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IDEALS IN DIRECT PRODUCTS OF COMMUTATIVE RINGS

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Abstract

Let R and S be commutative rings, not necessarily with identity. We investigate the ideals, prime ideals, radical ideals, primary ideals, and maximal ideals of $R \times S$. Unlike the case where R and S have an identity, an ideal (or primary ideal, or maximal ideal) of $R \times S$ need not be a 'subproduct' $I \times J$ of ideals. We show that for a ring R, for each commutative ring S every ideal (or primary ideal, or maximal ideal) is a subproduct if and only if R is an e-ring (that is, for $r \in R$, there exists $e_r \in R$ with $e_r r = r$) (or u-ring (that is, for each proper ideal A of R, $\sqrt{A} \neq R$)), the Abelian group $(R/R^2, +)$ has no maximal subgroups).

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Suppose that R and S are commutative rings with identity. It is well known that the ideals of $R \times S$ have the form $I \times J$ where I is an ideal of R and J is an ideal of S. It easily follows that the prime (primary, maximal) ideals of $R \times S$ have the form $P \times S$ or $R \times Q$ where P is a prime (primary, maximal) ideal of R or Q is a prime (primary, maximal) ideal of S.

Suppose that R and S are commutative rings not necessarily with identity. If I is an ideal of R and J is an ideal of S, then certainly $I \times J$ is an ideal of $R \times S$. (It is obvious that if $I \subseteq R$ and $J \subseteq S$ with $I \times J$ an ideal of $R \times S$, then I is an ideal of R and J is an ideal of S.) We call such an ideal $I \times J$ of $R \times S$, a *subproduct*. However, ideals of $R \times S$ need not be subproducts. For if A and B are non-zero Abelian groups, then $A \times B$ with the zero product is a commutative ring whose ideals are just the subgroups of $A \times B$. However, it is rare [2, Theorem 2] that every subgroup of $A \times B$ is a subproduct. For example, if $A = B = \mathbb{Z}_2$, then $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ is an ideal of $\mathbb{Z}_2 \times \mathbb{Z}_2$ that is not a subproduct.

A commutative ring R is an *e-ring* [3] if for each $r \in R$, there exists an $e_r \in R$ with $e_r r = r$. We show (Theorem 2) that a commutative ring R is an e-ring if and only if, for each commutative ring S, every ideal of $R \times S$ is a subproduct. Now every prime ideal of $R \times S$ has the form $P \times S$ where P is a prime ideal of R or $R \times Q$ where Q is a prime ideal of S (Theorem 6). However, a commutative ring R is a *u-ring* (for each proper ideal A of R, $\sqrt{A} \neq R$) [3] if and only if, for each commutative ring S, every primary ideal of $R \times S$ has the form $Q \times S$ where Q is a primary ideal of R or $R \times Q$ where Q is a primary ideal of S, or equivalently, each primary ideal of $R \times S$ is a subproduct (Theorem 9). Finally, we determine (Theorem 12) the commutative rings R with the property that, for each commutative ring S, each maximal ideal of $R \times S$ is a subproduct.

We start with the following simple proposition whose proof is left to the reader.

PROPOSITION 1. *Let R and S be commutative rings. Then the following conditions are equivalent (for an ideal A of $R \times S$).*

(1) *Every ideal of $R \times S$ (The ideal A of $R \times S$) is a subproduct.*
(2) *For each $r \in R$ and $s \in S$ (with $(r, s) \in A$), $((r, s)) = (r) \times (s)$.*
(3) *For each $r \in R$ and $s \in S$ (with $(r, s) \in A$), $(r, 0) \in ((r, s))$ $((r, 0) \in A)$.*
(4) *For each $r \in R$ and $s \in S$ (with $(r, s) \in A$), there exist $a \in R$, $b \in S$, and $n \in \mathbb{Z}$ with $r = ar + nr$ and $0 = bs + ns$.*

Of course, (3) of Proposition 1 is equivalent to $(0, s) \in ((r, s))$. Note that (4) is equivalent to $0 = (-a)\,r + (1 - n)\,r$ and $s = (-b)\,s + (1 - n)\,s$. Also note that if an ideal A of $R \times S$ is a subproduct, then $A = I \times J$ where $I = \{r \in R \mid (r, 0) \in A\}$ $(= \{r \in R \mid (r, s) \in A$ for some $s \in S\})$ and $J = \{s \in S \mid (0, s) \in A\}$ $(= \{s \in S \mid (r, s) \in A$ for some $r \in R\})$.

