ON (V^*) SETS AND PELCZYNSKI'S PROPERTY (V^*)

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Introduction. The concept of (V^*) set was introduced, as a dual companion of that of (V)-set, by Pelczynski in his important paper **[14]**. In the same paper, the so called properties (V) and (V^*) are defined by the coincidence of the (V) or (V^*) sets with the weakly relatively compact sets. Many important Banach space properties are (or can be) defined in the same way; that is, by the coincidence of two classes of bounded sets. In this paper, we are concerned with the study of the class of (V^*) sets in a Banach space, and its relationship with other related classes. To this general study is devoted Section I. A (as far as we know) new Banach space property (we called it property weak (V^*)) is defined, by imposing the coincidence of (V^*) sets and weakly conditionally compact sets. In this way, property (V^*) is decomposed into the conjunction of the weak (V^*) property and the weak sequential completeness. In Section II, we specialize to the study of (V^*) sets in Banach lattices. The main result in the section is that every order continuous Banach lattice has property weak (V^*), which extends previous results of E. and P. Saab (**[16]**). Finally, Section III is devoted to the study of (V^*) sets in spaces of Bochner integrable functions. We characterize a broad class of (V^*) sets in $L_1(\mu, E)$, obtaining similar results to those of Andrews **[1]**, Bourgain **[6]** and Diestel **[7]** for other classes of subsets. Applications to the study of properties (V^*) and weak (V^*) are obtained. Extension of these results to vector valued Orlicz function spaces are also given.

We shall try to follow the standard notations in Banach space theory, as in **[8]**, **[11]** and **[12]**. In order to prevent any doubt, we shall fix some terminology. If E is a Banach space, $B(E)$ will be its closed unit ball and E^* its topological dual. The word operator will always mean linear bounded operator, and $\mathscr{L}(E, F)$ will stand for the banach space of all operators from E into F. A subset B of a Banach space is called *weakly conditionally compact* if every sequence in B has a weakly Cauchy subsequence. A series $\sum x_n$ in E is said to be *weakly unconditionally Cauchy* (w.u.c. in short) if $\sum |x^*(x_n)| < \infty$ for every $x^* \in E^*$ $\left(\text{equivalently, if } \left\{\sum_\sigma x_n : \sigma \subset \mathbb{N} \text{ finite}\right\} \text{ is a bounded subset}\right)$. If A is a subset of the normed space E, $[A]$ will be the closed linear span of A. We shall denote by $\mathscr{B}(E)$, $\mathscr{W}\mathscr{C}(E)$ and $\mathscr{W}(E)$ the classes of bounded, weakly conditionally compact and weakly relatively compact subsets of E, respectively.

Given a finite positive measure space (Ω, Σ, μ) and a Young's function Φ with conjugate function Ψ (see **[19]**, p. 77 and ff.), for every strongly measurable function $f : \Omega \to E$ we shall write

$$M_\Phi(f) = \int \Phi(\|f\|)\, d\mu.$$

The Orlicz space $L_\Phi(\mu, E)$ is the vector space of all (classes of) strongly measurable functions f from Ω into E such that $M_\Phi(kf) < \infty$ for some $k > 0$ (if $\Phi(t) = t^p (1 \le p < \infty)$,

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$L_\Phi(\mu, E)$ is the usual Lebesgue space $L_p(\mu, E)$). In general, $L_\Phi(\mu, E)$ coincides with the set of all Bochner measurable functions $f:\Omega\to E$ such that

$$\|f\|_\Phi=\sup\left\{\int \|f\|\,\beta\,d\mu:\beta\in L_\Psi(\mu,\mathbb{K}),\ M_\Psi(\beta)\le 1\right\}<\infty.$$

This expression defines a Banach space norm in $L_\Phi(\mu, E)$. We have

$$L_\infty(\mu, E)\subset L_\Phi(\mu, E)\subset L_1(\mu, E),$$

with continuous inclusion. Finally, let us recall that Φ is said to satisfy the (Δ_2) condition if it is everywhere finite and

$$\limsup_{t\to\infty}\frac{\Phi(2t)}{\Phi(t)}<\infty.$$

In this case, the simple functions are dense in $L_\Phi(\mu, E)$.

