GENERAL RELATIVISTIC FLUID SPHERES WITH VARIABLE POLYTROPIC INDEX

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Abstract

In this paper we derive solutions of the field equations of general relativity for a compressible fluid sphere which obeys density-temperature and pressure-temperature relations which allow for a variation of the polytropic index throughout the sphere.

A comparison is made with results obtained recently by R. F. Tooper for general relativistic polytropic fluid spheres and it is shown that our model corresponds to a polytropic model of index slightly larger than two.

1. Introduction

Recently Tooper (1964) has discussed solutions of the field equations of general relativity for a compressible fluid sphere in gravitational equilibrium under the assumption that the fluid obeys a polytropic equation of state. He obtained solutions of the equilibrium equations in terms of the polytropic index n and the parameter σ which is the ratio of pressure to energy density at the centre of the sphere.

In the present paper we study a problem of the same type in which we adopt an equation of state in the form of pressure-temperature and density-temperature relations which allow for the variation of the polytropic index throughout the sphere. Various characteristics of the model are derived and the equilibrium equations are integrated for various values of the parameter σ. The equivalent polytropic index in this model varies from 2.7508 at the centre of the sphere to the value 1.5 at the boundary and it is shown that this leads to results similar to those for a polytropic model of index slightly larger than two.

2. Basic equations

The field equations, corresponding to the orthogonal metric

$$ds^2 = e^{\alpha} dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\theta^2) - e^{\gamma} dt^2 \tag{1}$$

can be written (e.g. Synge 1960)

(2) $$e^{\alpha} = 1-\kappa v/r$$

(3) $$\alpha_1+\gamma_1 = \kappa r\, e^{\alpha}(p+\mu)$$

(4) $$p_1+\tfrac{1}{2}\gamma_1(p+\mu) = 0$$

where

(5) $$v = \int_0^r r^2\mu\, dr$$

and

(6) $$\kappa = 8\pi G/c^4.$$

The subscript one indicates differentiation with respect to r and the non-zero components of the energy-momentum tensor are

(7) $$\begin{aligned} T_1^1 = T_2^2 = T_3^3 &= p \\ T_4^4 = -\mu &= -c^2\rho. \end{aligned}$$

Using equations (2) to (5) it is possible to derive the following system of differential equations:

(8) $$-v+r\frac{dv}{dr} - \frac{2p_1}{p+\mu}\, r^2\left(\frac{1}{\kappa} - \frac{v}{r}\right) = r^3(p+\mu),$$

(9) $$\frac{dv}{dr} = r^2\mu.$$

In order to proceed any further we have to introduce an equation of state and we adopt in this paper the following density-temperature and pressure-temperature relations

(10) $$\mu = \alpha c^2/L(x),$$

(11) $$p = \frac{\beta}{xL(x)},$$

where α and β are constants. In these expressions, $\theta = 1/x$ is known as the effective temperature and

$$L(x) = \frac{x}{K_2(x)} \exp\{-xH(x)\},$$

where

(12) $$H(x) = K_3(x)/K_2(x),$$

$K_2(x)$ and $K_3(x)$ being the modified Bessel functions of the second kind.

As pointed out by Synge (1957), the density-temperature and pressure-temperature-relations (10) and (11), due to the asymptotic properties of $L(x)$, reduce to the familiar pressure density relation

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$$p = \text{const. } \rho^{1+1/n}$$

with

$$n = 3 \quad \text{for very high effective temperatures}$$

and

$$n = 1.5 \text{ for low effective temperatures.}$$

From equation (11) it follows that

$$p_1 = \frac{-\beta}{x^2 L^2} \{L + xL'\} \frac{dx}{dr} \tag{13}$$

$$\text{or} \qquad p_1 = \frac{\beta H'}{L} \frac{dx}{dr},$$

since it follows from the definition of $L(x)$ that

$$xL' + L = -x^2 H' L \tag{14}$$

$$\text{where} \qquad L' = \frac{dL(x)}{dx} \quad \text{and} \quad H' = \frac{dH(x)}{dx}.$$

Using the well-known recurrence relations for the modified Bessel functions of the second kind $K_2(x)$ and $K_3(x)$, it can be shown that $H(x)$ satisfies the following differential equation,

$$H' = -\frac{5}{x} H + H^2 - 1,$$

and therefore it follows from (13) that

$$p_1 = \frac{\beta}{L} \left\{ -\frac{5}{x} H + H^2 - 1 \right\} \frac{dx}{dr}. \tag{15}$$

