t \in \mathbb{R}}$. In particular, $\widehat{D} = \{D(t)\}_{t \in \mathbb{R}} \in \mathscr{D}$, where $D(t) \equiv B$ for all t, and $B \subset H$ is a bounded set. In what follows, we will consider this set $\mathscr{D}$ to be fixed, so that the concepts of absorption and attraction in our analysis are always referred to it.

For $A, B \subset H$ we define the Hausdorff semidistance as

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

DEFINITION 2.1. Given $t_0 \in \mathbb{R}$, we say that $K(t) \subset H$ is attracting at time t_0 if for every $\widehat{D} = \{D(t)\} \in \mathscr{D}$ we have that $\lim_{\tau \to -\infty} \mathrm{dist}(S(t_0, \tau, D(\tau)), K(t_0)) = 0$.
A family $\widehat{K} = \{K(t)\}_{t \in \mathbb{R}}$ is attracting if $K(t_0)$ is attracting at time t_0, for all $t_0 \in \mathbb{R}$.

The previous concept considers a fixed final time and moves the initial time to $-\infty$. Note that this does not mean that we are going backwards in time, but we consider the state of the system at time t_0 starting at $\tau \to -\infty$. This is called *pullback attraction* in the literature (see, for example, [19, 26]).

DEFINITION 2.2. Given $t_0 \in \mathbb{R}$, we say that $B(t_0) \subset H$ is absorbing at time t_0 if for every $\widehat{D} = \{D(t)\} \in \mathscr{D}$ there exists $T = T(t, \widehat{D}) \in \mathbb{R}$ such that

$$S(t_0, \tau, D(\tau)) \subset B(t_0), \quad \text{for all } \tau \leq T.$$

A family $\widehat{B} = \{B(t)\}_{t \in \mathbb{R}}$ is absorbing if $B(t_0)$ is absorbing at time t_0, for all $t_0 \in \mathbb{R}$.

Note that every absorbing set at time t_0 is attracting.

DEFINITION 2.3. Let $\widehat{B} = \{B(t)\}_{t \in \mathbb{R}}$ be a family of subsets of H. This family is said to be invariant with respect to the process S if $S(t, \tau, B(\tau)) = B(t)$, for all $(\tau, t) \in \mathbb{R}^2$, $\tau \leq t$.

Note that this property is a generalisation of the classical property of invariance for semigroups. However, in this case we have to define the invariance with respect to a family of sets depending on a parameter.

We define the *omega-limit set* at time t_0 of $\widehat{D} \equiv \{D(t)\} \in \mathcal{D}$ as

$$\Lambda(\widehat{D}, t_0) = \bigcap_{s \leq t_0} \overline{\bigcup_{\tau \leq s} S(t_0, \tau, D(\tau))}.$$

From now on, we assume that there exists a family $\widehat{K} = \{K(t)\}_{t \in \mathbb{R}}$ of compact absorbing sets, that is, $K(t) \subset H$ is non-empty, compact and absorbing for each $t \in \mathbb{R}$. Note that, in this case, $\Lambda(\widehat{D}, t_0) \subset K(t_0)$, for all $\widehat{D} = \{D(t)\} \in \mathcal{D}$, $t_0 \in \mathbb{R}$. As in the autonomous case, it is not difficult to prove that under these conditions $\Lambda(\widehat{D}, t_0)$ is non-empty, compact and attracts $\widehat{D} = \{D(t)\} \in \mathcal{D}$ at time t_0. The proof is similar to that of [8, Lemma 1.1], where the set $\mathcal{D}$ consists only of bounded sets.

DEFINITION 2.4. The family of compact sets $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ is said to be the global attractor associated with the process S if it is invariant, attracting every $\widehat{D} = \{D(t)\} \in \mathcal{D}$ (for all $t_0 \in \mathbb{R}$) and minimal in the sense that if $\widehat{C} = \{C(t)\}_{t \in \mathbb{R}}$ is another family of closed attracting sets, then $\mathcal{A}(t) \subset C(t)$ for all $t \in \mathbb{R}$.

REMARK 2.5. Chepyzhov and Vishik [4] define the concept of kernel sections for nonautonomous dynamical systems which corresponds to our definition of a global nonautonomous attractor with $\widehat{D} = \{D(t) \equiv B\}_{t \in \mathbb{R}}$ where $B \subset H$ is bounded.

The general result on the existence of nonautonomous attractors is a generalisation of the abstract theory for autonomous dynamical systems (Temam [27], Hale [18]).

THEOREM 2.6. *Assume that there exists a family of compact absorbing sets. Then the family* $\widehat{\mathcal{A}} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$ *defined by*

$$\mathcal{A}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda(\widehat{D}, t)}$$

is the global nonautonomous attractor.

As the proof of this theorem repeats that of [8, Theorem 1.1] with slight modifications, we will omit it.

REMARK 2.7. All the general theory of nonautonomous attractors can be written in the framework of cocycles (see among others, Cheban *et al.* [3], Crauel and Flandoli [9], Kloeden and Schmalfuss [19], Schmalfuss [26]). We could have also followed this notation here, but we think that, in this case, it is clearer to keep the explicit dependence on time of the attractor, which in addition, allows a more straightforward comparison of the results in [4].

3. Dimension of nonautonomous attractors

In [4], Chepyzhov and Vishik prove a general result for the Hausdorff dimension of kernel sections $\mathscr{A}(t)$ associated with a process $\{S(t, \tau)\}$ generated by a nonautonomous differential equation. The main hypothesis is the uniform boundedness of the set $\bigcup_{t \in \mathbb{R}} \mathscr{A}(t)$. In applications, this is related to the existence of a uniform bound for the nonautonomous terms in the system. In our case, we allow these terms to be unbounded in t, so that their results are not suitable for our situation. However, we are able to prove a general result on the finite fractal dimensionality of the nonautonomous attractor.

