THE DIMENSIONS OF IRREDUCIBLE REPRESENTATIONS OF LINEAR GROUPS

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1. Introduction. The theory of the relationship between the symmetric group on a symbols, Σ_a, and the general linear group in n-dimensions, GL(n), was greatly developed by Weyl [**4**] who, in this connection, made use of tensor representations of GL(n).

The set of mixed tensors

$$T^{(\beta)_b}_{(\alpha)_a} = T^{\beta_1\beta_2\ldots\beta_b}_{\alpha_1\alpha_2\ldots\alpha_a}$$

forms the basis of a representation of GL(n) if all the indices may take the values $1, 2, \ldots, n$, and if the linear transformation

$$T^{(\beta)_b}_{(\alpha)_a} \rightarrow T'^{(\beta)_b}_{(\alpha)_a} = \prod_{i=1}^{a} (A)_{\alpha_i}^{\alpha_i'} \prod_{j=1}^{b} (A^{-1})^{\beta_j}_{\beta_j'} T^{(\beta')_b}_{(\alpha')_a}$$

is associated with every non-singular $n \times n$ matrix A. The representation is irreducible if the tensors are traceless and if the sets of covariant indices $(\alpha)_a$ and contravariant indices $(\beta)_b$ themselves form the bases of irreducible representations (IRs) of Σ_a and Σ_b, respectively. These IRs of Σ_a and Σ_b may be specified by Young tableaux $[\mu]_a$ and $[\nu]_b$ in the usual way [**4**]. It has been shown in a previous paper [**2**] that it is convenient to specify the corresponding IR of GL(n) by a composite tableau $[\nu; \mu]^b_a$.

The same composite tableau may be used to specify IRs of not only GL(n), but also of U(n), U$(n - m, m)$, SL(n), SU(n), and SU$(n - m, m)$. The tensorial bases of the corresponding IRs of these groups are only distinguished by the properties of the transformation matrix A. These linear groups are denoted collectively by L_n, and SL_n is used to denote those L_n for which $\det A = 1$.

Jahn and El Samra [**1**] have derived a very simple and useful formula for the dimension, $D_n(\nu; \mu)$, of the IR of each L_n specified by $[\nu; \mu]^b_a$. This formula fully exploits the composite tableau notation and takes the form

$$D_n(\nu; \mu) = \frac{N_n(\nu; \mu)}{H(\nu)H(\mu)}, \tag{1}$$

where $H(\nu)$ and $H(\mu)$ are the conventional hook length factors [**3**] associated with the tableaux $[\mu]_a$ and $[\nu]_b$, and $N_n(\nu; \mu)$ is a polynomial in n containing just $(a + b)$ factors. Two alternative schemes A and B were given for writing

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down $N_n(\nu;\mu)$. Scheme A was proved directly and the trivial derivation of scheme B from scheme A was also given. In this paper an alternative derivation of (1) is given which involves a direct proof of scheme B.

2. Derivation of $D_n(\nu;\mu)$. Every inequivalent finite-dimensional IR of L_n may be specified by a regular composite tableau of the form:

(2) $$[\nu;\mu]_a^b = [\nu_1\nu_2\ldots\nu_r;\mu_1\mu_2\ldots\mu_p]_a^b = (\nu_s'\ldots\nu_2'\nu_1';\mu_1'\mu_2'\ldots\mu_q')_a^b,$$

where

$$\sum_{i=1}^{p}\mu_i = \sum_{j=1}^{q}\mu_j' = a, \qquad \sum_{k=1}^{r}\nu_k = \sum_{l=1}^{s}\nu_l' = b,$$

and

$$\mu_1 \geqq \mu_2 \geqq \ldots \geqq \mu_p > 0, \qquad \mu_1' \geqq \mu_2' \geqq \ldots \geqq \mu_q' > 0,$$
$$\nu_1 \geqq \nu_2 \geqq \ldots \geqq \nu_r > 0, \qquad \nu_1' \geqq \nu_2' \geqq \ldots \geqq \nu_s' > 0,$$

with

$$p = \mu_1', \quad q = \mu_1, \quad r = \nu_1', \quad s = \nu_1,$$

subject only to the restriction

(3) $$p + r \leqq n.$$

The IR of the special linear group, SL_n, specified by $[\nu;\mu]_a^b$ is equivalent to the IR of SL_n specified by the conventional regular Young tableau $[\lambda]_c$, where