By substituting (15), (10) and (11) in (8) and (9) it can be shown that the problem reduces to the solution of the following system of differential equations:

$$\frac{dr}{dx} = \phi(r, v, H, L, x)$$

where

$$\phi = \frac{-2r^2 \left(\frac{1}{\kappa} - \frac{v}{r}\right)\left(-\frac{5}{x} H + H^2 - 1\right)}{\left(\frac{\alpha c^2}{\beta} + \frac{1}{x}\right)\left(\frac{\beta}{xL} r^3 + v\right)}$$

$$\frac{dv}{dx} = \frac{r^2 \alpha c^2}{L} \phi(r, v, H, L, x) \tag{16}$$

$$\frac{dH}{dx} = -\frac{5}{x}H + H^2 - 1$$

$$\frac{dL}{dx} = -xL\left(-\frac{5}{x}H + H^2 - 1\right) - \frac{L}{x}.$$

If we assume $\theta = 1$ at the centre of the sphere it follows from (10) and (11) that

(17)
$$\rho_c = \frac{\alpha}{L(1)}$$
$$p_c = \frac{\beta}{L(1)}$$

where

$$L(1) = \frac{1}{K_2(1)} \exp\{-K_3(1)/K_2(1)\}.$$

Introduce the notation

(18) $$l(x) = 1/L(x);$$

it follows (Watson 1922) that $l(1) = 128.5$.

If we now introduce

(i) the parameter

(19) $$\sigma = \frac{\beta}{c^2\alpha} = \frac{p_c}{\rho_c c^2}$$

(ii) the new variable ξ defined by

(20) $$r = R\xi \text{ where } R = \left\{\frac{c^2\sigma}{4\pi GL(1)\rho_c}\right\}^{\frac{1}{2}}$$

(iii) the new variable $\bar{v}$ defined by

(21) $$v = \alpha c^2 R^3 \bar{v}$$

we obtain, after a few lengthy but straightforward transformations, the following system of differential equations:

(22)
$$\frac{d\xi}{d\theta} = \chi$$
$$\frac{d\bar{v}}{d\theta} = \xi^2 l\chi$$
$$\frac{dH}{d\theta} = -\frac{1}{\theta^2}\{-5\theta H + H^2 - 1\}$$
$$\frac{dl}{d\theta} = -\frac{l}{\theta^3}\{-5H\theta + H^2 - 1\} - \frac{l}{\theta}$$

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where

$$\chi = \frac{(\xi^2 - 2\xi\bar{v}\sigma)(-5H\theta + H^2 - 1)}{\theta^2(1+\sigma\theta)(\sigma\theta l\xi^3 + \bar{v})}.$$

This system has to be integrated, subject to the following conditions:
(i) At the centre $\theta = 1$, $\xi = 0$, $\bar{v} = 0$,

$$H(1) = K_3(1)/K_2(1) = 4.370544,$$

$$l(1) = \frac{1}{L(1)} = K_2(1) \exp\{H(1)\} = 128.5.$$

(ii) At the boundary $\theta = 0$, i.e. $x = \infty$, $\xi = \xi_1$ and $\bar{v} = \bar{v}_1$, to be determined by numerical integration of the system (22).

By using the following asymptotic expansion (McLachlan 1955) for $K_n(x)$,

$$K_n(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \left\{1 + \frac{4n^2 - 1^2}{1!\,8x} + \frac{(4n^2-1)(4n^3-3^2)}{2!\,(8x)^2} + \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!\,(8x)^3} + \cdots\right\}$$

it can be shown that $H(x)$ has the following asymptotic expansion:

$$H = 1 + \tfrac{5}{2}\frac{1}{x} + \tfrac{15}{8}\left(\frac{1}{x}\right)^2 - \tfrac{15}{8}\left(\frac{1}{x}\right)^3 + \cdots$$

from which it follows that at the boundary of the sphere $H = 1$.

Also, for x large (Synge 1957)

$$L(x) = \sqrt{\frac{2}{\pi}}\, x^{\frac{1}{2}}$$

or

$$l(x) = \sqrt{\frac{\pi}{2}}\, x^{-\frac{1}{2}},$$

therefore $l = 0$, from which it follows that ρ and p both vanish at the boundary.

3. Various characteristics of the model

If the solution of the system (22) is known, i.e. if we know ξ, $\bar{v}$, H and l as functions of the effective temperature θ, then it is possible to derive the following characteristics of the model.