I. (V^*) sets.

DEFINITION. A subset A of a Banach space E is called a (V^*) *set* if for every w.u.c. series $\sum x_n^*$ in E^*, the following holds;

$$\lim_{n\to\infty}\sup\{|x_n^*(x)|:x\in A\}=0.$$

It is obvious that a (V^*) set is bounded. Also, every weakly relatively compact set is a (V^*) set (see Corollary 1.3 below). Let us denote by $\mathcal{V}^*(E)$ the family of all the (V^*) sets in E. Then, $\mathcal{W}(E)\subset\mathcal{V}^*(E)\subset\mathcal{B}(E)$. The Banach space E is said to have the *property* (V^*) of Pelczynski if every (V^*) set in E is relatively weakly compact; that is, if $\mathcal{W}(E)=\mathcal{V}^*(E)$. (V^*) sets and property (V^*) were introduced by Pelczynski in **[14]**, as a kind of dual property, and he proved also that $L_1(\mu)$ and reflexive spaces have property (V^*).

The following elementary properties of (V^*) are easily established.

(a) Subsets, linear combinations and closed absolutely convex hulls of (V^*) sets are (V^*) sets.

(b) If A is a (V^*) set in E and $T\in\mathcal{L}(E, F)$, then $T(A)$ is a (V^*) set in F.

(c) If every countable subset of A is a (V^*) set, then A is a (V^*) set.

The following proposition collects some useful characterizations of (V^*) sets.

PROPOSITION 1.1. *For a bounded subset A of a Banach space E, the following assertions are equivalent.*

(a) *A is a (V^*)* set.

(b) *Every weakly unconditionally Cauchy series $\sum x_n^*$ in E^* converges absolutely and uniformly on A, i.e.,*

$$\lim_{m\to\infty}\sup\left\{\sum_m^\infty |x_n^*(x)|:x\in A\right\}=0.$$

(c) *Every operator T from E into l_1 maps A into a relatively compact subset.*

(d) *A does not contain a sequence* (x_n) *equivalent to the unit basis of* l_1 *and such that the closed span* $[x_n]$ *is complemented in* E.

Proof. (a)$\Rightarrow$(b): If (b) does not hold, there exist an $\varepsilon > 0$, a subsequence $p_1 < q_1 < p_2 \ldots < p_n < q_n < \ldots$ of integers and a sequence (x_n) in A such that

$$\sum_{n=p_j}^{q_j} |x_n^*(x_j)| > \varepsilon.$$

But then there is a subset σ_j of $\{p_j, \ldots, q_j\}$ such that

$$\left| \sum_{\sigma_j} x_n^*(x_j) \right| > \frac{\varepsilon}{4}.$$

(See, for instance, [**15**], Lemma 6.3). If we put $y_j^* = \sum_{\sigma_j} x_n^*$, then $\sum y_j^*$ is a weakly unconditionally Cauchy series in E^*, such that $|y_j^*(x_j)| > \varepsilon/4$ for every $j \in \mathbb{N}$.

The equivalence between (b) and (c) follows from the well known characterization of the relatively compact subsets of l_1 and the one to one correspondence between operators from E into l_1 and weakly unconditionally Cauchy series in E^*, that associates to the series $\sum x_n^*$ in E^* the operator defined by $T(x) = (x_n^*(x)) \in l_1$ (see [**8**], Ch. VII). As (a) follows obviously from (b), it results that (a), (b) and (c) are equivalent.

But if $(x_n) \subset A$ is equivalent to the unit basis of l_1, P is a continuous projection onto $[x_n] = F$ and S is the canonical isomorphism between F and l_1, then $T = S \,.\, P \in \mathcal{L}(E, l_1)$, and $T(A)$ contains the unit basis of l_1. In particular, $T(A)$ is not relatively compact. This proves that (c) implies (d).

Finally, if there exists $T \in \mathcal{L}(E, l_1)$ such that $T(A)$ is not relatively compact. Theorem 1.4 of [**13**] assures the existence of a sequence (x_n) in A such that $(T(x_n))$ is equivalent to the l_1 unit basis and spans a closed complemented subspace. For every finitely non zero scalar sequence (λ_n), we have

$$m \sum |\lambda_n| \le \| \sum \lambda_n T(x_n) \| \le \|T\| \, \| \sum \lambda_n x_n \| \le \|T\| \, M \sum |\lambda_n|$$

for some positive constants m and M. This proves that (x_n) is also equivalent to the unit basis of l_1 and that T restricted to $F = [x_n]$ is an isomorphism. If Q is a continuous projection from l_1 onto $[T(x_n)]$, then $(T_{|F})^{-1} \,.\, Q \,.\, T$ is a continuous projection from E onto F. This proves that (c) follows from (d).

Remark 1.2. The equivalence (a)$\Leftrightarrow$(c) is due to Emmanuele [**10**]. Also (a)$\Leftrightarrow$(d) appears more or less implicitly in [**10**], quoting a result of Godefroy and Saab.