Let H be a Hilbert space and $\mathscr{A} \subset H$ be a compact subset of H. We first recall the definition of the Hausdorff and fractal dimensions of $\mathscr{A}$.

We shall denote by $B(a, r)$ a closed ball of radius r centred at a. Let $\mathscr{U}$ be a covering of $\mathscr{A}$ by a finite family of balls $B(x_i, r_i)$ such that $\sup_i(r_i) = \delta(\mathscr{U}) \leq \delta$. Then the d-dimensional Hausdorff measure of $\mathscr{A}$ is defined as follows:

$$\mu_H(\mathscr{A}, d) = \lim_{\delta \to 0} \mu_H(\mathscr{A}, d, \delta),$$

where

$$\mu_H(\mathscr{A}, d, \delta) = \inf_{\delta(\mathscr{U}) \leq \delta} \sum_i r_i^d,$$

where the inf is extended to all the possible coverings $\mathscr{U}$ of $\mathscr{A}$ such that $\delta(\mathscr{U}) \leq \delta$. It is known that there exists $d = d_H(\mathscr{A}) \in [0, +\infty]$ such that $\mu_H(\mathscr{A}, d) = 0$ for $d > d_H(\mathscr{A})$ and $\mu_H(\mathscr{A}, d) = \infty$ for $d < d_H(\mathscr{A})$. The value $d_H(\mathscr{A})$ is called the Hausdorff dimension of $\mathscr{A}$.

The fractal dimension of $\mathscr{A}$ is given by

$$d_f(\mathscr{A}) = \inf\{d > 0 \mid \mu_f(\mathscr{A}, d) = 0\},$$

where

$$\mu_f(\mathscr{A}, d) = \overline{\lim_{\epsilon \to 0}} \mu_f(\mathscr{A}, \varepsilon, d) = \overline{\lim_{\epsilon \to 0}} \epsilon^d n_\epsilon,$$

and n_ϵ is the minimum number of balls of radius $r = \epsilon$ which is necessary to cover $\mathscr{A}$. Since $\mu_H(\mathscr{A}, d) \leq \mu_f(\mathscr{A}, d)$ it is clear that $d_H(\mathscr{A}) \leq d_f(\mathscr{A})$, the converse being false in general (Eden *et al.* [14]).

Before proving our main result in this section, we will recall a technical lemma which will be repeatedly used in the proof (see [1, Lemma 1]).

LEMMA 3.1. *Let $B(a, \gamma) \subset \mathbb{R}^N$ be a closed ball of radius γ centred at a. For any $0 < \lambda < \gamma$ the minimum number of balls n_λ of radius λ which is necessary to cover $B(a, \gamma)$ is less than or equal to $(3\gamma/\lambda)^N$.*

We now consider a process $S(t, \tau, u) : \mathbb{R} \times \mathbb{R} \times H \to H, t \geq \tau$, having the family of global attractors $\widehat{\mathscr{A}} = \{\mathscr{A}(t)\}_{t \in \mathbb{R}}$.

THEOREM 3.2. *Suppose there exist constants $K_0, K_1, \theta > 0$ such that*

$$\|\mathscr{A}(t)\|^+ \leq K_0 |t|^\theta + K_1, \forall t \in \mathbb{R}, \tag{3.1}$$

where $\|\mathscr{A}(t)\|^+ = \sup_{y \in \mathscr{A}(t)} \|y\|$.

Also assume that for any $t \in \mathbb{R}$ there exist $T^* = T^*(t), l = l(t, T^*) \in [1, +\infty),$ $\delta = \delta(t, T^*) \in (0, 1/\sqrt{2})$ and $N = N(t)$, such that for any $u, v \in A(\tau), \tau \leq t - T^*,$

$$\|S(\tau + T^*, \tau, u) - S(\tau + T^*, \tau, v)\| \leq l\|u - v\|, \tag{3.2}$$
$$\|Q_N(S(\tau + T^*, \tau, u) - S(\tau + T^*, \tau, v))\| \leq \delta\|u - v\|, \tag{3.3}$$

where Q_N is the projector mapping H onto some subspace $H_N^\perp$ of codimension $N \in \mathbb{N}$. Then, for any $\eta = \eta(t) > 0$ such that $\sigma = \sigma(t) = (6\sqrt{2}l)^N(\sqrt{2}\delta)^\eta < 1$, the following inequality holds:

$$d_H(\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)) \leq N + \eta. \tag{3.4}$$

PROOF. Let us fix $t \in \mathbb{R}$ and choose $\eta > 0$ such that $\sigma < 1$. We also take an arbitrary $\tau \leq t - T^*$, and denote $\varepsilon(\tau) = 2(K_0|\tau|^\theta + K_1)$. Let $\mathscr{U}_0$ be a covering of $A(\tau)$ by one ball $B(a_1, \varepsilon(\tau)), a_1 \in A(\tau)$, of radius $\varepsilon(\tau)$ centred at a_1. Hence $\mathscr{A}(\tau) \subset B(a_1, \varepsilon(\tau))$.

Since $\mathscr{A}(\tau + T^*) = S(\tau + T^*, \tau, \mathscr{A}(\tau))$ and using condition (3.2) we have

$$\mathscr{A}(\tau + T^*) \subset B(S(\tau + T^*, \tau, a_1), l\varepsilon(\tau)).$$

Let us denote by $\mathscr{P}_N$ the orthoprojector onto the subspace H_N of dimension N which is orthogonal to $H_N^\perp$ (and then $\mathscr{P}_N \oplus Q_N = I, H_N \oplus H_N^\perp = H$). It is clear that $\mathscr{P}_N B(S(\tau + T^*, \tau, a_1), l\varepsilon(\tau)) \subset B^N(\mathscr{P}_N S(\tau + T^*, \tau, a_1), l\varepsilon(\tau))$, where $B^N(a, \beta)$ denotes a closed ball in H^N of radius β and centred at a.