We next characterize the commutative rings R with the property that for each commutative ring S, every ideal of $R \times S$ is a subproduct. Most of Theorem 2 appears in [1, Proposition 3.1].

THEOREM 2. *For a commutative ring R the following conditions are equivalent.*

(1) *R is an e-ring (that is, for each $r \in R$, there exists an $e_r \in R$ with $e_r r = r$).*
(2) *For each commutative ring S, each ideal of $R \times S$ is a subproduct.*
(3) *For all $n \geq 2$, each ideal of R^n has the form $I_1 \times \cdots \times I_n$ where I_i is an ideal of R.*
(4) *For some $n \geq 2$, each ideal of R^n is a subproduct as in (3).*
(5) *Every ideal of $R \times R$ is a subproduct.*

PROOF. (1) $\Rightarrow$ (2). Suppose that R is an e-ring. Let $r \in R$ and $s \in S$. Choose $e_r \in R$ with $e_r r = r$. Then $(r, 0) = (e_r, 0)\,(r, s) \in ((r, s))$. By Proposition 1, every ideal of $R \times S$ is a subproduct.

(2) $\Rightarrow$ (3). Assume the result for $n - 1$ and then take $S = R^{n-1}$.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1). By Proposition 1(4) with $R = S$ and $r = s \in R$, there exist $a, b \in R$ and $n \in \mathbb{Z}$ with $r = ar + nr$ and $0 = br + nr$. Hence $r = ar - br = (a - b)\,r$. So R is an e-ring. □

We next give a 'local' alternative approach to (1) $\Rightarrow$ (2) of the previous theorem.

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PROPOSITION 3. *Let R and S be commutative rings and let I be an ideal of $R \times S$. Let $\varphi : R \to R \times S/I$ ($\varphi(r) = (r, 0) + I$) be the natural map. If $\varphi(R)$ is an e-ring, then I is a subproduct.*

PROOF. Now $\varphi(R)$ an e-ring says that, for $\overline{(r, 0)} \in R \times S/I$, there exists $\overline{(e, 0)} \in R \times S/I$ with $\overline{(e, 0)}\ \overline{(r, 0)} = \overline{(r, 0)}$, or $(r - er, 0) \in I$. Let $(x, y) \in I$. So there exists $e \in R$ with $(x - ex, 0) \in I$. Then $(x, 0) = (x - ex, 0) + (e, 0)(x, y) \in I$. So by Proposition 1, I is a subproduct. □

COROLLARY 4. *Let R and S be commutative rings and I an ideal of $R \times S$. If $R \times S/I$ is an e-ring, then I is a subproduct.*

PROOF. If $R \times S/I$ is an e-ring, then so is its subring $\varphi(R)$ where $\varphi(R)$ is as defined in Proposition 3. Indeed, if $(e_1, e_2)(r, 0) = (r, 0)$, then $(e_1, 0)(r, 0) = (r, 0)$. □

COROLLARY 5. *Let R be an e-ring. Then for any commutative ring S, every ideal of $R \times S$ is a subproduct.*

PROOF. Let I be an ideal of $R \times S$. If R is an e-ring, then so is its homomorphic image $\varphi(R)$ in $R \times S/I$. By Proposition 3, I is a subproduct. □

We next determine the prime ideals of $R \times S$. Here the situation is the same as in the case where the rings have an identity.

THEOREM 6. *Let R and S be commutative rings. Then an ideal $\mathcal{P}$ of $R \times S$ is prime if and only if $\mathcal{P}$ has the form $P \times S$ where P is a prime ideal of R or $R \times Q$ where Q is a prime ideal of S.*

PROOF. ($\Leftarrow$) Clear. ($\Rightarrow$) Suppose that $\mathcal{P}$ is a prime ideal of $R \times S$. Now $(0 \times S)(R \times 0) \subseteq \mathcal{P}$, so either $0 \times S \subseteq \mathcal{P}$ or $R \times 0 \subseteq \mathcal{P}$. Suppose that $R \times 0 \subseteq \mathcal{P}$. It follows from Proposition 1 that $\mathcal{P} = R \times Q$ for some ideal Q of S. It is easily checked that Q must be prime. The case where $0 \times S \subseteq \mathcal{P}$ is similar. □

COROLLARY 7. *Let R and S be commutative rings. The radical ideals of $R \times S$ have the form $I \times J$ where I is a radical ideal of R and J is a radical ideal of S.*

PROOF. Let I be a radical ideal of $R \times S$. We may assume that $I \neq R \times S$. So I is an intersection of prime ideals, each of which is a subproduct. So $I = I_1 \times I_2$ is a subproduct where I_i is either the whole ring or an intersection of prime ideals. In either case I_i is a radical ideal. □

Our next goal is to characterize the commutative rings R with the property that for each commutative ring S, every primary ideal of $R \times S$ is a subproduct. We need the following lemma.