Corollary 1.3. *For any Banach space* E, *we have* $\mathcal{WC}(E) \subset \mathcal{V}^*(E)$, *i.e., every weakly conditionally compact set is a* (V^*) *set.*

Proof. Follows from Proposition 1.1(c) and the fact that weakly conditionally compact and relatively compact sets coincide in l_1, because of the Schur lemma.

Corollary 1.3 justifies the following definition.

Definition. A Banach space is said to have the *property weak* (V^*) if $\mathcal{V}^*(E) = \mathcal{WC}(E)$.

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REMARK 1.4. (a) By the definition, it is obvious that E has property (V^*) if and only if it is weakly sequentially complete and has the property weak (V^*).

(b) If E does not contain a copy of l_1, Rosenthal's theorem ([**8**], Ch. XI) asserts that $\mathcal{B}(E) = \mathcal{W}\mathcal{C}(E)$, and so E has property weak (V^*).

(c) In particular, c_0 has the weak (V^*) property, but it does not have property (V^*).

(d) In the next section, we shall prove that every order continuous Banach lattice has property weak (V^*).

COROLLARY 1.5. *For a Banach space E, the following assertions are equivalent:*

(a) $\mathcal{B}(E) = \mathcal{V}^*(E)$; *i.e., every bounded set in E is a* (V^*) *set.*

(b) *Every operator T from E into* l_1 *is (weakly) compact.*

(c) *E does not contain a complemented copy of* l_1.

Proof. This is just a reformulation of the equivalences (a)⇔(c)⇔(d) in Proposition 1.1.

COROLLARY 1.6. *If a Banach space E has the property weak* (V^*) *and contains a copy of* l_1, *it contains also a complemented copy of* l_1.

Proof. If not, Corollary 1.5 and definition of weak (V^*) property yield $\mathcal{B}(E) = \mathcal{V}^*(E) = \mathcal{W}\mathcal{C}(E)$, and so E does not contain a copy of l_1, by Rosenthal's l_1 theorem.

In particular, the above corollary shows that neither $C[0, 1]$ nor l_∞ have property weak (V^*).

The next result will be used later, and extends to (V^*) sets a well known property of (weakly) relatively compact subsets of a Banach space (see [**8**], Ch. XIII. Lemma 2, f.i.)

COROLLARY 1.7. *Let A be a bounded subset of a Banach space E. If for every* $\varepsilon > 0$ *there exists a* (V^*) *set* $A_\varepsilon \subset E$ *such that*

$$A \subset A_\varepsilon + \varepsilon B(E),$$

then A is a (V^*) *set.*

Proof. Let $T \in \mathcal{L}(E, l_1)$ with $\|T\| \leq 1$. Then

$$T(A) \subset T(A_\varepsilon) + \varepsilon T(B(E)) \subset T(A_\varepsilon) + \varepsilon B(E),$$

and $T(A_\varepsilon)$ is relatively compact. Then $T(A)$ is relatively compact by the aforementioned property of these sets.

If F is a subspace of a Banach space E, it is evident that every (V^*) set in F is also a (V^*) set in E, but the converse is not true. In fact, let F be an isomorphic copy of l_1 in l_∞. Then the unit ball $B(F)$ belongs to $\mathcal{V}^*(E)$ (by Proposition 1.1(d)), but it does not belong to $\mathcal{V}^*(F)$ (by Proposition 1.1(c), for example). However, the following result holds.

PROPOSITION 1.8. *Let E be a Banach space. The following assertions are equivalent.*

(a) $A \in \mathcal{V}^*(E)$.

(b) *For every separable and complemented subspace F of E,* $A \cap F$ *belongs to* $\mathcal{V}^*(F)$.

Proof. Let $A \in \mathcal{V}^*(E)$ and F a complemented subspace of E. Then every operator $T \in \mathcal{L}(F, l_1)$ extends to an operator $\bar{T} \in \mathcal{L}(E, l_1)$, and so $T(A \cap F) = \bar{T}(A \cap F) \subset \bar{T}(A)$ is relatively compact. It follows from Proposition 1.1(c) that $A \cap F \in \mathcal{V}^*(F)$.

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Conversely, if $A \notin \mathcal{V}^*(E)$, Proposition 1.1(d) yields a sequence $(x_n) \subset A$, equivalent to the l_1 unit basis and such that $F = [x_n]$ is complemented. Then clearly $A \cap F \notin \mathcal{V}^*(F)$.

COROLLARY 1.9. *If E has the separable complementation property, then E has property (weak) (V^*) if and only if every closed separable subspace of E has property (weak) (V^*).*

Proof. Let us consider only the case of the property (V^*), the rest being completely analogous. For the non trivial implication, let $A \in \mathcal{V}^*(E)$. By Eberlein's theorem, it is enough to prove that $A \cap F$ is weakly relatively compact for every closed separable subspace F of E. But the hypothesis implies the existence of a separable and complemented subspace M containing F. By Proposition 1.8, $A \cap M \in \mathcal{V}^*(M)$, and hence it is weakly relatively compact, and *a fortori* so is $A \cap F$.