In view of the preceding lemma we can cover $B^N(\mathscr{P}_N S(\tau + T^*, \tau, a_1), l\varepsilon(\tau))$ by balls $B^N(a_{1j}, (\delta/2)\varepsilon(\tau)), j = 1, \ldots, m_1, a_{1j} \in H_N$ and $m_1 = m_1(t) \leq (6l/\delta)^N$.

Let us denote $\mathscr{M}_{1j} = (\mathscr{P}_N^{-1} B^N(a_{1j}, (\delta/2)\varepsilon(\tau))) \cap \mathscr{A}(\tau + T^*)$. We take arbitrary $y_{1j} \in \mathscr{M}_{1j}$. We shall show that the set of balls $B(y_{1j}, \gamma\varepsilon(\tau)), j = 1, \ldots, m_1,$ $\gamma = \sqrt{2}\delta$ (note that we have assumed that $\gamma < 1$), is a new covering of $\mathscr{A}(\tau + T^*)$. Since

$$\mathscr{A}(\tau + T^*) \subset \bigcup_{j=1}^{m_1} \mathscr{M}_{1j},$$

it is sufficient to prove that $\mathscr{M}_{1j} \subset B(y_{1j}, \gamma\varepsilon(\tau))$, $\forall j$. Let $y \in \mathscr{M}_{1j}$. There exist $v_1, v_2 \in B(a_1, \varepsilon(\tau)) \cap \mathscr{A}(\tau)$ such that $S(\tau + T^*, \tau, v_1) = y$, $S(\tau + T^*, \tau, v_2) = y_{1j}$. Then $\|v_1 - v_2\| \leq \varepsilon(\tau)$ and in view of (3.3), $\|Q_N y - Q_N y_{1j}\| \leq \delta\varepsilon(\tau)$. On the other hand, $\|\mathscr{P}_N y - \mathscr{P}_N y_{1j}\| \leq \|\mathscr{P}_N y - a_{1j}\| + \|\mathscr{P}_N y_{1j} - a_{1j}\| \leq \delta\varepsilon(\tau)$. Hence

$$\|y - y_{1j}\| \leq \sqrt{(\delta\varepsilon(\tau))^2 + (\delta\varepsilon(\tau))^2} = \gamma\varepsilon(\tau).$$

We have obtained a covering $\mathscr{U}_1$ of $\mathscr{A}(\tau + T^*)$ by balls of radius $\gamma\varepsilon(\tau)$ such that the number of balls is m_1. Therefore $n_{\gamma\varepsilon(\tau)} \leq m_1 \leq (6l/\delta)^N$, where $n_{\gamma\varepsilon(\tau)}$ now denotes the minimum number of balls of radius equal to $\gamma\varepsilon(\tau)$ which is necessary to cover $\mathscr{A}(\tau + T^*)$. Then

$$\mu_f(\mathscr{A}(\tau + T^*), \gamma\varepsilon(\tau), d) = n_{\gamma\varepsilon(\tau)}(\gamma\varepsilon(\tau))^d \leq (6l/\delta)^N \left(\sqrt{2}\delta\varepsilon(\tau)\right)^d.$$

Taking $d = d(t) = N + \eta$, we get ·

$$\mu_f(\mathscr{A}(\tau + T^*), \gamma\varepsilon(\tau), N + \eta) \leq \left(\sqrt{2}\delta\right)^\eta \left(6\sqrt{2}l\right)^N \varepsilon(\tau)^{N+\eta} = \sigma\varepsilon(\tau)^{N+\eta}.$$

Suppose now that $\tau \leq t - 2T^*$. Take the covering $\mathscr{U}_1 = \{B(y_{1i}, \gamma\varepsilon(\tau))\}_{i=1}^{m_1}$ of $\mathscr{A}(\tau + T^*)$ and define $\mathscr{M}_i = S(\tau + 2T^*, \tau + T^*, A(\tau + T^*) \cap B(y_{1i}, \gamma\varepsilon(\tau))) \cap A(\tau + 2T^*)$, $i = 1, \ldots, m_1$.

Now, since $\mathscr{A}(\tau + 2T^*) = S(\tau + 2T^*, \tau + T^*, \mathscr{A}(\tau + T^*))$ and using (3.2), we have

$$\mathscr{A}(\tau + 2T^*) \subset \bigcup_{i=1}^{m_1} \mathscr{M}_i \subset \bigcup_{i=1}^{m_1} B(S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau)).$$

It is clear that $\mathscr{P}_N B(S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau)) \subset B^N(\mathscr{P}_N S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau))$, $\forall i$. In view of the preceding technical lemma, we can cover each $B^N(\mathscr{P}_N S(\tau + 2T^*, \tau + T^*, y_{1i}), l\gamma\varepsilon(\tau))$ by balls $B^N(a_{ij}, (\delta/2)\gamma\varepsilon(\tau))$, $j = 1, \ldots, n_i$, $a_{ij} \in H_N$ and $n_i = n_i(t) \leq (6l/\delta)^N$, $\forall i$.

Let us denote $\mathscr{M}_{ij} = (\mathscr{P}_N^{-1} B^N(a_{ij}, (\delta/2)\gamma\varepsilon(\tau))) \cap \mathscr{M}_i$. We take arbitrary $y_{ij} \in \mathscr{M}_{ij}$. We shall show that the set of balls $B(y_{ij}, \gamma^2\varepsilon(\tau))$, $i = 1, \ldots, m_1$, $j = 1, \ldots, n_i$, $\gamma = \sqrt{2}\delta$, is a new covering of $\mathscr{A}(\tau + 2T^*)$. Indeed, since $\mathscr{A}(\tau + 2T^*) \subset \bigcup_{ij} \mathscr{M}_{ij}$, it is sufficient to prove that $\mathscr{M}_{ij} \subset B(y_{ij}, \gamma^2\varepsilon(\tau))$, $\forall i, j$. Let $y \in \mathscr{M}_{ij}$. There exist $v_1, v_2 \in B(y_{1i}, \gamma\varepsilon(\tau)) \cap \mathscr{A}(\tau + T^*)$ such that $S(\tau + 2T^*, \tau + T^*, v_1) = y$, $S(\tau + 2T^*, \tau + T^*, v_2) = y_{ij}$. Then $\|v_1 - v_2\| \leq \gamma\varepsilon(\tau)$ and in view of (3.3), $\|Q_N y - Q_N y_{ij}\| \leq \delta\gamma\varepsilon(\tau)$. On the other hand, $\|\mathscr{P}_N y - \mathscr{P}_N y_{ij}\| \leq \|\mathscr{P}_N y - a_{ij}\| + \|\mathscr{P}_N y_{ij} - a_{ij}\| \leq \delta\gamma\varepsilon(\tau)$. Hence

