ON STRATIFICATIONS OF MITTAG-LEFFLER'S TRANSCENDENTS

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1. Guided by an observation of Hausdorff ([**4**]; reproduced by Whittaker and Robinson [**6**, pp. 177–178]), I pointed out a long time ago [**7**] that his " Fourier " treatment of certain products can be systematized so as to apply to an inclusive class of infinite convolutions. Recently I noticed [**8**] that an appropriate application of this method supplies the following curious result on gamma-quotients :

Corresponding to every index θ on the range $0 < \theta < 1$, there exists on the line $-\infty < t < \infty$ a *monotone* function $\mu = \mu_\theta = \mu_\theta(t)$ in terms of which the identity

$$\Gamma(z+1)/\Gamma(\theta z+1) = \int_{-\infty}^{\infty} e^{-zt}\, d\mu_\theta(t), \quad \text{where } d\mu_\theta(t) \geqq 0, \qquad (1)$$

holds on the half-plane $\operatorname{Re} z > -1$ and so, in particular, on the half-line $z \geqq 0$.

2. Choose $z = \beta x$, where $\beta > 0$ and $x \geqq 0$, put $a = \theta\beta$, and replace t by $-\log s$ in (1). Then, if $-\mu_\theta(-\log s)$ is denoted by $\lambda_\theta(s)$, where $0 < s < \infty$, it is seen that

$$\Gamma(\beta x+1)/\Gamma(\alpha x+1) = \int_0^{\infty} s^x\, d\lambda(s), \quad \text{where } d\lambda(s) \geqq 0,$$

holds for $0 \leqq x < \infty$, and for a certain function $\lambda(s) = \lambda(s\,;\ \alpha,\beta)$, whenever

$$0 < \alpha < \beta. \qquad (2)$$

Let x, s and λ be replaced by n, t and $\mu_{\alpha\beta}$ respectively. It then follows that, corresponding to any pair of indices α, β subject to (2), there exists on the half-line $0 \leqq t < \infty$ a *monotone* function $\mu = \mu_{\alpha\beta} = \mu_{\alpha\beta}(t)$ satisfying

$$\int_0^{\infty} t^n\, d\mu_{\alpha\beta}(t) = \Gamma(\beta n+1)/\Gamma(\alpha n+1) \quad (n = 0, 1, 2, \ldots;\ d\mu_{\alpha\beta}(t) \geqq 0). \qquad (3)$$

The relation (3) means that, subject to the proviso (2), the sequence $c_0, c_1, \ldots$, where $c_n = c_n(\alpha, \beta)$ is the quotient on the right of (3), is a moment sequence in the sense of Stieltjes (cf., e.g., [**2**, pp. 63–86]). It would be desirable to obtain a more direct (and, first of all, a more explicit) proof of this result, particularly because (3) contains a refinement of a related result of Hardy ([**3**], where references are given to the *Thèse* of Le Roy). In the latter regard, the situation will be clear from the following application of (3) to Mittag-Leffler's transcendent $E_\alpha(z)$.

3. If $\alpha > 0$ (or, for that matter, $\operatorname{Re} \alpha > 0$), the entire function $E_\alpha(z)$ is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} z^k/\Gamma(\alpha k+1) \qquad (4)$$

(cf. [**1**, p. 206]). It will be shown that, under the assumption (2), the function $E_\alpha(z)$ is a " stratification ", with a *non-negative* " weight-factor " $d\mu(t) = d\mu_{\alpha\beta}(t)$, of the function $E_\beta(zt)$,

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where $0 \leqq t < \infty$:

$$E_\alpha(z) = \int_0^\infty E_\beta(zt)\, d\mu_{\alpha\beta}(t) \quad (d\mu_{\alpha\beta}(t) \geqq 0). \qquad (5)$$

It is part of this statement, and it will be part of the proof, that the integral (5) is (absolutely) convergent at every point of the z-plane. But this is obvious from well-known properties of the function (4). It is also obvious that, in order to prove (5) for all z, it is sufficient to prove (5) for $z \geqq 0$, since (5) then follows for all z for reasons of analytic prolongation. In fact, the coefficients of the series defining E_β are positive.