For the property (V^*), the result above was proved by E. and P. Saab ([**16**], Proposition 3), in a different way.

These results are particular cases of the following more general situation. Suppose that E and F are Banach spaces, $J \in \mathcal{L}(E, F)$ and A is a bounded subset of E such that $J(A) \in \mathcal{V}^*(F)$. When does $A \in \mathcal{V}^*(E)$?. Our next result gives a complete answer.

PROPOSITION 1.10. *With the previous notations, the following assertions are equivalent.*

(a) *$A \in \mathcal{V}^*(E)$, whenever A is bounded and $J(A) \in \mathcal{V}^*(F)$.*

(b) *Every l_1-basis $(x_n) \subset E$ that spans a complemented subspace has a subsequence (y_k) such that $(J(y_k))$ is a l_1-basis in F which spans a complemented subspace.*

Proof. (a)$\Rightarrow$(b). Let (x_n) be a l_1 basis in E such that $[x_n]$ is complemented, and consider $A = \{x_n : n \in \mathbb{N}\}$. Obviously, A is bounded, but not a (V^*) set. Then, $J(A)$ cannot be a (V^*) set. Proposition 1.1(d) yields the result. The other implication is proved in a similar way.

PROPOSITION 1.11. *Assume the previous notations. If $J^*(F^*)$ is norm dense in E^*, then*

(a) *a bounded set $A \subset E$ belongs to $\mathcal{WC}(E)$ provided $J(A) \in \mathcal{WC}(F)$;*

(b) if *F has property weak (V^*), condition* (a) *of Proposition* 1.10 *holds, and E has property weak (V^*), too.*

Proof. (a). Let $A \subset E$ be bounded and such that $J(A) \in \mathcal{WC}(F)$. Let us show that $A \in \mathcal{WC}(E)$. In fact, if $(x_n) \subset A$, there is a subsequence (y_n) such that $(J(y_n))$ is weak Cauchy. Hence, for every $y^* \in F^*$, the limit

$$\lim_{n\to\infty} y^*(J(y_n)) = \lim_{n\to\infty} J^*(y^*)(y_n)$$

exists. Boundedness of (y_n) and the density of $J^*(F^*)$ in E^* implies then the existence of $\lim x^*(y_n)$, for every $x^* \in E^*$; i.e., (y_n) is weakly Cauchy.

(b) follows from (a) and Corollary 1.3.

II. (V^*) sets in Banach lattices. Recall that, given an order continuous Banach lattice E with a weak unit, there exists a probability space (Ω, Σ, μ) such that E is an order ideal of $L_1(\mu)$ and

$$L_\infty(\mu) \subset E \subset L_1(\mu),$$

with continuous inclusions ([**12**], Th. 1.b.14).

PROPOSITION 2.1. *Let E be an order continuous Banach lattice with a weak unit. With the previous notations, a bounded subset $K \subset E$ is weakly conditionally compact if and only if*

$$\lim_{\mu(A)\to 0} \sup\{|x^*(\chi_A f)| : f \in K\} = 0, \tag{†}$$

for every x^* in E^*.

Proof. The dual space E^* is identified with the set of all μ measurable functions g such that $\int fg\,d\mu < \infty$ for every $f \in E$, by means of the correspondence $g \mapsto x^*$, $x^*(f) = \int fg\,d\mu$. ([**12**], 1.b.14). With this identification, for every $g \in E^*$, the operator $T_g : E \mapsto L_1(\mu)$ defined by $T_g(f) = fg$ is continuous. Then, if K is weakly conditionally compact, $T_g(K)$ is weakly conditionally compact in $L_1(\mu)$, and so uniformly integrable. Hence, condition (†) holds.

Conversely, suppose that $K \subset E$ is bounded and satisfies (†). As $L_\infty(\mu) = L_1(\mu)^*$ is (identified to a) subspace of E^*, taking $x^* = 1$ in (†) we get

$$\lim_{\mu(A)\to 0} \left| \int_A f\,d\mu \right| = 0, \quad \text{uniformly in } f \in K,$$

and so K is uniformly integrable. Let $(f_n) \subset K$. Passing to a subsequence if necessary, we can suppose that (f_n) converges weakly in $L_1(\mu)$. Let $g \in E^*$, $A_n = \{\omega \in \Omega : |g(\omega)| \le n\}$ and $g_n = g\chi_{A_n}$. Clearly $(g_n) \to g$ in the weak* topology of E^*, and $(A_n) \uparrow \chi_\Omega$. Given $\varepsilon > 0$, let $\delta > 0$ be such that $\mu(A) < \delta$ implies that

$$\left| \int_A gf\,d\mu \right| < \varepsilon, \quad \text{for every } f \in K.$$

