ON COMPACTIFICATION OF MAPPINGS

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If X and Y are Tychonoff spaces then the continuous function f mapping X onto Y is said to be compact (perfect, or proper) if it is closed and point inverses are compact. If h is a continuous function mapping X onto Y then by a compactification of h we mean a pair (X^*, h^*) where X^* is Tychonoff and contains X as a dense subspace, and where $h^*\colon X^* \to Y$ is a compact extension of h. The idea of a mapping compactification first appeared in (**7**). In (**1**) it was shown that any compactification of X determines a compactification of h, and that any compactification of h can be determined in this way. This idea was then developed in (**2**) and (**3**).

Throughout this paper all topological spaces are assumed to be Tychonoff. We consider throughout a fixed continuous function f mapping X onto Y.

A *compactification* of X is a compact Hausdorff space containing as a dense subspace a homeomorphic image of X. We assume X to be a subspace of each of its compactifications. If a is a partition of βX, the Stone-Cech compactification of X, then a is said to be an *upper semicontinuous* (*u.s.c.*) decomposition of βX if

(*a*) the members (*blocks*) of a are compact subsets of βX and

(*b*) whenever V is open in βX and contains the block A of a, there exists an open set W in βX such that $A \subseteq W \subseteq V$ and W contains any block of a that it meets. (Notice that since βX is normal the partition a of βX is u.s.c. if and only if the associated canonical quotient mapping is closed.) Refinement of partitions imposes a natural partial order upon the set of u.s.c. decompositions of βX: $a \leqq b$ if and only if given $A \in a$, there exists $B \in b$ such that $A \subseteq B$. The u.s.c. decompositions of βX with this partial order form a complete lattice (**4**). For our purposes we need only note that

$$\bigwedge\{a_i\colon\ i \in I\} = \{\bigcap\{A_i\colon\ i \in I\}\colon\ A_i \in a_i,\ i \in I\}.$$

A natural partial order also exists on the set of compactifications of X. We say $aX \leqq bX$ if and only if there exists a continuous function h mapping bX onto aX such that $h \mid X$ is the identity. The compactifications aX, bX are said to be equivalent if $aX \leqq bX$ and $bX \leqq aX$ and we consider equivalent compactifications to be the same compactification. Similarly, if (X^*, f^*) and (X', f') are compactifications of f, we say $(X^*, f^*) \leqq (X', f')$ if and only if there is a continuous function h mapping X' onto X^* such that $h \mid X$ is the identity and such that $f' = f^* . h$. Again we say that (X^*, f^*) and (X', f')

are equivalent if $(X^*,f^*) \leqq (X',f')$ and $(X',f') \leqq (X^*,f^*)$, and by considering equivalent compactifications to be the same compactification we have a partial order defined on the compactifications of f. In (6) it was shown that there is a natural bijection between the set of compactifications of X and the set of u.s.c. decompositions of βX which contain the points of X as blocks. If a is a u.s.c. decomposition of βX with this property then we denote by aX the corresponding compactification of X and by q_a the corresponding mapping from βX onto aX. (In fact q_a is the canonical quotient mapping associated with the partition a.) In (5) it was shown that $aX \leqq bX$ if and only if $b \leqq a$.

We denote by $\bar{f}$: $\beta X \to \beta Y$ the continuous extension of f: $X \to Y \subseteq \beta Y$, and by u we mean the u.s.c. decomposition $\{\bar{f}^{-1}(y): y \in \beta Y\}$ of βX. If a is a u.s.c. decomposition of βX containing the points of X as blocks, and if $a \leqq u$, then denote by p_a the restriction of q_a to $\bar{f}^{-1}(Y)$. We then define f_a and f_a^* by

$$f_a = \bar{f}.q_a^{-1}: aX \to \beta Y$$

$$f_a^* = \bar{f}.p_a^{-1}: f_a^{-1}(Y) \to Y.$$

It is a routine matter to show that f_a and f_a^* are continuous functions onto βY and Y respectively. Notice that $f_a^{-1}(Y) = q_a(\bar{f}^{-1}(Y)) = p_a(\bar{f}^{-1}(Y))$ and f_a^* is the restriction of f_a to this set. Notice also that f_a, being a continuous function defined on a compact space, is compact.