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LEMMA 8. *Let R and S be commutative rings.*

(1) *If $A \neq R$ is an ideal with $\sqrt{A} = R$, then A is primary.*
(2) *If Q is a primary ideal of $R \times S$ with $\sqrt{Q} \neq R \times S$, then either $Q = Q_1 \times S$ where Q_1 is a primary ideal of R or $Q = R \times Q_2$ where Q_2 is a primary ideal of S.*

PROOF. (1) Suppose that $ab \in A$ where $a, b \in R$. Then $\sqrt{A} = R$ gives $b^n \in A$ for some $n \geq 1$ regardless of whether $a \in A$ or not. (2) Now $\sqrt{Q}$ is a prime ideal of $R \times S$, so by Theorem 6 either $\sqrt{Q} = P \times S$ where P is a prime ideal of R or $\sqrt{Q} = R \times P$ where P is a prime ideal of S. Without loss of generality we may assume that $\sqrt{Q} = P \times S$. Let $x \in R - P$; so $(x, 0) \notin \sqrt{Q}$. Let $s \in S$. Then $(0, s)\,(x, 0) = (0, 0) \in Q$ and $(x, 0) \notin \sqrt{Q}$, so $(0, s) \in Q$ since Q is primary. Hence $0 \times S \subseteq Q$. So by Proposition 1, $Q = Q_1 \times S$ for some ideal Q_1 of R which is easily seen to be primary. □

Concerning the condition in Lemma 8(2) that $\sqrt{Q} \neq R \times S$, a primary ideal A of $R \times S$ with $\sqrt{A} = R \times S$ may or may not be a subproduct. For example, $\{(\bar{0}, \bar{0})\}$ and $\{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ are both primary ideals of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with radical $\mathbb{Z}_2 \times \mathbb{Z}_2$, but the first is a subproduct (but not of the form given in Lemma 8(2)), while the second is not.

THEOREM 9. *For a commutative ring R the following conditions are equivalent.*

(1) *R is a u-ring (that is, if $A \neq R$ is an ideal of R, then $\sqrt{A} \neq R$).*
(2) *For each commutative ring S, each primary ideal of $R \times S$ has the form $Q_1 \times S$ where Q_1 is a primary ideal of R or $R \times Q_2$ where Q_2 is a primary ideal of S.*
(3) *For each commutative ring S, each primary ideal of $R \times S$ is a subproduct.*
(4) *Each primary ideal of $R \times R$ has the form $Q \times R$ or $R \times Q$ where Q is a primary ideal of R.*
(5) *Each primary ideal of $R \times R$ is a subproduct.*

PROOF. (1) $\Rightarrow$ (2). Let Q be a primary ideal of $R \times S$. If $\sqrt{Q} \neq R \times S$, the result follows from Lemma 8(2). So suppose that $\sqrt{Q} = R \times S$. Let $A = \{a \in R \mid (a, 0) \in Q\}$, an ideal of R. For $r \in R$, $(r, 0) \in R \times S = \sqrt{Q}$, so $(r^n, 0) \in Q$ for some $n \geq 1$, and hence $r^n \in A$. So $\sqrt{A} = R$. Since R is a u-ring, $A = R$. So $R \times 0 \subseteq Q$. By Proposition 1 $Q = R \times Q_2$ for some ideal Q_2 of S, necessarily primary.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1), Suppose that R is not a u-ring, so there is an ideal $A \subsetneq R$ with $\sqrt{A} = R$. So for each ideal $B \supseteq A \times A$ of $R \times R$, $\sqrt{B} = R \times R$. So by Lemma 8(1), B is primary. So by hypothesis, B is a subproduct. So each ideal of $R/A \times R/A$ is a subproduct. By Theorem 2, R/A is an e-ring. Let $0 \neq x \in R/A$. Then there is an $e \in R/A$ with $ex = x$. Since $\sqrt{A} = R$, there is an $n \geq 1$ with $e^n = 0$. But then $x = ex = e^2x = \cdots = e^n x = 0$, a contradiction. □

We next characterize the commutative rings R with the property that, for each commutative ring S, the maximal ideals of $R \times S$ are subproducts. Of course a

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subproduct of $R \times S$ is a maximal ideal if and only if it has the form $M \times S$ where M is a maximal ideal of R or $R \times N$ where N is a maximal ideal of S.

