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ON EDGE-COLORABILITY OF CARTESIAN PRODUCTS OF GRAPHS*

BY

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In an article P. E. Himelwright and J. E. Williamson [3] proved a theorem on 1-factorability of Cartesian product of two graphs. With a very short proof we prove a more general theorem which immediately implies their theorem as a corollary. We will follow the notations and definitions of [1], [2] and [3].

THEOREM. *If* $\chi_1(G)=\Delta(G)$, *then* $\chi_1(G\times H)=\Delta(G)+\Delta(H)$.

Proof. $G\times H$, which is isomorphic with $H\times G$, contains $|V(H)|$ disjoint "horizontal" copies $G_1, G_2, \ldots, G_{|V(H)|}$ of G, and $|V(G)|$ disjoint "vertical" copies $H_1, H_2, \ldots, H_{|V(G)|}$ of H. A horizontal copy G_i and a vertical copy H_j have only one vertex (u_i, v_j) in common.

By a theorem of Vizing (see [4] p. 245) we have

$$\Delta(G\times H)\leq\chi_1(G\times H)\leq\Delta(G\times H)+1.$$

But, $\Delta(G\times H)=\Delta(G)+\Delta(H)$. Therefore it is enough to show that $\chi_1(G\times H)\leq \Delta(G)+\Delta(H)$.

To see this, color the edges of each horizontal copy properly and identically with colors $\{1, 2, \ldots, \Delta(G)=\chi_1(G)\}$ and each vertical copy properly and identically with colors $\{\Delta(G)+1, \Delta(G)+2, \ldots, \Delta(G)+\chi_1(H)\}$. If $\chi_1(H)=\Delta(H)$ then we are done. If $\chi_1(H)=\Delta(H)+1$, then take any edge $e=[(u_i, v_k), (u_j, v_k)]$ in any horizontal copy G_k, which is colored in color number 1. Each end ventex of e in the copies H_i or H_j is joined to at most $\Delta(H)$ vertical edges. Therefore there is at least one color missing at both ends. We color the edge e the missing color. In this manner, color 1 is removed, and we have colored $G\times H$ in just $\Delta(G)+\Delta(H)$ colors $\{2, 3, \ldots, \Delta(G)+\Delta(H)+1\}$.

Behzad and Mahmoodian [2] discussed the topological invariants of $G\times H$ in terms of those of G and H. It is shown (page 159), that if both $\chi_1(G)$ and $\chi_1(H)$ assume the right side of the Vizing inequalities (i.e., $\chi_1(G)=\Delta(G)+1, \chi_1(H)=\Delta(H)+1$), then $\chi_1(G\times H)$ can assume either side of the inequalities with the proper G and H. The above theorem shows that if at least one of

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$\chi_1(G)$ or $\chi_1(H)$ assumes the left side of the Vizing inequalities then so does $\chi_1(G\times H)$.

Now the following corollary is the theorem of Himelwright and Williamson:

COROLLARY. *If G is a 1-factorable graph and H is a regular graph, then $G\times H$ is a 1-factorable graph.*

Proof. The 1-factorability of G implies $\chi_1(G)=\Delta(G)$. Then by the above theorem $\chi_1(G\times H)=\Delta(G\times H)$, and since $G\times H$ is regular, it is 1-factorable.

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