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Picard's Iterations for Integral Equations of Mixed Hammerstein Type

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Abstract. A new semilocal convergence result for the Picard method is presented, where the main required condition in the contraction mapping principle is relaxed.

1 Introduction

Numerical methods are used for approximating the solution of nonlinear integral equations of mixed Hammerstein type

$$\phi(x) = f(x) + \lambda \sum_{j=1}^{m} \int_a^b K_j(x,t) H_j(\phi(t))\, dt, \tag{1.1}$$

where $-\infty < a < b < +\infty$, f, H_j and K_j, for $j = 1, 2 \dots, m$, are known functions and ϕ is a solution to be determined [5]. There are different numerical methods to approximate solutions of equation (1.1) when the functions H_j are linear [4,9]. If the functions H_j are nonlinear, the more usual numerical methods are the collocation-type methods or similar ones [3, 7]. These kinds of methods have two principal characteristics: equation (1.1) is discretized and the associated nonlinear finite system is solved by applying numerical methods to approximate the solutions, and, by interpolation, the solution is approximated.

Our goal in this paper is to find a solution of the nonlinear integral equation defined in (1.1) and approximate it by the direct application of iterative processes. The first iteration that comes to mind is the famous Newton's method [1,2]:

$$z_{n+1} = z_n - [F'(z_n)]^{-1} F(z_n), \quad n = 0, 1, 2, \dots,$$

but the existence of the operator $[F'(z)]^{-1}$ is needed, and it is not easy to calculate for some equations. Then we discard iterations in which the operator $[F'(z)]^{-1}$ appears. But it is usually possible to reformulate a given equation as a fixed point problem of the type $z = G(z)$, where G is an operator that maps a Banach space into itself [6]. For example, if F is a given operator that maps a Banach space into itself and we want to solve $F(z) = 0$, we can consider $G(z) = z - F(z)$, and it is evident that $z = G(z)$

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is satisfyed by z if and only if z is a solution of $F(z) = 0$. Thus we may approach the problem of solving an equation of the form $F(z) = 0$ by transforming it into a fixed point problem and then using the following iterative method, which is known as Picard's iteration,

$$z_{n+1} = G(z_n) = z_n - F(z_n), \quad n = 0, 1, 2, \dots \tag{1.2}$$

to generate successive approximations to the solution sought [8].

However, according to the contraction mapping principle [10], to prove the convergence of this iteration, the conditon that the operator G is a contraction is needed. This is a problem that reduces the number of equations that can be solved by the method, as we can see for the following nonlinear Bratu equation [4]

$$\phi(x) = \frac{1}{10} \int_0^1 K(x, t)\, \mathrm{e}^{\phi(t)}\, dt, \tag{1.3}$$

where the kernel K is the Green function

$$K(x, t) = \begin{cases} (1 - x)t & t \le x, \\ (1 - t)x & x \le t, \end{cases}$$

which is a mixed Hammerstein equation of type (1.1). For this nonlinear integral equation, the corresponding operator $G(\phi) = \frac{1}{10} \int_0^1 K(x, t)\, \mathrm{e}^{\phi(t)}\, dt$ is not a contraction, and the contraction mapping principle does not then guarantee the convergence of Picard's method to a solution of (1.3).

After that we note that relaxing the convergence conditions is important in proving the convergence of the Picard iteration. So we present an alternative to the contraction mapping principle, so that Picard's process can be applied to approximate a solution of (1.3).

Throughout the paper we denote

$$\overline{B(x, r)} = \{y \in X\, ;\, \|y - x\| \le r\} \quad \text{and} \quad B(x, r) = \{y \in X\, ;\, \|y - x\| < r\},$$

and the space of continuous functions on the interval $[a, b]$ is equipped with the max-norm

$$\|h\| = \max_{s \in [a,b]} |h(s)|, \quad h \in C[a, b].$$

2 Preliminaries

To simplify what is being proposed we introduce some new notation for (1.1). Let $F\colon C[a, b] \to C[a, b]$ be defined by

$$[F(\phi)](x) = \phi(x) - f(x) - \lambda \sum_{j=1}^{m} \int_a^b K_j(x, t) H_j(\phi(t))\, dt. \tag{2.1}$$