(i) *Metric*

From (2) it follows after a few obvious substitutions that

(23) $$e^{-\alpha} = 1 - \frac{2}{\xi}\bar{v}\sigma.$$

Since the metric must go over into the Schwarzschild metric with

$$e^{\gamma} = e^{-\alpha} = 1 - \frac{2GM}{c^2 r}$$

outside the sphere, we must have on the boundary (where $\theta = 0$)

$$e^{\gamma} = e^{-\alpha} = 1 - \frac{2GM}{c^2 R\xi_1},$$

or using (23),

(24) $$\frac{2GM}{c^2 R\xi_1} = \frac{2\bar{v}(\xi_1)\sigma}{\xi_1}.$$

From (4) it follows that

$$\gamma = \gamma(\xi_1) + 2\sigma \int_0^{\theta} \frac{(-5H\theta + H^2 - 1)}{(1+\sigma\theta)\theta^2}\, d\theta$$

or, using the condition at the boundary (24),

(25) $$\gamma = ln\left(1 - \frac{2\sigma\bar{v}(\xi_1)}{\xi_1}\right) + 2\sigma \int_0^{\theta} \frac{(-5H\theta + H^2 - 1)}{(1+\sigma\theta)\theta^2}\, d\theta.$$

(ii) *Pressure and density*

The values of these follow from (10) and (11) and are given by:

(26) $$\frac{\rho}{\rho_c} = \frac{l(\theta)}{l(1)}$$

(27) $$\frac{p}{p_c} = \frac{\theta l(\theta)}{l(1)}.$$

(iii) *Concentration towards the centre*

Following Tooper, we define the average density as follows:

(28) $$\bar{\rho} = \frac{3M}{4\pi r_1^3}, \quad \text{where } r_1 = R\xi_1.$$

From

$$\frac{GM}{c^2 r_1} = \frac{\bar{v}(\xi_1)\sigma}{\xi_1}$$

and (20), i.e.

$$R^2 = \frac{c^2 \sigma l(1)}{4\pi\rho_c G},$$

it follows that

(29)
$$\frac{\rho_c}{\bar{\rho}} = \frac{\xi_1^3 l(1)}{3\bar{v}(\xi_1)}.$$

4. Numerical integration

The system of equations (22) was integrated numerically for various values of σ in the range 0.05 to 0.70, starting from the centre and using the following initial expansions:

$$\begin{aligned} \xi &= (h/\theta_2)^{\frac{1}{2}} \\ \bar{v} &= \frac{l_0}{3}(h/\theta_2)^{\frac{3}{2}} \\ l &= l_0 - l_2(h/\theta_2) \\ H &= H_0 - H_2(h/\theta_2), \end{aligned}$$

where

$$\begin{aligned} l_0 &= l(1) \\ H_0 &= H(1) \\ \theta_2 &= \frac{-(1+\sigma)l_0(\sigma+\frac{1}{3})}{2(-5H_0+H_0^2-1)} \\ H_2 &= \frac{(1+\sigma)l_0(\sigma+\frac{1}{3})}{2} \\ l_2 &= \frac{-(5-H_0)H_0 l_0^2(1+\sigma)(\sigma+\frac{1}{3})}{2(-5H_0+H_0^2-1)}, \end{aligned}$$

h being the initial step length.

In Table I we give the various characteristics of the model for several values of σ, such as (i) the values ξ_1 and $\bar{v}_1$ at the boundary, (ii) the maximum value of e^{α} and the point $\bar{r}/\bar{R}$ at which it occurs, (iii) the minimum value of e^{γ} and (iv) the concentration towards the centre $\rho_c/\bar{\rho}$.

The relative radius $\bar{r}/\bar{R}$ is given by

$$\bar{r}/\bar{R} = \frac{\int_1^{\theta} e^{\alpha/2}\chi\, d\theta}{\int_1^{0} e^{\alpha/2}\chi\, d\theta}.$$

In Fig. 1 we give the values of p/p_c, ρ/ρ_c, θ, $e^{\alpha}/(e^{\alpha})_{\max}$, $M_r/M = \bar{v}/\bar{v}_1$ and e^{γ} as functions of the relative radius $\bar{r}/\bar{R}$ in the case where $\sigma = 0.5$.