LEMMA 10. *Let R be a commutative ring. If M is a maximal ideal of R that is not prime, then $R^2 \subseteq M$. Thus $\overline{M} = M/R^2$ is a maximal subgroup of $(R/R^2, +)$. Conversely, if $R \neq R^2$ and $\overline{M} = M/R^2$ is a maximal subgroup of R/R^2 where $R^2 \subseteq M \subsetneq R$ with M a (maximal) subgroup of $(R, +)$, then M is a maximal ideal of R that is not prime.*

PROOF. Suppose that M is a maximal ideal of R that is not prime. Choose $a, b \in R$ with $ab \in M$ but $a \notin M$ and $b \notin M$. Then since M is maximal, $(M, a) = R = (M, b)$. So $R^2 = (M, a)(M, b) \subseteq M$. Since the ring R/R^2 has the zero product, additive subgroups are the same thing as ideals. Thus M/R^2 is a maximal subgroup of R/R^2. The converse is immediate. □

LEMMA 11. *Let R and S be commutative rings with $R = R^2$. Then every maximal ideal of $R \times S$ has the form $N_1 \times S$ or $R \times N_2$ where N_1 (N_2) is a maximal ideal of R (S).*

PROOF. Let M be a maximal ideal of $R \times S$. If M is prime, then M has the desired form by Theorem 6 and the remarks preceding Lemma 10. So we may suppose that M is not prime. Then by Lemma 10, $(R \times S)^2 \subseteq M$. But since $R^2 = R$, $R \times S^2 = (R \times S)^2 \subseteq M$. Hence by Proposition 1, M is a subproduct necessarily of the form $R \times N_2$ where N_2 is a maximal ideal of S. □

THEOREM 12. *For a commutative ring R the following conditions are equivalent.*

(1) *The Abelian group $(R/R^2, +)$ has no maximal subgroups.*
(2) *For each commutative ring S, every maximal ideal of $R \times S$ has the form $M \times S$ or $R \times N$ where M (N) is a maximal ideal of R (S).*
(3) *For each commutative ring S, every maximal ideal of $R \times S$ is a subproduct.*
(4) *Every maximal ideal of $R \times R$ has the form $M \times R$ or $R \times M$ where M is a maximal ideal of R.*
(5) *Every maximal ideal of $R \times R$ is a subproduct.*
(6) *Every maximal ideal of R is prime.*
(7) *Every maximal ideal of $R \times R$ is prime.*

PROOF. We have already remarked that (2) ⇔ (3) and (4) ⇔ (5).

(1) ⇒ (2). Suppose that $R \times S$ has a maximal ideal $\mathcal{M}$ not of the form $M \times S$ or $R \times N$ where M is a maximal ideal of R and N is a maximal ideal of S. So $R^2 \neq R$ and $S^2 \neq S$ by Lemma 11 and $R^2 \times S^2 = (R \times S)^2 \subseteq \mathcal{M}$ by Lemma 10 since $\mathcal{M}$ cannot be prime by Theorem 6. Hence $T = (R \times S)/\mathcal{M}$ is a simple Abelian group. Now the natural map $R/R^2 \times S/S^2 \to T$ is an epimorphism. Since T is a simple Abelian group, the natural map $R/R^2 \to R/R^2 \times S/S^2 \to T$ is either onto or the zero map. Since $(R/R^2, +)$ has no maximal subgroups, the map must be the zero map.

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Hence $R \times 0 \subseteq \mathcal{M}$. So by Proposition 1, $\mathcal{M}$ is a subproduct and hence has the form $R \times N$ for some maximal ideal N of S.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are clear.