Theorem 1. *If aX is a compactification of X then $(f_{a\wedge u}^{-1}(Y), f_{a\wedge u}^*)$ is a compactification of f and each compactification of f can be described in this way.*

Proof. $a \wedge u$ is a u.s.c. decomposition of βX containing the points of X as blocks and $a \wedge u \leqq u$. Then $f_{a\wedge u}^*$, being the restriction of $f_{a\wedge u}$ to $f_{a\wedge u}^{-1}(Y)$ is a compact mapping, and moreover $f_{a\wedge u}^*$ is an extension of f, since for each $x \in X$ $p_{a\wedge u}^{-1}(x) = x$.

Suppose now that (X',f') is a compactification of f. Then $\beta X'$ is a compactification of X which we denote by bX. If $\bar{f}'$: $\beta X = bX \to \beta Y$ denotes the continuous extension of f', then since $\bar{f}'.q_b$ is equal to $\bar{f}$ on X, $\bar{f}'.q_b = \bar{f}$ on βX and so $b \leqq u$. Then f_b is defined and, again since the extension of f to bX is unique, $f_b = \bar{f}'$ on bX, and so if we can show that $f_b^{-1}(Y) = X'$ then the proof is complete. Suppose then that $x \in \beta X' \backslash f'^{-1}(y)$. Then since $f'^{-1}(y)$ is compact x has a compact neighbourhood N disjoint from $f'^{-1}(y)$. Then $N \cap X'$ is closed and non-empty and so $f'(N \cap X')$ is closed, non-empty, and does not contain y. Then since $\bar{f}'$ is continuous

$$\bar{f}'(\mathrm{cl}\,(N \cap X')) \subseteq \mathrm{cl}\,(\bar{f}'(N \cap X')) = f'(N \cap X')$$

and so $\bar{f}'(x) \neq y$. Hence $X' = f'^{-1}(Y) = \bar{f}'^{-1}(Y) = f_b^{-1}(Y)$ and this completes the proof.

Notice that if (X^*,f^*) is a compactification of X then the content of Theorem 1 is to allow us to consider X^* to be a subspace of any compactification aX producing f^*.

Theorem 2. *If* (X^*, f^*) *and* (X', f') *are compactifications of* f *then*

$$(X^*, f^*) \leqq (X', f')$$

if and only if $\beta X^* = bX \leqq aX = \beta X'$.

Proof. We saw in the proof of Theorem 1 that (X^*, f^*) and (X', f') respectively are determined by bX and aX.

If $bX \leqq aX$ then h defined by $h = p_b . p_a^{-1}$ is a function onto X^* and since p_a is continuous and p_b is closed, h is continuous. But then

$$f' = \bar{f} . p_a^{-1} = f^* . p_b . p_a^{-1} = f^* . h.$$

Conversely, if $(X^*, f^*) \leqq (X', f')$ then there is a continuous function h mapping X' onto X^* and such that $h \mid X$ is the identity. If we extend h to the continuous function $\bar{h}$: $\beta X' = aX \rightarrow bX = \beta X^*$ then $\bar{h}$ maps onto βX^* and so $bX \leqq aX$.

If p is a partition on the set S and if $A \subseteq S$ then by $p \mid S$ we mean the partition $\{P \cap A \colon P \in p\}$.