Let us choose $r \in \mathbb{N}$ so that $\mu(\Omega \backslash A_r) < \delta$. Then, for every m, $n \in \mathbb{N}$ we have

$$\left| \int (f_n - f_m) g\,d\mu \right| \le \left| \int_{\Omega\backslash A_r} (f_n - f_m) g\,d\mu \right| + \left| \int (f_n - f_m) g_r\,d\mu \right| \le 2\varepsilon + \left| 2\int (f_n - f_m) g\,d\mu \right|.$$

g_r defines a continuous linear form on $L_1(\mu)$. Hence, there exists and n_0 such that if n, $m \ge n_0$, then $|\int (f_n - f_m) g_r\,d\mu| < \varepsilon$, and so

$$\left| \int (f_n - f_m) g\,d\mu \right| \le 3\varepsilon,$$

which proves that (f_n) is weakly Cauchy in E.

The next theorem is the main result of this section.

THEOREM 2.2. *Every closed subspace of an order continuous Banach lattice has property weak (V^*).*

Proof. Let E be a closed subspace of the order continuous Banach lattice F, and let $K \in \mathcal{V}^*(E)$. In order to prove $K \in \mathcal{WC}(E)$, we can suppose K countable and hence, without loss of generality, E separable. But then there is a band F_0 with weak unit in F that contains E ([**12**], Proposition 1.a.9). It is enough to prove that $K \in \mathcal{WC}(F_0)$. But, with the notations of Proposition 2.1, every $g \in F_0^*$ defines a continuous operator $T_g : F_0 \mapsto L_1(\mu)$ by $T_g(f) = fg$. Then, $T_g(K)$ is a (V^*) set in $L_1(\mu)$, and hence weakly relatively compact. In particular, $T_g(K) = \{fg : f \in K\}$ is uniformly integrable, which is condition (†) in Proposition 2.1.

Part (b) in the next corollary is a slight generalization of the main result in [**16**].

COROLLARY 2.3. (a) *A closed subspace of an order continuous Banach lattice has property* (V^*) *if and only if it is weakly sequentially complete.*

(b) *A complemented subspace of a Banach lattice has property* (V^*) *if and only if it does not contain an isomorphic copy of* c_0.

Proof. (a) is an immediate consequence of Theorem 2.2; (b) is true by (a) and [**12**, 1, c.7] which states that a complemented subspace of a Banach lattice is weakly sequentially complete if it does not contain an isomorphic copy of c_0.

COROLLARY 2.4. *Let E be a σ-order complete Banach lattice. Then the following assertions are equivalent:*

(a) *E has property weak* (V^*),

(b) *E does not contain a copy of* l_∞,

(c) *E is order continuous.*

Proof. (a)⇒(b), because l_∞ does not have the property weak (V^*) (see Corollary 1.6). (b)⇒(c) follows from [**12**], Proposition 1.a.7, and (c)⇒(d) by Theorem 2.2.

COROLLARY 2.5. ([**18**], Theorem 16). *Let E be an order continuous Banach lattice. If E contains a copy of* l_1, *it contains also a complemented copy of* l_1.

Proof. This follows immediately from Theorem 2.2 and Corollary 1.6.

III. (V^*) **sets in Bocher integrable function spaces.** One of the most interesting problems in the study of vector valued function spaces is the characterization of different classes of subsets in terms of the corresponding classes in the base space. In the case of $L_1(\mu, E)$, the space of E-valued Bochner integrable functions on a finite, complete, positive measure space (Ω, Σ, μ), the attention has been focussed to weakly compact and weakly conditionally compact subsets, mainly. We shall be interested in the study of (V^*) sets. Following Batt and Hiermayer [**2**], given a class $\mathcal{H}$ of bounded sets, we define a uniformly integrable subset K of $L_1(\mu, E)$ to be a $\delta\mathcal{H}$-set if for all $\delta > 0$, there exists a set $H_\delta \in \mathcal{H}(E)$ and for each $f \in K$, a measurable set Ω_f such that $\mu(\Omega_f) < \delta$ and $f(\Omega \backslash \Omega_f) \subset H_\delta$. Bourgain proved in [**6**] that every $\delta\mathcal{W}\mathcal{C}$-set belongs to $\mathcal{W}\mathcal{C}(L_1(\mu, E))$. The corresponding result for weakly relatively compact sets was proved by Diestel in [**7**], and for Dunford–Pettis sets by Andrews in [**1**]. Here we shall prove the analogous result for $\delta\mathcal{V}^*$-sets. In the first place, we note the following result.