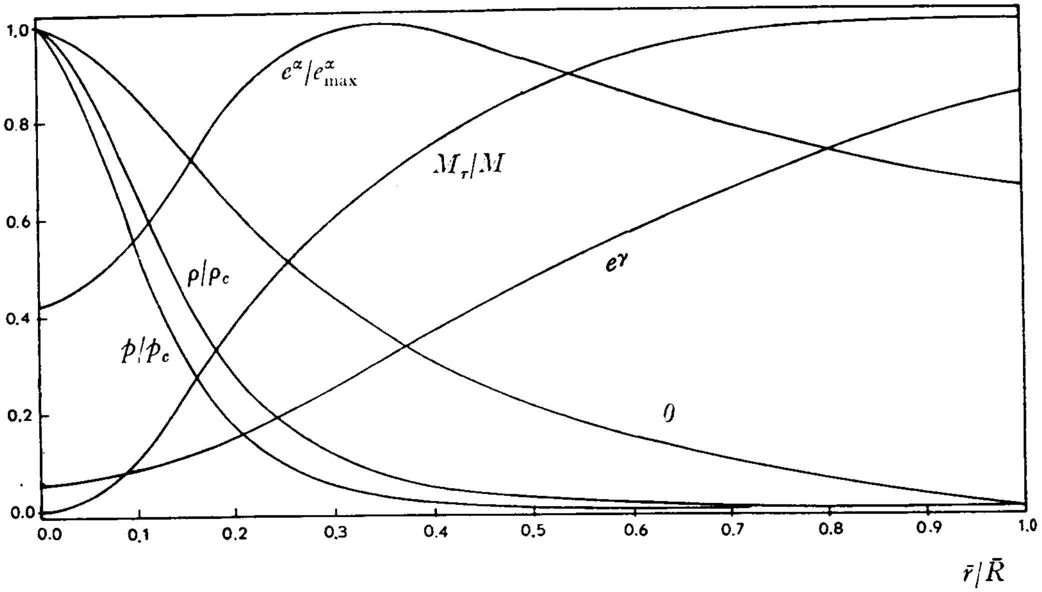

Figure 1. Values of p/p_c, ρ/ρ_c, θ, $e^{\alpha}/(e^{\alpha})_{\max}$, $M_r/M = \bar{v}/\bar{v}_1$ and e^{γ} as functions of the relative radius $\bar{r}/\bar{R}$ in the case where $\sigma = 0.5$.

5. Conclusions

If we compare our system of differential equations with the general relativistic Emden equation derived by Tooper (1964) we see that our model has the "equivalent polytropic index"

$$n = -1-(-5\theta H+H^2-1)/\theta^2$$

and this varies from 2.7508 at the centre of the sphere to 1.5 on the boundary.

A comparison with the results derived by Tooper shows that the present model is equivalent to a polytropic model with polytropic index slightly larger than $n = 2$.

This is shown in Table II in which we compare the central condensation of polytropic models $n = 2$ and $n = 2.5$ with the results derived in this paper.

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TABLE I

Various characteristics of the model

σ	ξ_1	$\bar{v}_1$	Max of e^{α}	Corresponding $\bar{r}/\bar{R}$	Minimum of e^{γ}	$\rho_c/\bar{\rho}$
0.05	0.68923	0.90731	1.2151	0.5457	0.62417	15.370
0.10	0.64315	0.68220	1.4109	0.50604	0.41366	16.610
0.15	0.61221	0.53786	1.5861	0.48235	0.28660	18.172
0.20	0.59152	0.43948	1.7413	0.46092	0.20547	20.059
0 25	0.57811	0.36919	1.8779	0.44189	0.15134	22.290
0.30	0.57008	0.31705	1.9976	0.42039	0.11394	24.890
0.35	0.56615	0.27719	2.1022	0.40148	0.08735	27.885
0.40	0.56544	0.24594	2.1935	0.38510	0.06799	31.309
0.45	0.56729	0.22092	2.2729	0.36970	0.05363	35.197
0.50	0.57122	0.20053	2.3421	0.35365	0.04278	39.588
0.55	0.57686	0.18364	2.4021	0.34009	0.03447	44.523
0.60	0.58390	0.16947	2.4543	0.32741	0.02802	50.033
0.65	0.59212	0.15742	2.4995	0.31558	0.02296	56.171
0.70	0.60133	0.14707	2.5387	0.30309	0.01895	62.973

TABLE II

Comparison of the central condensation $\rho_c/\bar{\rho}$ with the one obtained for polytropic models

σ	Polytropic model $n = 2$	Present model	Polytropic model $n = 2.5$
0.1	12.99	16.610	31.18
0.2	15.57	20.059	46.10
0.3	19.27	24.890	74.35
0.4	24.41	31.309	129.6
0.5	31.42	39.588	242.3
0.6	40.90	50.033	480.4

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