(4) $$[\lambda]_c = [\lambda_1\lambda_2\ldots\lambda_t]_c = (\lambda_1'\lambda_2'\ldots\lambda_u')_c$$

with

$$\lambda_g = \begin{cases} s + \mu_g & \text{if } g = 1, 2, \ldots, p, \\ s & \text{if } g = p+1, p+2, \ldots, n-r, \\ s - \nu_{n-g+1} & \text{if } g = n-r+1, n-r+2, \ldots, n-\nu_s', \end{cases}$$

and

$$\lambda_h' = \begin{cases} n - \nu'_{s-h+1} & \text{if } h = 1, 2, \ldots, s, \\ \mu'_{h-s} & \text{if } h = s+1, s+2, \ldots, s+q, \end{cases}$$

so that

$$t = n - \nu_s', \quad u = s + q, \quad c = ns - b + a.$$

Schematically, it is convenient to represent typical tableaux $[\nu;\mu]_a^b$ and $[\lambda]_c$ by diagrams (a) and (b) of Figure 1. The IRs of L_n specified by these two tableaux are of the same dimension, so that

(5) $$D_n(\nu;\mu) = D_n(\lambda).$$

Quite generally, for any tableau $[\lambda]_c$ the dimension of the corresponding IR of L_n is given by (see [**3**])

(6) $$D_n(\lambda) = G_n(\lambda)/H(\lambda),$$

where $G_n(\lambda)$ and $H(\lambda)$ are the products of the contents and the hook lengths of the boxes of $[\lambda]_c$. The content of the box in the gth row and the hth column of $[\lambda]_c$ is defined to be $(n - g + h)$, and the hook length of this box is defined to be $(1 + \lambda_g - h + \lambda_h' - g)$, so that

$$G_n(\lambda) = \prod_{g,h}^{t,u} (n - g + h) \tag{7}$$

and

$$H(\lambda) = \prod_{g,h}^{t,u} (1 + \lambda_g - h + \lambda_h' - g). \tag{8}$$

The content $G_n(\lambda)$ may be formed by taking the product of the c numbers in the array produced by inserting in each box of the tableau its corresponding content. It is clear from this array that

$$G_n(\lambda) = G_n(\rho)G_{n+s}(\mu)G_{n-p}(\sigma), \tag{9}$$

where the tableaux $[\rho]_d$ and $[\sigma]_e$ are defined by:

$$[\rho]_d = [\rho_1\rho_2 \ldots \rho_p]_d = (\rho_1'\rho_2' \ldots \rho_s')_d \tag{10}$$

with

$$\rho_i = s, \quad i = 1, 2, \ldots, p, \qquad \rho_l' = p, \quad l = 1, 2, \ldots, s,$$

so that $d = ps$, and

$$[\sigma]_e = [\sigma_1\sigma_2 \ldots \sigma_{t-p}]_e = (\sigma_1'\sigma_2' \ldots \sigma_s')_e \tag{11}$$

with

$$\sigma_m = \begin{cases} s & \text{if } m = 1, 2, \ldots, n - p - r, \\ s - \nu_{n-p-m+1} & \text{if } m = n - p - r + 1, n - p - r + 2, \ldots, n - p - \nu_s', \end{cases}$$

$$\sigma_l' = n - p - \sigma'_{s-l+1}, \qquad l = 1, 2, \ldots, s,$$

so that $e = ns - ps - b$. Schematically, the tableaux $[\mu]_a$, $[\rho]_d$, and $[\sigma]_e$ together form the tableau $[\lambda]_c$ as shown in diagram (c) of Figure 1. The factor $G_n(\rho)$ is then given explicitly by

$$G_n(\rho) = \prod_{i,l}^{p,s} (n - i + l). \tag{12}$$

It is convenient to reverse the order of the elements in each row of the rectangular array corresponding to (12) to yield

$$G_n(\rho) = \begin{cases} (n-1+s) & \ldots & (n-1+l) & \ldots & (n+1) & (n) \\ (n-2+s) & \ldots & (n-2+l) & \ldots & (n) & (n-1) \\ \vdots & & \vdots & & \vdots & \vdots \\ (n-i+s) & \ldots & (n-i+l) & \ldots & (n-i+2) & (n-i+1) \\ \vdots & & \vdots & & \vdots & \vdots \\ (n-p+s) & \ldots & (n-p+l) & \ldots & (n-p+2) & (n-p+1) \end{cases} \tag{13}$$