$$\|y - y_{ij}\| \leq \sqrt{(\delta\gamma\varepsilon(\tau))^2 + (\delta\gamma\varepsilon(\tau))^2} = \gamma^2\varepsilon(\tau).$$

We have obtained a covering $\mathcal{U}_2$ of $\mathscr{A}(\tau + 2T^*t)$ by balls of radius $\gamma^2 \varepsilon(\tau)$ such that the number of balls is $m_2 = m_2(t) = \sum_{i=1}^{m_1} n_i$. Therefore

$$n_{\gamma^2 \varepsilon(\tau)} \leq \sum_{i=1}^{m_1} n_i \leq m_1 (6l/\delta)^N \leq (6l/\delta)^{2N}.$$

Let $k \in \mathbb{N}$. If we suppose that $\tau \leq t - kT^*$, we can obtain, in the same way as before, a sequence of coverings $\mathcal{U}_j$, $j = 1, 2, \ldots, k$ of the sets $\mathscr{A}(\tau + j T^*)$ by balls of radius $\gamma^j \varepsilon(\tau)$ and such that the number of balls is less than or equal to $(6l/\delta)^{jN}$. Therefore $n_{\gamma^j \varepsilon(\tau)} \leq (6l/\delta)^{jN}$, where $n_{\gamma^j \varepsilon(\tau)}$ now denotes the minimum number of balls of radius equal to $\gamma^j \varepsilon(\tau)$ which is necessary to cover $\mathscr{A}(\tau + j T^*)$.

Hence choosing $\tau = t - kT^*$ we obtain

$$\mu_f \left(\mathscr{A}(t), \gamma^k \varepsilon(\tau), N + \eta \right) \leq \left(\sqrt{2}\delta \right)^{k\eta} \left(6\sqrt{2}l \right)^{kN} \varepsilon(\tau)^{N+\eta}$$
$$\leq \sigma^k \left(K_1 + K_0 |(t - kT^*)|^\theta \right)^{N+\eta}.$$

This implies that $\overline{\lim}_{\alpha \to 0} \mu_f (\mathscr{A}(t), \alpha, d) = 0$, for $d = N + \eta$. Indeed, as for k large enough the sequence

$$r(k) = \gamma^k \varepsilon(t - kT^*) = \gamma^k \left(K_1 + K_0 |t - kT^*|^\theta \right)$$

is decreasing, we have that for any $\alpha > 0$ small enough, one can find some $k \in \mathbb{N}$ such that $r(k) \leq \alpha < r(k - 1)$. It is clear that $n_\alpha \leq n_{r(k)} \leq (6l/\delta)^{kN}$. Then

$$\overline{\lim_{\alpha \to 0}} \mu_f (\mathscr{A}(t), \alpha, N + \eta) = \overline{\lim_{\alpha \to 0}} n_\alpha \alpha^{N+\eta} \leq \lim_{k \to \infty} (6l/\delta)^{kN} (r(k - 1))^{N+\eta}$$
$$= \lim_{k \to \infty} \sigma^k \left(\frac{K_1 + K_0 |t - (k - 1)T^*|^\theta}{\gamma} \right)^{N+\eta} = 0.$$

Hence $d_H(\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)) \leq N + \eta$.

COROLLARY 3.3. *Let conditions* (3.1)–(3.3) *hold. Then*

$$d_H(\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)) \leq N(t) \left(1 - \frac{\log 6\sqrt{2}l(t, T^*)}{\log \sqrt{2}\delta(t, T^*)} \right). \tag{3.5}$$

REMARK 3.4. Note that if the constants T^*, N, l and δ do not depend on t, then the estimate is uniform for all $\mathscr{A}(t)$.

4. Applications to a nonautonomous partial differential equation

Consider now the nonautonomous partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(t, u) = h(t), \\ u \mid_{\partial \Omega} = 0, \\ u(\tau) = u_\tau, \end{cases} \tag{4.1}$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $h(\cdot) \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))$, Ω is a bounded open subset of $\mathbb{R}^n$ and there exist $r \geq 0$, $p \geq 2$, $c_i > 0$, $i = 1, \ldots, 7$, such that

$$c_1|u|^p - c_2 \leq f(t, u)u \leq c_3|u|^p + c_4, \tag{4.2}$$

$$f_u(t, u) \geq -c_5, \tag{4.3}$$

$$\|h(t)\|_{L^2} \leq c_6|t|^r + c_7, \tag{4.4}$$

for all $u, t \in \mathbb{R}$.

Denote $H = L^2(\Omega)$ with norm $\| \cdot \|$, $V = H^1_0(\Omega)$. For a norm in another space X we shall use the notation $\| \cdot \|_X$.