For the latter reason, if α and z in (4) are replaced by β and zt respectively, where $z \geqq 0$ is fixed and $t \geqq 0$ varies, the legitimacy of a term-by-term integration in (5) is assured (since $d\mu_{\alpha\beta}(t)$, too, is claimed to be non-negative). This means that (5) is equivalent to

$$E_\alpha(z) = \sum_{k=0}^{\infty} C_k(\alpha, \beta) z^k,$$

where

$$C_k(\alpha, \beta) = \int_0^\infty t^k\, d\mu_{\alpha\beta}(t) / \Gamma(\beta k + 1).$$

But if this is compared with (4), it follows that (5) is equivalent to (3).

4. Since $E_1(z) = e^z$, by (4), and since the proviso (2) of (5) reduces to $0 < \alpha < 1$ or to $\beta > 1$ according as $\beta = 1$ or $\alpha = 1$, it follows from (5) that, for every complex z,

$$E_\alpha(z) = \int_0^\infty e^{zt}\, d\phi_\alpha(t), \quad \text{where} \quad d\phi_\alpha(t) \geqq 0, \text{ if } 0 < \alpha < 1, \qquad (6)$$

and

$$e^z = \int_0^\infty E_\alpha(zt)\, d\psi_\alpha(t), \quad \text{where } d\psi_\alpha(t) \geqq 0, \text{ if } \alpha > 1. \qquad (7)$$

The relation (6), being valid not only on the half-line $-\infty < z \leqq 0$, seems to improve on a result of Feller (cf. [**1**, p. 207]), but actually is equivalent to it (by virtue of a convergence theorem of Landau; cf., e.g., [**5**, pp. 88–89]). I do not know whether also the counterpart (7) of (6) occurs in the literature.

Integral relations formally similar to, but actually quite different from, (6) and (7) follow by choosing, on the one hand, either $\beta = \frac{1}{2}$ and $0 < \alpha < \frac{1}{2}$ or $\alpha = \frac{1}{2}$ and $\beta > \frac{1}{2}$, and, on the other hand, either $\beta = 2$ and $0 < \alpha < 2$ or $\alpha = 2$ and $\beta > 2$. The four formulae in question result if the relation $E_1(z) = e^z$, used in (6) and (7), is replaced by the well-known relations

$$E_{\frac{1}{2}}(z) = 2\pi^{-\frac{1}{2}} \exp(-z^2) \operatorname{Erfc}(-z), \quad E_2(z) = \cosh(z^{\frac{1}{2}}),$$

(cf. [**1**, p. 206]), which readily follow from (4).

REFERENCES

1. Bateman Manuscript Project, *Higher transcendental functions* (New York-Toronto-London, 1955), vol. III, chapter 18.

2. E. Borel, *Leçons sur les séries divergentes*, 2nd ed. (Paris, 1928).

3. G. H. Hardy, Notes on some points in the integral calculus XLVI, *Messenger of Mathematics*, **46** (1916), 175–182.

4. F. Hausdorff, Beiträge zur Wahrscheinlichkeitsrechnung, *Berichte über die Verhandlungen der Königl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, **53** (1901), 152–178.

5. A. E. Ingham, *The distribution of prime numbers* (Cambridge Tracts, No. 30, 1932).

6. E. T. Whittaker and G. Robinson, *The calculus of observations*, 4th ed. (London-Glasgow, 1944).

7. A Wintner, On a class of Fourier transforms, *American J. Math.*, 58 (1936), 45–90.

8. A Wintner, On Heaviside's and Mittag-Leffler's generalizations of the exponential function, the symmetric stable distributions of Cauchy-Lévy, and a property of the gamma-function ; to appear in the Fréchet jubilee volume of the *Journal de Mathématique* (1958).

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