Similarly, the hook length factor $H(\lambda)$ may be formed by taking the product of the c numbers in the array produced by inserting in each box of the tableau its corresponding hook length. From this array,

$$H(\lambda) = F_n(\rho)H(\mu)H(\sigma), \tag{14}$$

where $H(\mu)$ and $H(\sigma)$ are the hook length factors of the tableaux $[\mu]_a$ and $[\sigma]_e$, respectively, and $F_n(\rho)$ is given in terms of $[\mu]_a$ and $[\nu]^b$ by:

$$F_n(\rho) = \prod_{g,h}^{p,s} (1 + s + \mu_g - g + n - \nu'_{s-h+1} - h).$$

Hence

$$F_n(\rho) = \prod_{i,l}^{p,s} (n - i + l + \mu_i - \nu_l'). \tag{15}$$

Once again, reversing the order of the elements in each row of the rectangular array corresponding to (15) yields

$$F_n(\rho) = \begin{cases} (n-1+s+\mu_1-\nu_s') & \dots & (n-1+l+\mu_1-\nu_l') & \dots & (n+1+\mu_1-\nu_2') & (n+\mu_1-\nu_1') \\ (n-2+s+\mu_2-\nu_s') & \dots & (n-2+l+\mu_2-\nu_l') & \dots & (n+\mu_2-\nu_2') & (n-1+\mu_2-\nu_1') \\ \vdots & & \vdots & & \vdots & \vdots \\ (n-i+s+\mu_i-\nu_s') & \dots & (n-i+l+\mu_i-\nu_l') & \dots & (n-i+2+\mu_i-\nu_2') & (n-i+1+\mu_i-\nu_1') \\ \vdots & & \vdots & & \vdots & \vdots \\ (n-p+s+\mu_p-\nu_s') & \dots & (n-p+1+\mu_p-\nu_l') & \dots & (n-p+2+\mu_p-\nu_2') & (n-p+1+\mu_p-\nu_1') \end{cases} \tag{16}$$

Substituting (9) and (14) into (6) and making use of the generality of (6) then yields

$$D_n(\lambda) = G_n(\rho)G_{n+s}(\mu)D_{n-p}(\sigma)/F_n(\rho)H(\mu). \tag{17}$$

The IR of L_{n-p} specified by $[\sigma]_e$ is the complement of the adjoint of the IR of L_{n-p} specified by $[\nu]^b$, so that these two IRs have the same dimension. Moreover, an IR and its adjoint have the same dimension. Therefore

$$D_{n-p}(\sigma) = D_{n-p}(\nu),$$

and again making use of the generality of (6) yields

$$D_n(\lambda) = G_n(\rho)G_{n+s}(\mu)G_{n-p}(\nu)/F_n(\rho)H(\mu)H(\nu). \tag{18}$$

The form of (18) indicates that it is convenient to introduce the tableaux $[\theta]_{a+d}$ and $[\phi]^{b+d}$ defined by:

$$[\theta]_{a+d} = [\theta_1\theta_2 \dots \theta_p]_{a+d} = (\theta_1'\theta_2' \dots)_{a+d} \tag{19}$$

with

$$\theta_i = s + \mu_i, \qquad i = 1, 2, \dots, p,$$

and

$$[\phi]^{b+d} = [\phi_1\phi_2 \dots]^{b+d} = (\phi_s' \dots \phi_2'\phi_1')^{b+d} \tag{20}$$

with

$$\phi_l' = p + \nu_l', \qquad l = 1, 2, \dots, s.$$

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Schematically, $[\theta]_{a+d}$ and $[\phi]^{b+d}$ are composed of $[\rho]_d$ and $[\rho]^d$ together with $[\mu]_a$ and $[\nu]^b$, respectively, as shown in diagram (d) of Figure 1. With these definitions,

$$G_n(\theta) = G_n(\rho)G_{n+s}(\mu) \quad \text{and} \quad G_n(\phi) = G_n(\rho)G_{n-p}(\nu)$$

so that using (5) and (18) we have

$$D_n(\nu;\mu) = G_n(\theta)G_n(\phi)/H(\mu)H(\nu)G_n(\rho)F_n(\rho). \tag{21}$$

In writing $G_n(\theta)$ and $G_n(\phi)$ as arrays of numbers it is convenient to order the elements so that

$$G_n(\theta) = \left\{ \begin{array}{llllll} (n+\mu_1+s-1) & (n+\mu_1+s-2) & \dots\ \dots & \dots & (n+1) & (n) \\ (n+\mu_2+s-2) & (n+\mu_2+s-3) & \dots\ \dots & \dots & (n) & (n-1) \\ \vdots & \vdots & & & & \\ (n+\mu_i+s-i) & (n+\mu_i+s-i-1) & \dots & \dots & (n-i+2) & (n-i+1) \\ \vdots & \vdots & & & & \\ (n+\mu_p+s-p) & (n+\mu_p+s-p-1) & \dots & (n-p+2) & (n-p+1) & \end{array} \right. \tag{22}$$