THEOREM 4.1. *For any* $\tau, T \in \mathbb{R}$, $T > \tau$, $u_\tau \in L^2(\Omega)$ *there exists a unique solution* $u(\cdot) \in C([\tau, T], H) \cap L^2(\tau, T; V) \cap L^p(\tau, T; L^p(\Omega))$. *Moreover, for all* $u_\tau, v_\tau \in L^2(\Omega)$, $t \in [\tau, T]$ *it holds that*

$$\|u(t) - v(t)\| \leq e^{c_5(t-\tau)}\|u_\tau - v_\tau\|. \tag{4.5}$$

PROOF. The existence of a solution for any $u_\tau \in L^2(\Omega)$ was proved in [6, Theorem 2.1]. The uniqueness property and (4.5) can be obtained in exactly the same way as in [21, Theorem 1.1] or [6, Theorem 3.1].

Denote $S(t, \tau, u_\tau) = u(t)$, where $u(\tau) = u_\tau$, which is a process. We denote by $SE(H)$ the class of families of bounded sets $\widehat{B} = \{B(\tau)\}_{\tau \in \mathbb{R}}$ $(B(\tau) \subset H)$ such that

$$\lim_{\tau \to \infty} \frac{\max \{\log \|B(\tau)\|^+, 0\}}{\tau} = 0, \tag{4.6}$$

that is, the class of sets with subexponential growth on the time variable. In this case $\mathscr{D} = SE(H)$ (see the notation in Section 2).

LEMMA 4.2. *For any* $t \in \mathbb{R}$, *there exists a bounded set* $B_0(t)$ *in* H *such that for any family* $\widehat{B} \in SE(H)$ *and any* $t_0 < t$, *there exists* $T = T(\widehat{B}, t_0) < t_0$ *such that*

$$S(t_1, \tau, B(\tau)) \subset B_0(t), \quad \forall \tau \leq T, \ \forall t_1 \in [t_0, t]. \tag{4.7}$$

PROOF. Multiplying (4.1) by $u(s) = S(s, \tau, u_\tau)$, $u_\tau \in B(\tau)$, and using (4.2) and (4.4) we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{ds}\|u\|^2 + \|\nabla u\|^2 + c_1\|u\|_{L^p}^p &\leq c_2\mu(\Omega) + \|u\|\|h(s)\| \\
&\leq c_2\mu(\Omega) + \frac{\lambda_1}{2}\|u\|^2 + \frac{1}{2\lambda_1}\|h(s)\|^2 \\
&\leq c_2\mu(\Omega) + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2\lambda_1}(c_6|s|^r + c_7)^2, \quad (4.8)
\end{aligned}
$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω in $\mathbb{R}^n$. Therefore

$$
\begin{aligned}
\frac{d}{ds}\|u\|^2 + \lambda_1\|u\|^2 + 2c_1\|u\|_{L^p}^p &\leq \frac{d}{ds}\|u\|^2 + \|\nabla u\|^2 + 2c_1\|u\|_{L^p}^p \\
&\leq 2c_2\mu(\Omega) + \frac{1}{\lambda_1}(c_6|s|^r + c_7)^2.
\end{aligned}
$$

By the Gronwall lemma

$$
\|u(t_1)\|^2 \leq e^{-\lambda_1(t_1-\tau)}\|u_\tau\|^2 + \int_\tau^{t_1} e^{-\lambda_1(t_1-s)}\left(2c_2\mu(\Omega) + \frac{1}{\lambda_1}(c_6|s|^r + c_7)^2\right)ds,
$$

so that the ball

$$
B_0(t) = \left\{ y \in H : \|y\| \leq \sqrt{K(t) + \alpha} \right\},
$$

with $\alpha > 0$, $K(t) = \int_{-\infty}^t e^{-\lambda_1(t-s)}(2c_2\mu(\Omega) + (c_6|s|^r + c_7)^2/\lambda_1)\, ds$, satisfies (4.7). Indeed, in view of condition (4.6), we can find $T(\widehat{B}, t_0)$ such that $e^{-\lambda_1(t_1-\tau)}\|u_\tau\|_H^2 \leq \alpha$, $\forall \tau \leq T(\widehat{B}, t_0), \forall u_\tau \in B(\tau), \forall t_0 \leq t_1 \leq t$.

COROLLARY 4.3. *For any* $u_\tau \in B(\tau)$, $\tau < T(\widehat{B}, t_0)$,

$$
\int_{t_0}^t \left(\|\nabla u\|^2 + 2c_1\|u\|_{L^p}^p\right)ds \leq R(t_0, t),
$$

where

$$
R(t_0, t) = 2c_2\mu(\Omega)(t - t_0) + \frac{1}{\lambda_1}\int_{t_0}^t (c_6|s|^r + c_7)^2\, ds + K(t) + \alpha.
$$

PROOF. It is a consequence of Lemma 4.2 and (4.8).

LEMMA 4.4. *For any* $t \in \mathbb{R}$, $t_0 < t$ *there exists a set* $B_1(t_0, t)$ *bounded in* V *and compact in* H *such that for any* $\widehat{B} \in SE(H)$ *there exists* $T = T(\widehat{B}, t_0) < t_0$ *such that* $\forall \tau \leq T$,

$$
S(t, \tau, B(\tau)) \subset B_1(t_0, t). \qquad (4.9)
$$

PROOF. Multiplying (4.1) by $-\Delta u$, where $u(r) = S(r, \tau, u_\tau)$, $u_\tau \in B(\tau)$, and integrating by parts we have

$$\frac{1}{2}\frac{d}{dr}\|\nabla u\|^2 + \|\Delta u\|^2 + (f_u(r, u)\nabla u, \nabla u) = (h, \Delta u) \leq \frac{1}{2}\|h(r)\|^2 + \frac{1}{2}\|\Delta u\|^2.$$

Using (4.3) we obtain

$$\frac{d}{dr}\|\nabla u\|^2 \leq \frac{d}{dt}\|\nabla u\|^2 + \|\Delta u\|^2 \leq \|h(r)\|^2 + 2c_5\|\nabla u\|^2. \qquad (4.10)$$