(4) $\Rightarrow$ (1). Suppose that $(R/R^2, +)$ has a maximal subgroup N, so $(R/R^2)/N \approx \mathbb{Z}_p$ for some prime p. Then $((R/R^2) \times (R/R^2))/N \times N \approx ((R/R^2)/N) \times ((R/R^2)/N) \approx \mathbb{Z}_p \times \mathbb{Z}_p$. Now $\langle(\bar{1}, \bar{1})\rangle$ is a maximal subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, by the correspondence theorem, $(R/R^2) \times (R/R^2) \approx (R \times R)/R^2 \times R^2$ has a maximal subgroup not of the form $(R/R^2) \times N'$ or $N' \times (R/R^2)$ for some maximal subgroup N' of R/R^2. Hence $R \times R$ has a maximal ideal that is not of the form $R \times M$ or $M \times R$ for some maximal ideal M of R, a contradiction.

(1) $\Leftrightarrow$ (6) by Lemma 10.

(7) $\Rightarrow$ (5) by Theorem 6.

(6) $\Rightarrow$ (7). Let $\mathcal{M}$ be a maximal ideal of $R \times R$. By (6) $\Rightarrow$ (1) $\Rightarrow$ (4) $\mathcal{M} = M \times R$ or $R \times M$ where M is a maximal ideal of R. But by hypothesis M is prime and hence so are $M \times R$ and $R \times M$. □

REMARK 13. Observe that the proof of Theorem 12 shows that a non-zero Abelian group A (R/R^2 in Theorem 12) has a maximal subgroup if and only if $A \times A$ has a maximal subgroup and then $A \times A$ has a maximal subgroup that is not a subproduct.

However, we cannot conclude from Theorem 12 that if R is a ring for which $(R/R^2, +)$ has no maximal subgroups, then every ideal of $R \times R$ is contained in a maximal ideal of the form $M \times R$ or $R \times M$ for some maximal ideal M of R. For if $R^2 \subsetneq R$, then $R^2 \times R$ is a proper ideal of $R \times R$ that is not contained in a maximal ideal of the form $M \times R$ (and hence is contained in no maximal ideal). For example, if we take $R = \mathbb{Z}_{p^\infty}$ with the zero product, then $R^2 = 0$ and $R \times R$ has no maximal ideals. Hence $\mathbb{Z}_{p^\infty} \times \mathbb{Z}_{p^\infty}$ vacuously satisfies the condition that each maximal ideal has the form $M \times \mathbb{Z}_{p^\infty}$ or $\mathbb{Z}_{p^\infty} \times M$. One implication of the following result follows from Theorem 12 and the preceding remarks.

THEOREM 14. *Let R be a commutative ring. Then each proper ideal of $R \times R$ is contained in a maximal ideal of the form $M \times R$ or $R \times M$ for some maximal ideal of M of R if and only if $R = R^2$ and each proper ideal of R is contained in a maximal ideal of R.*

PROOF. ($\Rightarrow$) Suppose that each proper ideal of $R \times R$ is contained in a maximal ideal of the form $M \times R$ or $R \times M$ for some maximal ideal M of R. By the above remarks, $R = R^2$. If A is a proper ideal of R, then $A \times R$ is contained in a maximal ideal of $R \times R$ of the form $M \times R$ where M is a maximal ideal of R. Then M is a maximal ideal of R containing A.

($\Leftarrow$) Let A be a proper ideal of $R \times R$. Let $A_1 = \{r \in R \mid (r, 0) \in A\}$. Suppose that $\sqrt{A} = R \times R$. Then for $r \in R$, $(r^n, 0) \in A$ for some $n \geq 1$, so $r^n \in A_1$. Thus $\sqrt{A_1} = R$. Thus $A_1 = R$. For if not, then $A_1 \subseteq M$ for some maximal ideal M of R. Then $R = R^2$ gives that M is prime (see the proof of Lemma 10).

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So $\sqrt{A_1} \subseteq \sqrt{M} = M \subsetneq R$, a contradiction. Likewise $A_2 = \{r \in R \mid (0, r) \in A\} = R$. So $A = R \times R$, a contradiction. Thus $\sqrt{A} \neq R \times R$. Hence $A \subseteq \mathcal{P}$ for some prime ideal $\mathcal{P}$ of $R \times R$. Without loss of generality, we can assume that $\mathcal{P} = P \times R$ where P is a prime ideal of R. By hypothesis $P \subseteq M$ for some maximal ideal M of R. But then $A \subseteq M \times R$, a maximal ideal of $R \times R$. □

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