and

$$G_n(\phi) = \left\{ \begin{array}{lllllll} (n-\nu_s'+s-p) & \dots & (n-\nu_l'+l-p) & \dots & (n-\nu_2'+2-p) & (n-\nu_1'+1-p) \\ (n-\nu_s'+s-p+1) & \dots & (n-\nu_l'+l-p+1) & \dots & (n-\nu_2'+3-p) & (n-\nu_1'+2-p) \\ \vdots & & \vdots & & \vdots & \vdots \\ & & \vdots & & \vdots & \vdots \\ (n+s-2) & & & & & \\ (n+s-1) & & & & & \\ & & (n+l-2) & & & \vdots \\ & & (n+l-1) & & (n) & \\ & & & & (n+1) & \\ & & & & & (n-1) \\ & & & & & (n) \end{array} \right. \tag{23}$$

It is to be noted that the rows of the arrays $G_n(\theta)$, $F_n(\rho)$, and $G_n(\rho)$, and the columns of the arrays $G_n(\phi)$, $F_n(\rho)$, and $G_n(\rho)$ are labelled by indices i and l, respectively, ranging over the values $1, 2, \ldots, p$ and $1, 2, \ldots, s$. Moreover, they are counted in the same way as the rows and columns of $[\mu]_a$ and $[\nu]^b$, respectively, that is from top to bottom and from right to left.

The notation used in (2) to define the composite tableau $[\nu;\mu]_a^b$ may be extended slightly so that $\mu_j' = 0$ for $j > q$ and $\nu_k = 0$ for $k > r$. Then if

$$l > \nu_{\mu_i} \geqq 0, \tag{24}$$

it follows that the lth column of $[\nu]^b$ does not intersect the μ_ith row of $[\nu]^b$. Hence

$$\mu_i > \nu_l', \tag{25}$$

and therefore the ith row of $[\mu]_a$ does intersect the ν_l'th column of $[\mu]_a$ so that

$$i \leqq \mu_{\nu_l'}'. \tag{26}$$

Similarly, if

(27) $$i > \mu_{\nu_l'}' \geqq 0,$$

it follows that

(28) $$\mu_i < \nu_l'$$

and

(29) $$l \leqq \nu_{\mu_i}.$$

Furthermore, if

(30) $$l \leqq \nu_{\mu_i} \quad \text{and} \quad i \leqq \mu_{\nu_l'}',$$

it follows that

$$\mu_i \leqq \nu_l' \quad \text{and} \quad \nu_l' \leqq \mu_i,$$

and therefore

(31) $$\mu_i = \nu_l'.$$

It is then possible to decompose the rectangle representing $[\rho]_d$ into three distinct regions α, β, and γ as shown in diagram (e) of Figure 1, where the regions α and β are defined by the tableaux

(32) $$[\alpha] = [\alpha_1\alpha_2 \ldots \alpha_p] = (\alpha_1'\alpha_2' \ldots)$$

with

$$\alpha_i = s - \nu_{\mu_i}, \qquad i = 1, 2, \ldots, p,$$

and

(33) $$[\beta] = [\beta_1\beta_2 \ldots] = (\beta_s' \ldots \beta_2'\beta_1')$$

with

$$\beta_l' = p - \mu_{\nu_l'}', \qquad l = 1, 2, \ldots, s.$$

The important result which follows from (25) and (28) is that the regions α and β are non-intersecting, and from (31) all the elements of γ are such that $\mu_i = \nu_l'$. The arrays (13) and (16) may therefore be decomposed similarly to yield

$$G_n(\rho) = G_n^\alpha(\rho)G_n^\beta(\rho)G_n^\gamma(\rho) \quad \text{and} \quad F_n(\rho) = F_n^\alpha(\rho)F_n^\beta(\rho)F_n^\gamma(\rho),$$

where the various factors are formed by taking the product of the appropriate elements corresponding to the regions of the array signified by the superscripts. By virtue of (31), $G_n^\gamma(\rho) = F_n^\gamma(\rho)$ so that

(34) $$G_n(\rho)F_n(\rho) = A_n(\rho)B_n(\rho),$$

where

$$A_n(\rho) = F_n^\alpha(\rho)G_n^{\beta+\gamma}(\rho) \quad \text{and} \quad B_n(\rho) = F_n^\beta(\rho)G_n^{\alpha+\gamma}(\rho).$$

It is straightforward, using (13) and (16), to write down arrays corresponding to $A_n(\rho)$ and $B_n(\rho)$. It is convenient to order the terms of $A_n(\rho)$ in the same way as in (13) and (16), but to order the terms of $B_n(\rho)$ in a way corresponding to a reversal of the order of the terms in the columns of (13) and (16).