Denote $a_1 = a_1(t) = \int_{t_0}^t \|h(r)\|^2 \, dr$, $a_2 = a_2(t) = \exp(2c_5(t - t_0))$. Assume that $t_0 \leq s \leq r \leq t$ and multiply (4.10) by $\exp(-2c_5(r - t_0))$. Then

$$\frac{d}{dr}\left(e^{-2c_5(r-t_0)}\|\nabla u\|^2\right) \leq \|h(r)\|^2 e^{-2c_5(r-t_0)} \leq \|h(r)\|^2.$$

Integrating over (s, t) we obtain

$$\|\nabla u(t)\|^2 \leq e^{2c_5(t-s)}\|\nabla u(s)\|^2 + e^{2c_5(t-t_0)}\int_s^t \|h(r)\|^2 \, dr \leq \left(\|\nabla u(s)\|^2 + a_1\right)a_2.$$

Finally, integrating with respect to s over (t_0, t) and using Corollary 4.3 we have

$$(t - t_0)\|\nabla u(t)\|^2 \leq (R(t_0, t) + a_1(t - t_0))a_2, \quad \forall \tau < T(\widehat{B}, t_0),$$

so that, taking into account the compact embedding $V \subset H$, the closure in H of the set

$$B_1(t_0, t) = \left\{u \in H_0^1(\Omega) : \|\nabla u\|^2 \leq \left(\frac{R(t_0, t)}{t - t_0} + a_1\right)a_2\right\}$$

is the desired set.

THEOREM 4.5. *The process S has the global attractor $\widehat{\mathscr{A}} = \{\mathscr{A}(t)\}_{t\in\mathbb{R}}$. Moreover, there exist $K_0, K_1, \theta > 0$ such that*

$$\|\mathscr{A}(t)\|^+ \leq K_0|t|^\theta + K_1, \quad \forall t \in \mathbb{R}, \qquad (4.11)$$

so that $\widehat{\mathscr{A}} \in SE(H)$.

PROOF. The existence of the global attractor is a consequence of Lemma 4.4 and Theorem 2.6.

Further, we note that, choosing t_0 in Lemma 4.4 such that $t - t_0 = l > 0$, we have

$$\|B_1(t_0, t)\|_H^+ \leq \beta \sup_{y \in B_1(t_0, t)} \|\nabla y\| \leq \beta \left((R(t_0, t)/l + a_1) a_2\right)^{1/2}$$

$$\leq \beta \left(\left(\nu + \int_{t_0}^t (c_6|s|^r + c_7)^2 \, ds\right) a_2\right)^{1/2},$$

where

$$\nu = \frac{1}{l} \left(2c_2\mu(\Omega)l + \frac{1}{\lambda_1} \int_{t_0}^t (c_6|s|^r + c_7)^2 \, ds + K(t) + \alpha\right).$$

From the definition of $K(t)$, we are ensured the existence of R_1, R_2, $\zeta > 0$ such that

$$\|K(t)\|^+ \leq R_1|t|^\zeta + R_2, \quad \forall t.$$

Hence, since $\mathscr{A}(t) \subset B_1(t_0, t) \; \forall t$, estimation (4.11) follows.

THEOREM 4.6. *Suppose there exists a positive and nondecreasing function $\xi(t)$ defined for all $t \in \mathbb{R}$ and such that for all $\tau \leq t$, u, $v \in \mathbb{R}$,*

$$|f(\tau, u) - f(\tau, v)| \leq \xi(t)|u - v|. \tag{4.12}$$

Then there exist L_1, $L_2 > 0$ depending on Ω and n such that

$$d_H(\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)) \leq \max\left\{L_1(\xi(t))^{n/2}, L_2(c_5)^{n/2}\right\}. \tag{4.13}$$

PROOF. The proof is similar to that of [1, Theorem 7]. Let us first prove condition (3.2). We take a fixed $t \in \mathbb{R}$, and arbitrary solutions $u(\cdot)$, $v(\cdot)$. It is easy to obtain in a standard way that

$$\frac{1}{2}\frac{d}{ds}\|u(s) - v(s)\|^2 + \|\nabla(u(s) - v(s))\|^2$$
$$+ (f(s, u(s)) - f(s, v(s)), u(s) - v(s)) = 0, \quad \text{for all } s \in \mathbb{R}.$$

In view of (4.3),

$$\frac{d}{ds}\|u(s) - v(s)\|^2 \leq 2c_5\|u(s) - v(s)\|^2.$$

Let us now take $T^* > 0$ to be determined later on and depending on t. Gronwall's lemma implies that for any $\tau \in \mathbb{R}$,

$$\|u(\tau + T^*) - v(\tau + T^*)\|^2 \leq e^{2c_5 T^*}\|u(\tau) - v(\tau)\|^2, \tag{4.14}$$

so that (3.2) holds with $l(t, T^*) = e^{c_5 T^*}$.

Denote $m(s) = u(s) - v(s)$. Multiplying

$$\frac{dm(s)}{ds} - \Delta m(s) + f(s, u(s)) - f(s, v(s)) = 0$$

by $Q_N m(s)$, we get

$$\frac{1}{2}\frac{d}{ds}\|Q_N m(s)\|^2 + \|\nabla Q_N m(s)\|^2 + (f(s, u(s)) - f(s, v(s)), Q_N m(s)) = 0.$$

Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N \to \infty$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. Since $\|\nabla Q_N m(s)\| \ge \lambda_{N+1}\|Q_N m(s)\|^2$, and using conditions (4.12) and (4.14), we obtain for $s \le \tau + T^* \le t$,

$$\frac{d}{ds}\|Q_N m(s)\|^2 \le -2\lambda_{N+1}\|Q_N m(s)\|^2 + 2\xi(t)\|m(s)\|^2$$

$$\le -2\lambda_{N+1}\|Q_N m(s)\|^2 + 2\xi(t)e^{2c_5(s-\tau)}\|u(\tau) - v(\tau)\|^2.$$