Corollary 3. *If the compactifications* (X^*, f^*) *and* (X', f') *of* f *are produced by the compactifications* aX *and* bX *respectively, then* $(X^*, f^*) \leqq (X', f')$ *if and only if*

$$b \wedge u \mid \bar{f}^{-1}(Y) \leqq a \wedge u \mid \bar{f}^{-1}(Y).$$

Proof. (X^*, f^*) is produced by the compactifications $a \wedge uX$ and $cX = \beta X^*$ and thus $a \wedge u \mid \bar{f}^{-1}(Y) = c \mid \bar{f}^{-1}(Y)$. Similarly, $b \wedge u \mid \bar{f}^{-1}(Y) = d \mid \bar{f}^{-1}(Y)$ where $dX = \beta X'$. Then $(X^*, f^*) \leqq (X', f')$ if and only if $cX \leqq dX$ and this is so if and only if $d \leqq c$. However, since $cX = \beta X^*$, c is the minimal u.s.c. decomposition of βX such that $c \mid \bar{f}^{-1}(Y) = a \wedge u \mid \bar{f}^{-1}(Y)$ and similarly d is minimal such that $d \mid \bar{f}^{-1}(Y) = b \wedge u \mid \bar{f}^{-1}(Y)$. Thus $d \leqq c$ if and only if $b \wedge u \mid \bar{f}^{-1}(Y) \leqq a \wedge u \mid \bar{f}^{-1}(Y)$.

Corollary 4. *The compactifications* aX, bX *determine equivalent compactifications of* f *if and only if in* $\bar{f}^{-1}(Y)$ *the partitions*

$$\{q_a^{-1}(t) \cap \bar{f}^{-1}(y) \colon t \in aX,\ y \in Y\}$$

and $\{q_b^{-1}(t) \cap \bar{f}^{-1}(y) \colon t \in bX,\ y \in Y\}$ *are equal.*

Corollary 5. *The compactifications of* f *with the usual partial order form a complete upper semilattice. They form a complete lattice if and only if for some compactification* sX *of* X, $s \wedge u \mid (\bar{f}^{-1}(Y) \backslash X) = u \mid (\bar{f}^{-1}(Y) \backslash X)$.

Proof. If $\{(X_i, f_i) \colon i \in I\}$ is a family of compactifications of f then

$$\{b_i X = \beta X_i \colon i \in I\},$$

being a family of compactifications of X, has a supremum, aX say. Then for each $i \in I$, $a \leqq b_i \leqq u$ and so from Corollary 3 we see that (X_a^*, f_a^*) is an upper bound for $\{(X_i, f_i) \colon i \in I\}$. Moreover, if (X', f') is an upper bound

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for $\{(X_i, f_i): i \in I\}$ then for each $i \in I$, $b_i X \leqq bX = \beta X$ and so $aX \leqq bX$. Then $b \leqq a$ and so $(X_a^*, f_a^*) \leqq (X', f')$. Thus the compactifications of f form a complete upper semilattice.

This semilattice will be a complete lattice if and only if it has a smallest member.

Now, if there is a smallest compactification of f, determined by the compactification sX of X say, where $s \leqq u$, then if $s \mid \bar{f}^{-1}(Y) \backslash X \neq u \mid \bar{f}^{-1}(Y) \backslash X$ there exist points x, y in $\bar{f}^{-1}(Y) \backslash X$ belonging to the same block in u but belonging to different blocks in s. But then the partition m produced from s by joining the blocks containing x and y is again u.s.c. and is strictly less than s on $\bar{f}^{-1}(Y)$. It follows from Corollary 3 that $(X_m^*, f_m^*) < (X_s^*, f_s^*)$ contradicting the fact that (X_s^*, f_s^*) is minimal.

Conversely, if sX is such that $s \mid \bar{f}^{-1}(Y) \backslash X = u \mid \bar{f}^{-1}(Y) \backslash X$ then for any compactification aX of X, $a \wedge u \leqq u$ and so from Corollary 3

$$(X_s^*, f_s^*) \leqq (X_a^*, f_a^*).$$

In particular if X is locally compact the compactifications of f form a complete lattice, since in this case the one-point compactification of X satisfies the requirements of sX in Corollary 5.

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