From the expression, (16), and the definition (32), it follows that the product of the terms in the ith row of $F_n^\alpha(\rho)$ may be evaluated by taking the product of the array of numbers formed by inserting in the box at the foot of each of the last $(s - \nu_{\mu_i})$ columns of $[\nu]^b$, counted from the right, the factor $(n - i + \mu_i +$ the number of the column $-$ the length of that column). In the kth row of $[\nu]^b$, these numbers are thus distributed over the last $(\nu_k - \nu_{k+1})$ boxes which contain $\times$ in diagram (a) of Figure 2. In terms of the label k, it follows that $F_n^\alpha(\rho)$ may be written in the form:

$$F_n^\alpha(\rho) = \prod_{i=1}^{p} \prod_{k=1}^{\mu_i - 1} \frac{(n - i - k + \mu_i + \nu_k)!}{(n - i - k + \mu_i + \nu_{k+1})!}. \tag{35}$$

Similarly, the product of the terms in the lth column of $F_n^\beta(\rho)$ may be evaluated by taking the product of the array of numbers formed by inserting in the box at the right-hand end of each of the last $(p - \mu'_{\nu_l'})$ rows of $[\mu]_a$, counted from the top, the factor $(n + l - \nu_l' -$ the number of the row $+$ the length of that row). In the jth column of $[\mu]_a$ these numbers are distributed over the last $(\mu_j' - \mu'_{j+1})$ boxes which contain $\times$ in diagram (b) of Figure 2. Hence, in terms of the label j, it follows that:

$$F_n^\beta(\rho) = \prod_{l=1}^{s} \prod_{j=1}^{\nu_l' - 1} \frac{(n + l + j - \nu_l' - \mu_j')!}{(n + l + j - \nu_l' - \mu'_{j+1})!}. \tag{36}$$

In the same way from (13) together with (32) and (33) it is clear that

$$G_n^{\beta+\gamma}(\rho) = \prod_{i=1}^{p} \frac{(n - i + \nu_{\mu_i})!}{(n - i)!} \tag{37}$$

and

$$G_n^{\alpha+\gamma}(\rho) = \prod_{l=1}^{s} \frac{(n + l - 1)!}{(n + l - \mu_{\nu_l'}')!}. \tag{38}$$

From the arrays (22) and (23) and the expressions (37) and (38) it can be seen that

$$G_n(\theta) = E_n(\theta) G_n^{\beta+\gamma}(\rho) \quad \text{and} \quad G_n(\phi) = E_n(\phi) G_n^{\alpha+\gamma}(\rho)$$

with

$$E_n(\theta) = \prod_{i=1}^{p} \frac{(n - i + \mu_i + \nu_1)!}{(n - i + \nu_{\mu_i})!} \tag{39}$$

and

$$E_n(\phi) = \prod_{l=1}^{s} \frac{(n + l - \mu_{\nu_l'}')!}{(r + l - 1 - \nu_l' - \mu_1')!}. \tag{40}$$

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The arrays $E_n(\theta)$ and $E_n(\phi)$ are obtained from $G_n(\theta)$ and $G_n(\phi)$ by retaining the first $(s + \mu_i - \nu_{\mu_i})$ terms in the ith row of (22) and the first $(p + \nu_l' - \mu_{\nu_l'}')$ terms in the lth column of (23), respectively.

The product of the terms in the ith row of $E_n(\theta)$ may be evaluated in exactly the same way as the product of the terms in the ith row of $F_n^\alpha(\rho)$ by taking the product of an array of numbers associated with diagram (a) of Figure 2. The only difference is that the ith row of $E_n(\theta)$ includes an additional μ_i terms arising from numbers placed in the boxes which contain ○ in the diagram. These boxes are not of course part of the tableau $[\nu]^b$. Hence

$$(41) \qquad E_n(\theta) = F_n^\alpha(\rho)\prod_{i=1}^{p}\prod_{k=1}^{\mu_i}(n - i - k + \mu_i + \nu_k + 1).$$

Similarly, the product of the terms in the lth column of $E_n(\phi)$ may be evaluated in exactly the same way as the product of the terms in the lth column of $F_n^\beta(\rho)$ using diagram (b) of Figure 2. The only difference is that the lth column of $E_n(\phi)$ includes an additional ν_l' terms arising from numbers placed in the boxes which contain ○ in the diagram. These boxes are not part of the tableau $[\mu]_a$. Hence

$$(42) \qquad E_n(\phi) = F_n^\beta(\rho)\prod_{l=1}^{s}\prod_{j=1}^{\nu_l'}(n + l + j - \nu_l' - \mu_j' - 1).$$