Multiplying both sides by $e^{2\lambda_{N+1}(s-\tau)}$, we have

$$\frac{d}{ds}\left(\|Q_N m(s)\|^2 e^{2\lambda_{N+1}(s-\tau)}\right) \le 2\xi(t)e^{2(c_5+\lambda_{N+1})(s-\tau)}\|u(\tau) - v(\tau)\|^2.$$

Integrating over $(\tau, \tau + T^*)$ we get

$$\|Q_N m(\tau + T^*)\|^2 e^{2\lambda_{N+1}(T^*)} \le \|u(\tau) - v(\tau)\|^2 \left(1 + \frac{\xi(t)}{c_5 + \lambda_{N+1}}\left(e^{2(c_5+\lambda_{N+1})T^*} - 1\right)\right).$$

Hence

$$\|Q_N m(\tau + T^*)\|^2 \le \|u(\tau) - v(\tau)\|^2 \left(\frac{c_5 + \lambda_{N+1} - \xi(t)}{c_5 + \lambda_{N+1}}e^{-2\lambda_{N+1}T^*} + \frac{\xi(t)}{c_5 + \lambda_{N+1}}e^{2c_5 T^*}\right)$$

$$\le \|u(\tau) - v(\tau)\|^2 \left[e^{-2\lambda_{N+1}T^*} + \frac{\xi(t)}{c_5 + \lambda_{N+1}}e^{2c_5 T^*}\right]$$

$$= \delta^2(t, T^*, N)\|u(\tau) - v(\tau)\|^2.$$

Choosing appropriate $N = N(t)$ and T^*, we obtain $\delta(t) = \delta(t, T^*, N) < 1/\sqrt{2}$. Then condition (3.3) is satisfied. It follows from Theorems 4.5 and 3.2 that

$$d_f(\mathscr{A}(t)) \le N + \eta,$$

where η is given by the condition $\left(6\sqrt{2}l(t)\right)^N \left(\sqrt{2}\delta(t)\right)^\eta = \sigma(t) < 1$.

We shall further prove (4.13). It is well-known (see [2, page 201], [22, page 136]) that $\lambda_N = O(N^{2/n})$, as $N \to \infty$, so that there exists $D > 0$ such that $\lambda_N/N^{2/n} \ge D$, $\forall N \in \mathbb{N}$. If we put $\eta = N$ then

$$\sigma^2(t) = (12\delta(t)l(t))^{2N} = 12^2 \left(e^{-2\lambda_{N+1}T^*+2c_5 T^*} + \frac{\xi(t)}{c_5 + \lambda_{N+1}}e^{4c_5 T^*}\right).$$

Denote $\gamma = 12$. We have to choose T^* and λ_{N+1} in such a way that

$$e^{-2(\lambda_{N+1}-c_5)T^*} = \frac{1}{2\gamma^2}, \tag{4.15}$$

$$\frac{\xi(t)}{c_5 + \lambda_{N+1}} e^{4c_5 T^*} \leq \frac{1}{2\gamma^2 + \alpha}, \tag{4.16}$$

where $\alpha > 0$.

It follows from (4.15) that $T^* = (\log 2\gamma^2)/(2(\lambda_{N+1} - c_5))$. Hence (4.16) will be satisfied if the following inequality holds:

$$(\lambda_{N+1} - c_5) \log\left(\frac{c_5 + \lambda_{N+1}}{(2\gamma^2 + \alpha)\xi(t)}\right) \geq 4c_5 \log \sqrt{2}\gamma.$$

Using the inequality $\lambda_{N+1} \geq D(N + 1)^{2/n}$ we get

$$(\lambda_{N+1} - c_5) \log\left(\frac{c_5 + \lambda_{N+1}}{(2\gamma^2 + \alpha)\xi(t)}\right) \geq \left((N + 1)^{2/n}D - c_5\right) \log\left(\frac{c_5 + D(N + 1)^{2/n}}{(2\gamma^2 + \alpha)\xi(t)}\right).$$

Choosing $N = N(t)$ such that $D(N + 1)^{2/n} \geq 5c_5$ and

$$\frac{c_5 + D(N + 1)^{2/n}}{(2\gamma^2 + \alpha)\xi(t)} \geq \sqrt{2}\gamma,$$

the inequality (4.16) holds. Hence it is sufficient to choose N satisfying

$$N \geq \max\left\{(D_1\xi(t))^{n/2} - 1, (5c_5/D)^{n/2} - 1\right\},$$

where $D_1 = c\sqrt{2}\gamma(2\gamma^2 + \alpha)/D$. We take $N = \max\left\{[(D_1\xi(t))^{n/2}], [(5c_5/D)^{n/2}]\right\}$, where $[x]$ denotes the integer part of x, and then $N \leq \max\left\{(D_1\xi(t))^{n/2}, (5c_5/D)^{n/2}\right\}$.

Finally, Theorem 3.2 implies that

$$d_f(\mathscr{A}(t)) \leq 2N \leq 2\max\left\{(D_1\xi(t))^{n/2}, (5c_5/D)^{n/2}\right\}$$
$$= \max\left\{L_1(\xi(t))^{n/2}, L_2(c_5)^{n/2}\right\},$$

where $L_1 = 2(D_1)^{n/2}$, $L_2 = 2(5/D)^{n/2}$.

From the previous result we can also obtain a uniform bound in t for the fractal dimension of the attractors.