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Define the Nemytskii operator $\mathcal{H}_j\colon C[a,b] \to C[a,b]$ by

$$\mathcal{H}_j(\phi)(t) = H_j(\phi(t)), \quad t \in [a,b], \phi \in C[a,b],$$

and $H_j\colon \mathbb{R} \to \mathbb{R}$. Define the linear integral operator $\mathcal{K}_j\colon C[a,b] \to C[a,b]$ by

$$\mathcal{K}_j(\psi)(x) = \int_a^b K_j(x,t)\psi(t)\,dt, \quad x \in [a,b],\ \psi \in C[a,b].$$

Then the equation being solved is $F(\phi) = 0$ with

$$F(\phi) = \phi - f - \lambda \sum_{j=1}^{m} \mathcal{K}_j(\mathcal{H}_j(\phi)).$$

3 A Semilocal Convergence Result

The derivative of F at ϕ is given by $F'(\phi)\colon C[a,b] \to C[a,b]$, where

$$F'(\phi)\psi = \psi - \lambda \sum_{j=1}^{m} [\mathcal{K}_j(\mathcal{H}'_j(\phi))]\psi, \quad \psi \in C[a,b],$$

and

$$\mathcal{H}'_j(\phi)\psi(x) = H'_j(\phi(x))\psi(x), \quad x \in [a,b], j = 1,2,\dots,m.$$

Then, we consider (1.2) to approximate a solution ϕ^* of (1.1). To analyse the convergence of (1.2) it suffices to see that it is a Cauchy sequence. First, some properties that this sequence satisfies are given. For that, we consider Taylor's formula and (1.2) to write

$$\begin{aligned}(3.1)\quad F(\phi_n)(x) &= F(\phi_{n-1})(x) + \int_0^1 F'(\phi_{n-1} + \tau(\phi_n - \phi_{n-1}))(\phi_n - \phi_{n-1})(x)\,d\tau \\ &= -\lambda \sum_{j=1}^{m} \int_0^1 \int_a^b K_j(x,t) H'_j(\phi_{n-1}(t) + \tau(\phi_n(t) - \phi_{n-1}(t))) \\ &\qquad \times (\phi_n(t) - \phi_{n-1}(t))\,dt d\tau.\end{aligned}$$

In the following lemma, we give a bound for $\|\phi_{n+1} - \phi_n\|$.

Lemma 3.1 *If $\phi_0 \in C[a,b]$, $\phi_n \in B(\phi_0, R)$, for all $n \in \mathbb{N}$, and $|H'_j(s)| \le \omega_j(|s|)$, $s \in [a,b]$, where ω_j are nondecreasing positive real functions $\omega_j\colon \mathbb{R}_+ \to \mathbb{R}_+$, then*

$$\|\phi_{n+1} - \phi_n\| \le N(R)\|\phi_n - \phi_{n-1}\|, \quad n \ge 1,$$

where $N(R) = |\lambda| \sum_{j=1}^{m} M_j \omega_j(\|\phi_0\| + R)$ and $M_j = \max_{[a,b]} \int_a^b |K_j(x,t)|\,dt$, for $j = 1,2,\dots,m$.

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Proof Observing that $\|\phi_{n+1} - \phi_n\| \leq \|F(\phi_n)\|$ and taking into account (3.1), it follows that

$$\|F(\phi_n)\| \leq \Big(|\lambda| \sum_{j=1}^{m} \int_0^1 \Big[\Big(\max_{x\in[a,b]} \int_a^b |K_j(x,t)|\, dt \Big) \times \Big(\max_{t\in[a,b]} |H_j'(\phi_{n-1}(t) + \tau(\phi_n(t) - \phi_{n-1}(t)))| \Big) \Big]\, d\tau \Big) \|\phi_n - \phi_{n-1}\|$$

and consequently

$$\|F(\phi_n)\| \leq \Big(|\lambda| \sum_{j=1}^{m} M_j \omega_j(\|\phi_0\| + R) \Big) \|\phi_n - \phi_{n-1}\| \tag{3.2}$$

since $|H_j'(s)| \leq \omega_j(|s|)$, for $j = 1, 2, \dots, m$, and $\phi_{n-1}(t) + \tau(\phi_n(t) - \phi_{n-1}(t)) \in B(\phi_0, R)$. ■

Next, we give the semilocal convergence result for sequence (1.2).