Thus

$$G_n(\theta) = A_n(\rho)P_n^\mu(\nu;\mu) \quad \text{and} \quad G_n(\phi) = B_n(\rho)S_n^\nu(\nu;\mu)$$

with

$$(43) \qquad P_n^\mu(\nu;\mu) = \prod_{i=1}^{p}\prod_{k=1}^{\mu_i}(n + 1 - i - k + \mu_i + \nu_k)$$

and

$$(44) \qquad S_n^\nu(\nu;\mu) = \prod_{l=1}^{s}\prod_{j=1}^{\nu_l'}(n - 1 + l + j - \nu_l' - \mu_j').$$

These results together with (21) and (34) yield

$$(45) \qquad D_n(\nu;\mu) = \frac{P_n^\mu(\nu;\mu)S_n^\nu(\nu;\mu)}{H(\mu)H(\nu)},$$

that is

$$(46) \quad D_n(\nu;\mu) = \prod_{i,j}^{p,q}\prod_{l,k}^{r,s}\frac{(n + 1 - i - j + \mu_i + \nu_j)(n - 1 + k + l - \mu_k' - \nu_l')}{(1 - i - j + \mu_i + \mu_j')(1 - k - l + \nu_k + \nu_l')}.$$

Comparison with (1) indicates that

$$(47) \qquad N_n(\nu;\mu) = S_n^\nu(\nu;\mu)P_n^\mu(\nu;\mu),$$

and substituting into this expression (47), the product of the arrays corresponding to (43) and (44) yields exactly that form of $N_n(\nu;\mu)$ defined by the scheme B of Jahn and El Samra [**1**].

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It should be noted that if $[\nu]^b = [0]^0$, the factor $S_n(\nu;\mu)/H(\nu)$ must be replaced by 1, and since in this case

$$\nu_k = 0 \quad \text{for } k = 1, 2, \ldots, \mu_i \quad \text{with } i = 1, 2, \ldots, p,$$

a simple reordering of the terms in the rows of the array corresponding to (43) yields the identity

$$P_n^{\mu}(0;\mu) = G_n(\mu),$$

so that

(48) $$D_n(0;\mu) = D_n(\mu).$$

Similarly

$$S_n^{\nu}(\nu;0) = G_n(\nu),$$

so that

(49) $$D_n(\nu;0) = D_n(\nu).$$

This indicates that the formula (6) may be considered to be a special case of (45). However, in contrast to the fact that $G_n(\lambda)$ determines the corresponding tableaux $[\lambda]_c$ uniquely [**3**], $N_n(\nu;\mu)$ does not determine the corresponding tableau $[\nu;\mu]_a^b$ uniquely. For example,

$$N_n(\nu;\mu) = (n-5)(n-4)(n-2)(n-1)^2n^2(n+1)(n+2)(n+3)(n+4)$$

if $[\nu;\mu]_a^b$ is given by any one of the four distinct tableaux $[2^3; 31^2]_5^6$, $[31^2; 2^3]_6^5$, $[2^2; 321^2]_7^4$ or $[321^2; 2^2]_4^7$.

3. Example. As an example, it is instructive to calculate the dimension of the IR of L_n specified by the composite tableau:

$$[\nu;\mu]_a^b = [431; 2^21]_5^8 = (12^23; 32)_5^8.$$

The other tableaux specified in this paper in terms of $[\nu;\mu]_a^b$ are then given by:

$$\begin{aligned}
[\lambda]_c &= [6^254^{n-6}31]_{4n-3} = ((n-1)(n-2)^2(n-3)32)_{4n-3},\\
[\rho]_d &= [4^3]_{12} = (3^4)_{12},\\
[\sigma]_e &= [4^{n-6}31]_{4n-20} = ((n-4)(n-5)^2(n-6))_{4n-20},\\
[\theta]_{a+d} &= [6^25]_{17} = (3^52)_{17},\\
[\phi]^{b+d} &= [4^431]^{20} = (45^26)^{20},\\
[\alpha] &= [1^2]_2 = (2)_2,\\
[\beta] &= [31^2]^5 = (1^23)^5.
\end{aligned}$$

Thus

$$G_n(\rho) = \begin{array}{|c|cc|c|}
\hline
(n+3) & (n+2) & (n+1) & (n) \\
(n+2) & (n+1) & (n) & (n-1) \\
\hline
(n+1) & (n) & (n-1) & (n-2) \\
\hline
\end{array},$$

$$F_n(\rho) = \begin{array}{|c|cc|c|}
\hline
(n+4) & (n+2) & (n+1) & (n-1) \\
(n+3) & (n+1) & (n) & (n-2) \\
\hline
(n+1) & (n-1) & (n-2) & (n-4) \\
\hline
\end{array},$$