COROLLARY 4.7. *There exists a positive constant K depending on n, Ω, c_5 and $\xi(\cdot)$ (but not on t) such that $d_H(\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)) \leq K$, for all $t \in \mathbb{R}$.*

PROOF. Fix some $t^* \in \mathbb{R}$. Since $\xi(t)$ is non-decreasing, Theorem 4.6 gives

$$d_f(\mathscr{A}(t)) \leq \max \left\{ L_1(\xi(t))^{n/2}, L_2(c_5)^{n/2} \right\}$$
$$\leq \max \left\{ L_1(\xi(t^*))^{n/2}, L_2(c_5)^{n/2} \right\}, \text{ for all } t \leq t^*.$$

On the other hand, note that (4.14) implies that $S(t + T, t)$ is Lipschitz with constant $e^{c_5 T}$, for all $T > 0$. Then, by [23, Proposition 13.2] we get

$$d_f(\mathscr{A}(t + T)) = d_f(S(t + T, t)\mathscr{A}(t)) \leq d_f(\mathscr{A}(t)), \tag{4.17}$$

so that

$$d_f(\mathscr{A}(t)) \leq \max \left\{ L_1(\xi(t^*))^{n/2}, L_2(c_5)^{n/2} \right\} = K, \text{ for all } t \in \mathbb{R}. \tag{4.18}$$

REMARK 4.8. We note that (4.18) is satisfied for all $t^* \in \mathbb{R}$. Hence the best estimate is obtained by the limit $K = \lim_{t^* \to -\infty} \max \left\{ L_1(\xi(t^*))^{n/2}, L_2(c_5)^{n/2} \right\}$, which exists because the function is non-decreasing and bounded below by 0.

Acknowledgements

References

[1] F. Balibrea and J. Valero, "On dimension of attractors of differential inclusions and reaction-diffusion equations", *Discrete Contin. Dyn. Syst.* **5** (1999) 515–528.

[2] H. Brezis, *Análisis funcional* (Alianza, Madrid, 1984). (Translated from *Analyse fonctionelle* (Masson, Paris, 1983).)

[3] D. N. Cheban, P. E. Kloeden and B. Schmalfuß, "The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems", *Nonlinear Dynam. Systems Theory* **2** (2002) 9–28.

[4] V. V. Chepyzhov and M. I. Vishik, "A Hausdorff dimension estimate for kernel sections of nonautonomous evolution equations", *Indiana Univ. Math. J.* **42** (1993) 1057–1076.

[5] V. V. Chepyzhov and M. I. Vishik, "Attractors of nonautonomous dynamical systems and their dimension", *J. Math. Pures Appl.* **73** (1994) 279–333.

[6] V. V. Chepyzhov and M. I. Vishik, "Trajectory attractors for reaction-diffusion systems", *Topol. Methods Nonlinear Anal.* **7** (1996) 49–76.

[7] P. Constantin, C. Foias and R. Temam, "Attractors representing turbulent flows", *Mem. Amer. Math. Soc.* **53**, No. 314 (1985).

[8] H. Crauel, A. Debussche and F. Flandoli, "Random attractors", *J. Dynam. Differential Equations* **9** (1997) 307–341.

[9] H. Crauel and F. Flandoli, "Attractors for random dynamical systems", *Probab. Theory Related Fields* **100** (1994) 365–393.

[10] H. Crauel and F. Flandoli, "Hausdorff dimension of invariant sets for random dynamical systems", *J. Dynam. Differential Equations* **10** (1998) 449–474.

[11] C. M. Dafermos, "Semiflows associated with compact and uniform processes", *Math. Systems Theory* **8** (1974/75) 142–149.

[12] A. Debussche, "On the finite dimensionality of random attractors", *Stochastic Anal. Appl.* **15** (1997) 473–491.

[13] A. Debussche, "Hausdorff dimension of a random invariant set", *J. Math. Pures Appl.* **77** (1998) 967–988.

[14] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential attractors for dissipative evolution equations*, RAM: Research in Applied Mathematics 37 (Masson, Paris, 1994).

[15] A. Eden and J. M. Rakotoson, "Exponential attractors for a doubly nonlinear equation", *J. Math. Anal. Appl.* **185** (1994) 321–339.

[16] F. Flandoli and B. Schmalfuß, "Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise", *Stochastics Stochastics Rep.* **59** (1996) 21–45.

[17] F. Flandoli and B. Schmalfuß, "Weak solutions and attractors for three-dimensional Navier-Stokes equations with nonregular force", *J. Dynam. Differential Equations* **11** (1999) 355–398.

[18] J. Hale, *Asymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs 25 (Amer. Math. Soc., Providence, RI, 1988).

[19] P. E. Kloeden and B. Schmalfuß, "Asymptotic behaviour of nonautonomous difference inclusions", *Systems Control Lett.* **33** (1998) 275–280.

[20] O. A. Ladyzhenskaya, "Some comments to my papers on the theory of attractors for abstract semigroups", *Zap. Nauchn. Sem. LOMI* **182** (1992) 102–112 (English translation *J. Soviet Math.* **62** (1992) 1789–1794), (in Russian).

[21] M. Marion, "Attractors for reaction-diffusion equations: existence and estimate of their dimension", *Appl. Anal.* **25** (1987) 101–147.

[22] G. Metivier, "Valeurs propres d'operateurs définis par la restriction de systèmes variationelles a des sous-espaces", *J. Math. Pures Appl.* **57** (1978) 133–156.

[23] J. C. Robinson, *Infinite-dimensional dynamical systems* (Cambridge Univ. Press, Cambridge, 2002).

[24] K. R. Schenk-Hoppé, "Random attractors—general properties, existence and applications to stochastic bifurcation theory", *Discrete Contin. Dynam. Systems* **4** (1998) 99–130.

[25] B. Schmalfuß, "Backward cocycles and attractors of stochastic differential equations", in *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behaviour* (eds. V. Reitmann, T. Riedrich and N. Koksch), (Technical University of Dresden, Dresden, 1992) 185–192.

[26] B. Schmalfuß, "Attractors for the nonautonomous dynamical systems", in *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999)* (eds. B. Fiedler, K. Gröger and J. Sprekels), (World Sci. Publishing, River Edge, NJ, 2000) 684–689.

[27] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics* (Springer, New York, 1988).

[28] J. Valero, "Finite and infinite dimensional attractors of multivalued reaction-diffusion equations", *Acta Math. Hungar.* **88** (2000) 239–258.