PROPOSITION 3.1. *Let E be a Banach space. Every* (V^*) *set* $K \subset L_1(\mu, E)$ *is uniformly integrable.*

Proof. In [**2**], Proposition 2.2, it is proved that a bounded subset $K \subset L_1(\mu, E)$ is uniformly integrable if and only if $T(K)$ is relatively weakly compact for all $T \in \mathcal{L}(L_1(\mu, E), c_0)$ such that there exists $g \in L_\infty(\mu, E^*)$, $\|g\|_\infty \le 1$, and $(A_n) \subset \Sigma$, $(A_n) \downarrow \emptyset$, in such a way that $T(f) = (\int_{A_k} \langle g, f \rangle \, d\mu)$. If we write $h_n = g\chi_{A_n \backslash A_{n+1}}$, then $\sum h_n$ is w.u.c. in $L_\infty(\mu, E^*)$. If K is a (V^*) set, we have, by Proposition 1 (b),

$$\lim_{n\to\infty} \sup\left\{\sum_{n}^{\infty} \left| \int \langle h_m, f \rangle \, d\mu \right| : f \in K\right\} = 0.$$

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In particular,

$$\left|\sum_n^{\infty} \int \langle h_m, f\rangle \, d\mu\right| = \left|\int_{A_n} \langle g, f\rangle \, d\mu\right|$$

converges to 0, uniformly in K, which proves that $T(K)$ is relatively compact in c_0.

The following result is analogous to Theorem 2 in [**1**], but for (V^*) sets instead of Dunford–Pettis sets.

THEOREM 3.2. *Let K be a bounded and uniformly integrable set of $L_1(\mu, E)$ such that for every $\delta > 0$ there is a set $A \in \Sigma$ with $\mu(A) < \delta$, so that for each $\omega \notin Aa(V^*)$ set $V(\omega) \subset E$ exists satisfying $f(\omega) \in V(\omega)$ for all $f \in K$. Then K is a (V^*) set.*

Proof. By Proposition 1.1, it will be enough to prove that $T(K)$ is relatively compact, for every operator $T : L_1(\mu, E) \mapsto l_1$. Let T be such an operator. Taking c_0 as a norming subspace of l_1^*, Theorem §13.8 in [**9**] provides a function $g : \Omega \mapsto \mathscr{L}(E, l_1)$ such that:

(a) for every $a \in c_0$ and every $f \in L_1(\mu, E)$, the function $\langle gf, a\rangle \in L_1(\mu)$ and

$$\langle T(f), a\rangle = \int \langle gf, a\rangle \, d\mu.$$

(b) $|g| = \|g(.)\| \in L_\infty(\mu)$ and $\| \, |g| \, \|_\infty = \|T\|$.

Let (e_n) be the unit basis in c_0 and let us write $g_n(\omega) = e_n \circ g(\omega) \in E^*$. Then, for every $f \in L_1(\mu, E)$ and $\omega \in \Omega$,

$$g(\omega)f(\omega) = \langle f(\omega), g_n(\omega)\rangle \in l_1,$$

and

$$\sum |\langle f(\omega) \, . \, g_n(\omega)\rangle| \leq |g| \, (\omega) \, \|f(\omega)\| \leq \|T\| \, \|f(\omega)\|.$$

In particular, for every $x \in F$ we have

$$\sum |\langle x, g_n(\omega)\rangle| \leq \|T\| \, \|x\|,$$

and hence, for every finite subset $\sigma \subset \mathbb{N}$,

$$\left\|\sum_\sigma g_n(\omega)\right\| = \sup\left\{\left|\left\langle x, \sum_\sigma g_n(\omega)\right\rangle\right| : x \in E, \, \|x\| \leq 1\right\} \leq \|T\|,$$

which proves that $\sum g_n(\omega)$ is a w.u.c. series in E^*, for every ω in Ω.