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where the boundaries of the regions α and β are indicated. It follows that

$$\frac{G_n(\theta)}{A_n(\rho)} = \frac{\begin{array}{llllll} (n+5) & (n+4) & (n+3) & (n+2) & (n+1) & (n) \\ (n+4) & (n+3) & (n+2) & (n+1) & (n) & (n-1) \\ (n+2) & (n+1) & (n) & (n-1) & (n-2) & \end{array}}{\begin{array}{llll} (n+4) & (n+2) & (n+1) & (n) \\ (n+3) & (n+1) & (n) & (n-1) \\ (n+1) & (n) & (n-1) & (n-2) \end{array}}$$

and cancelling the terms in the corresponding rows then yields

$$P_n^{\mu}(\nu;\mu) = \left\{\begin{array}{ll} (n+5) & (n+3) \\ (n+4) & (n+2) \\ (n+2) & \end{array}\right.$$

Similarly,

$$\frac{G_n(\phi)}{B_n(\rho)} = \frac{\begin{array}{llll} (n) & (n-2) & (n-3) & (n-5) \\ (n+1) & (n-1) & (n-2) & (n-4) \\ (n+2) & (n) & (n-1) & (n-3) \\ (n+3) & (n+1) & (n) & (n-2) \\ & (n+2) & (n+1) & (n-1) \\ & & & (n) \end{array}}{\begin{array}{llll} (n+1) & (n-1) & (n-2) & (n-4) \\ (n+2) & (n+1) & (n) & (n-2) \\ (n+3) & (n+2) & (n+1) & (n-1) \end{array}}.$$

Cancelling the terms in the corresponding columns then yields

$$S_n^{\nu}(\nu;\mu) = \left\{\begin{array}{llll} (n) & (n-2) & (n-3) & (n-5) \\ & (n) & (n-1) & (n-3) \\ & & & (n) \end{array}\right. .$$

Finally, combining these results to form the numerator of (45) and inserting the hook length factors in the denominator of (45) then yields [**1**; **4**]

$$D_n(\nu;\mu) = \frac{\begin{array}{llllll} (n) & (n-2) & (n-3) & (n-5) & (n+5) & (n+3) \\ & (n) & (n-1) & (n-3) & (n+4) & (n+2) \\ & & & (n) & (n+2) & \end{array}}{\begin{array}{cccccc} 1 & 3 & 4 & 6 & 4 & 2 \\ & 1 & 2 & 4 & 3 & 1 \\ & & & 1 & 1 & \end{array}}.$$

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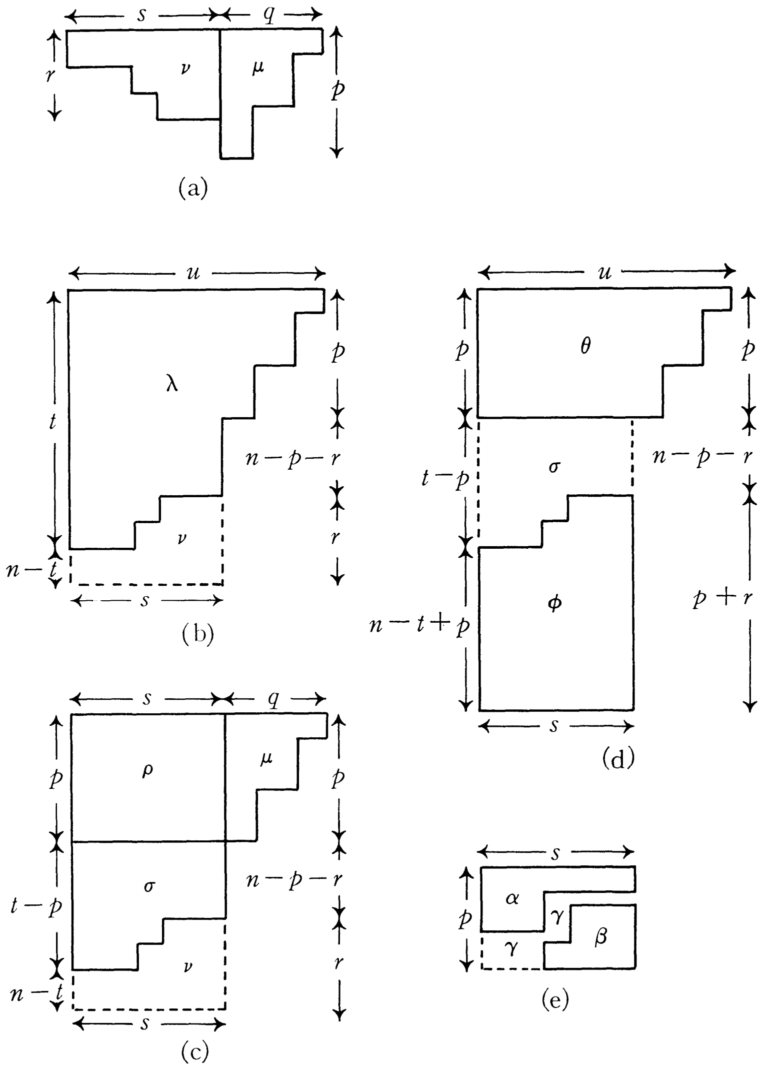