Theorem 3.2 *Let $\phi_0 \in C[a,b]$ and make the assumptions of Lemma* 3.1. *Let us suppose that R exists, the smallest positive root of the following scalar equation in the z variable:*

$$\Big(1 - |\lambda| \sum_{j=1}^{m} M_j \omega_j(\|\phi_0\| + z) \Big) z - \eta = 0, \tag{3.3}$$

where $\eta \geq \|F(\phi_0)\|$. Then the Picard iteration converges to a zero ϕ^* of operator (2.1).

Proof To study the convergence of iteration (1.2) it suffices to see that it is a Cauchy sequence. Firstly, observe that the smallest positive root R of (3.3) satisfies

$$\eta \sum_{i=0}^{n} N(R)^i < \eta \sum_{i=0}^{\infty} N(R)^i = \frac{\eta}{1 - N(R)} = R, \tag{3.4}$$

since $N(R) < 1$ as a consequence of R is a positive root of equation (3.3).

From $\|F(\phi_0)\| \leq \eta$ and (3.4), it follows $\|\phi_1 - \phi_0\| \leq \eta < R$, and therefore $\phi_1 \in B(\phi_0, R)$. Next, from Lemma 3.1, we have $\|\phi_2 - \phi_1\| \leq N(R)\|\phi_1 - \phi_0\|$, and consequently, $\|\phi_2 - \phi_0\| \leq \|\phi_2 - \phi_1\| + \|\phi_1 - \phi_0\| \leq (1 + N(R))\eta < R$ and $\phi_2 \in B(x_0, R)$.

Now, by induction, applying Lemma 3.1, using the hypotheses and (3.4), it is easy to prove that

$$\|\phi_{n+1} - \phi_n\| \leq N(R)^n \|\phi_n - \phi_0\|$$

and

$$\|\phi_{n+1} - \phi_0\| \leq \Big(\sum_{i=0}^{n} N(R)^i \Big) \|x_1 - x_0\| \eta < R.$$

Then $\phi_{n+1} \in B(\phi_0, R)$.

Finally, (1.2) is a Cauchy sequence, since

$$\begin{aligned}\|\phi_{n+p} - \phi_n\| &\le \|\phi_{n+p} - \phi_{n+p-1}\| + \|\phi_{n+p-1} - \phi_{n+p-2}\| + \cdots + \|\phi_{n+1} - \phi_n\| \\ &\le \frac{1 - N(R)^p}{1 - N(R)} N(R)^n \eta.\end{aligned}$$

Therefore $\{\phi_n\}$ converges to a limit ϕ^* such that $F(\phi^*) = 0$ by letting $n \to \infty$ in (3.2). ■

4 Example

As we have indicated in the introduction, the contraction mapping principle cannot be applied to equation (1.3), but Theorem 3.2 can. In consequence, we can use the Picard iteration to approximate a solution of (1.3).

Note that solving equation (1.3) is equivalent to solving $F(x) = 0$, where

$$\begin{aligned}&F : C[0,1] \to C[0,1] \\ &[F(\phi)](x) = \phi(x) - \frac{1}{10}\int_0^1 K(x,t)\, e^{\phi(t)}\, dt.\end{aligned} \tag{4.1}$$

For operator (4.1) we have

$$M = 1/8, \quad \omega(z) = e^z, \quad \eta = \|\phi_0\| + e^{\|\phi_0\|}/80,$$

and choosing $\phi_0 = 1$, equation (3.3) is reduced to

$$(1 - e^{1+z}/80)z - (1 + e/80) = 0.$$

The smallest positive root of the previous scalar equation is $R = 1.15962\ldots$. Consequently, by Theorem 3.2, Picard's iteration converges to a zero ϕ^* of operator (4.1).

Moreover, the theoretical significance of the Picard method should also be noted, as it can be used to draw conclusions about the existence of a solution, and about the region in which it is located, without finding the solution itself. This is sometimes more important than the actual knowledge of the solution. So equation (1.3) has a solution ϕ^* in

$$\{u \in C[0,1]\,;\, \|u - 1\| \le 1.15962\ldots\}.$$

The solution $\phi^*(x) = 0.0508548x - 0.0508548x^2$ of equation (1.3) is approximated by Picard's method after five iterations if seven significance decimal figures are used.

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