Now, if $T(K)$ is not relatively compact in l_1, there is an $\varepsilon > 0$, a subsequence (n_k) of $\mathbb{N}$, and a sequence $(f_k) \subset K$, such that for every k

$$\sum_{n_k}^{\infty} \left| \int \langle f_k(\omega), g_n(\omega)\rangle \, d\mu \right| \geq \varepsilon. \tag{*}$$

Let us choose $\delta > 0$ such that $A \in \Sigma$ and $\mu(A) < \delta$ implies $\int_A \|f\| \, d\mu < \varepsilon/2 \, \|T\|$, for every $f \in K$. Choose now a set $B \in \Sigma$ with $\mu(B) < \delta$ and such that for each $\omega \notin B$ there is a (V^*) set $V(\omega)$ satisfying $f(\omega) \in V(\omega)$ for all $f \in K$. Since $\sum g_n(\omega)$ is w.u.c. in E^*, Proposition

1.1(b) proves that

$$\lim_{k\to\infty} \sum_{n_k}^{\infty} |\langle f_k(\omega), g_n(\omega)\rangle| = 0, \quad \text{for } \omega \notin B.$$

Since the sequence

$$h_k(\omega) = \sum_{n_k}^{\infty} |\langle f_k(\omega), g_n(\omega)\rangle|$$

is uniformly integrable, Vitali's convergence theorem shows that $\lim \int_{\Omega\setminus B} h_k \, d\mu = 0$. Choose a sufficiently large k so that $\int_{\Omega\setminus B} h_k \, d\mu < \varepsilon/2$. Then

$$\begin{aligned} \sum_{n_k}^{\infty} \left| \int \langle f_k(\omega), g_n(\omega)\rangle \, d\mu \right| &\leq \int_B \sum_{n_k}^{\infty} |\langle f_k(\omega), g_n(\omega)\rangle| \, d\mu + \int_{\Omega\setminus B} \sum_{n_k}^{\infty} |\langle f_k(\omega), g_n(\omega)\rangle| \, d\mu \\ &\leq \|T\| \int_B \|f_k(\omega)\| \, d\mu + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which contradicts (*). This contradiction proves that K is a (V^*) set.

COROLLARY 3.3. *Every $\delta\mathcal{V}^*$ set in $L_1(\mu, E)$ is a (V^*)-set.*

Proof. Let K be a $\delta\mathcal{V}^*$-set. Then there exists a sequence (V_n) of (V^*) sets in E, and a null sequence (ε_n) of positive scalars such that $K \subset K_n + \varepsilon_n B(L_1(\mu, E))$ for every $n \in \mathbb{N}$, where

$$K_n = \{f \in L_1(\mu, E) : f(\omega) \in V_n \text{ for almost all } \omega \in \Omega\}.$$

It follows from Theorem 3.2 that each K_n is a (V^*) set in $L_1(\mu, E)$, and so by Corollary 1.7, K is a (V^*) set in $L_1(\mu, E)$, too.

The preceding corollary, together with Proposition 1.1(d) enables us in some cases to obtain complemented copies of l_1 in E, whenever $L_1(\mu, E)$ (or, more generally, $L_\Phi(\mu, E)$) contains a complemented copy of l_1. See [5], and also [4] for related results.

In general, not every (V^*) set is necessarily a $\delta\mathcal{V}^*$-set. In fact, Bourgain constructs in [6] a Banach space with a monotone unconditional basis, and a probability space (Ω, Σ, μ) such that there is a weakly null sequence (f_n) (in fact, equivalent to the unit l_2 basis) in $L_1(\mu, E)$ that is not a $\delta\mathcal{W}\mathcal{C}$-set. E has in a natural way an order continuous Banach lattice structure ([12, p. 2]), and so it has the weak (V^*) property by Theorem 2.2. Hence, (f_n) is not a $\delta\mathcal{V}^*$-set in $L_1(\mu, E)$. By a modification of Bourgain's example, Batt and Hiermeyer gave in [2] a weakly sequentially complete Banach lattice F with the same properties: there is a weakly null sequence in $L_1(\mu, F)$ that is not a $\delta\mathcal{W}$-set, and hence neither a $\delta\mathcal{V}^*$-set (because F has the property (V^*), by Corollary 2.3). On the other hand, we have the following general fact.

PROPOSITION 3.4. *If μ is purely atomic, for any class of bounded sets $\mathcal{H}$, closed under finite unions and continuous linear images, every uniformly integrable set in $\mathcal{H}(L_1(\mu, E))$ is a $\delta\mathcal{H}$-set.*

Proof. Let (α_n) be the countable set of atoms on which μ is concentrated, and write $A_k = \{\alpha_n : n > k\}$. Then, $\mu(A_k) \to 0$. Let H be a set in $\mathcal{H}(L_1(\mu, E))$ and $\delta > 0$. Choose

$k \in \mathbb{N}$ such that $\mu(A_k) < \delta$, and take $H_\delta = \bigcup_1^k \{f(\alpha_i) : f \in H\} \in \mathcal{H}(E)$ by hypothesis. Then $\Omega_f = \Omega \backslash \{\alpha_1, \ldots, \alpha_k\}$ for every $f \in H$ works in the definition of $\delta\mathcal{H}$-set.