Schematic representations of typical tableaux (a) $[\nu;\ \mu]_a^b$, (b) $[\lambda]_c$, (c) $[\mu]_a$, $[\rho]_d$, and $[\sigma]_e$, (d) $[\theta]_{a+d}$ and $[\phi]^{b+d}$, (e) $[\alpha]$, $[\beta]$. The outline of each tableau is given. A complete tableau is obtained by dividing the interior region into rows and columns of undotted or dotted boxes. For example, $[\nu;\ \mu]_a^b$ consists of p rows and q columns of undotted boxes, and r rows and s columns of dotted boxes. These are arranged so that the ith row, counted from the top of the region μ, and the jth column, counted from the left of the region μ, contain μ_i and μ_j' undotted boxes, respectively, whilst the kth row, counted from the top of the region ν, and the lth column, counted from the right of the region ν, contain ν_k and ν_l' dotted boxes, respectively.

FIGURE 1

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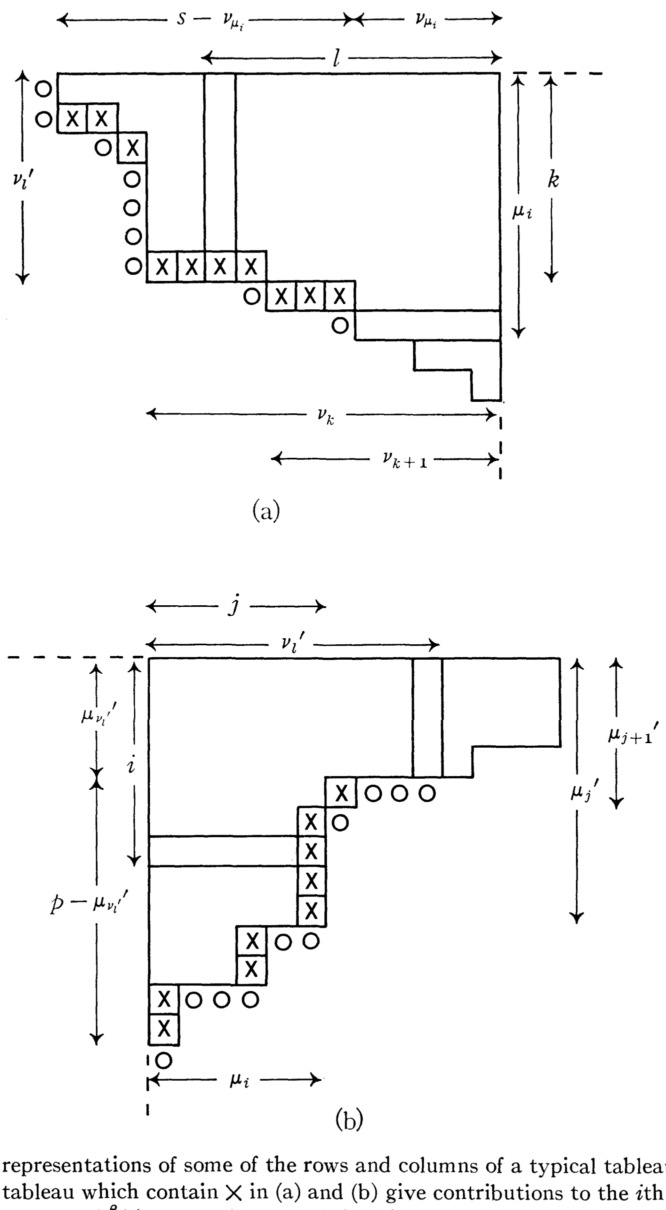

Schematic representations of some of the rows and columns of a typical tableau $[\nu; \mu]_a^b$. The boxes of this tableau which contain X in (a) and (b) give contributions to the ith row of $F_n^\alpha(\rho)$ and the lth column of $F_n^\beta(\rho)$, respectively, and the additional boxes outside the tableau which contain O in (a) and (b) give contributions to the ith row of $P_n^\mu(\nu; \mu)$ and the lth column of $S_n^\nu(\nu; \mu)$, respectively.

FIGURE 2

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