PROPOSITION 3.5. *Suppose that $\delta\mathcal{V}^*$-sets and (V^*) sets coincide on $L_1(\mu, E)$. Then, if E has property (V^*) (respectively, weak (V^*)), the same happens to $L_1(\mu, E)$.*

Proof. Let $K \in \mathcal{V}^*(L_1(\mu, E))$, and hence a $\delta\mathcal{V}^*$-set by hypothesis. If E has property (V^*), then K is a $\delta\mathcal{W}$-set, and so it belongs to $\mathcal{W}(L_1(\mu, E))$ by Diestel's result [**7** Theorem 8]. The case of weak (V^*) property is proved in the same way, but using now Bourgain's result [**6**, Proposition 13].

The proposition above shows the interest in knowing when $\delta\mathcal{V}^*$-sets and (V^*) coincide on $L_1(\mu, E)$. Our next result gives some answers in that direction.

PROPOSITION 3.6. *$\delta\mathcal{V}^*$ – sets and (V^*) sets coincide in $L_1(\mu, E)$ in the following cases:*
(a) *μ is purely atomic,*
(b) *E does not contain copies of l_1,*
(c) *$E = L_1(\nu)$, where ν is a positive σ-finite measure.*

Proof. (a) follows from Proposition 3.4, and (b) and (c) follow from Theorem 14 in [**6**], where it is proved that under such hypothesis, $\mathcal{W}\mathcal{C}$ and $\delta\mathcal{W}\mathcal{C}$-sets coincide on $L_1(\mu, E)$. It suffices to note that in both cases E and $L_1(\mu, E)$ have the property weak (V^*), by Remark 1.4(b), Theorem 2.2 and Corollary 9 in [**6**].

The following theorem gives more conditions under which $L_1(\mu, E)$ inherits property (V^*) or weak (V^*) from E. Part (b) extends slightly a previous result of E. and P. Saab [**16**, Theorem 6].

THEOREM 3.7. (a) *If E is a closed subspace of an order continuous Banach lattice, then $L_1(\mu, E)$ has property weak (V^*).*

(b) *If E is a weakly sequentially complete closed subspace of an order continuous Banach lattice (for instance, if E is complemented in a Banach lattice and does not contain a copy of c_0), then $L_1(\mu, E)$ has property (V^*).*

Proof. If E is a closed subspace of an order continuous Banach lattice F, then $L_1(\mu, E)$ is a closed subspace of $L_1(\mu, F)$, which is order continuous under the induced order. Hence, (a) follows from Theorem 2.2. As for (b), it follows from (a) and the fact that $L_1(\mu, E)$ is weakly sequentially complete if E is ([**17**, Theorem 11)].

The proof of Theorem 3.2 cannot be extended directly for the spaces $L_p(\mu, E)$, $1 < p < \infty$, because in this case there is not such a good representation theorem for operators on those spaces. However, under some additional assumptions, we can get the desired extension, even in a more general context. In fact, let Φ be a Young's function, with conjugate Ψ, and consider the corresponding Orlicz space $L_\Phi(\mu, E)$ (see introduction). The following result is proved in [**3**].

LEMMA 3.8. ([**3**], Proposition 1 and Corollary 4). *With the notations above, suppose Φ and Ψ both satisfy the (Δ_2) condition and J stands for the canonical inclusion map from $L_\Phi(\mu, E)$ into $L_1(\mu, E)$. Then we have:*
(a) *$J^*(L_1(\mu, E)^*)$ is norm dense in $L_\Phi(\mu, E)^*$,*

(b) *a bounded subset $K \subset L_\Phi(\mu, E)$ is weakly relatively compact if and only if $J(K)$ is weakly relatively compact.*

With Lemma 3.8 at hand, we can now prove the extensions claimed.

THEOREM 3.9. *Let Φ be a Young's function such that both Φ and its conjugate satisfy the (Δ_2) condition, and suppose $L_1(\mu, E)$ has property weak (V^*).*

(a) *Every $\delta\mathcal{V}^*$-set in $L_\Phi(\mu, E)$ is a (V^*) set.*

(b) *$\delta\mathcal{V}^*$-sets and (V^*) sets coincide in $L_\Phi(\mu, E)$ if they coincide in $L_1(\mu, E)$.*

Proof. (a) follows from Corollary 3.3, Lemma 3.8(a) and Proposition 1.11(b). (b) is immediate.

THEOREM 3.10. *Let Φ be a Young's function such that both Φ and its conjugate satisfy the (Δ_2) condition. Then, if $L_1(\mu, E)$ has property (V^*) (resp., weak (V^*)), so does $L_\Phi(\mu, E)$.*

Proof. This follows immediately from Lemma 3.8 and Proposition 1.1.

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