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LOCALLY CONVEX HYPERSURFACES

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1. Introduction. Let M be an n-dimensional connected topological manifold. Let $\xi : M \to \mathbf{R}^{n+1}$ be a continuous map with the following property: to each $x \in M$ there is an open set $x \in U_x \subset M$, and a convex body $K_x \subset \mathbf{R}^{n+1}$ such that $\xi(U_x)$ is an open subset of ∂K_x and such that $\xi | U_x : U_x \to \partial K_x$ is a homeomorphism onto its image. We shall call such a mapping ξ a *locally convex immersion* and, along with Van Heijenoort [**8**] we shall call $\xi(M)$ a *locally convex hypersurface* of $\mathbf{R}^{n+1}$. Note that we do not assume that ξ is $1 - 1$ or a homeomorphism onto its image or that $\xi(M)$ is closed in $\mathbf{R}^{n+1}$. We may define on M a metric induced by ξ as follows: if $x, y \in M$

$$d(x, y) = \inf\{\text{length}(\xi \circ \gamma) | \gamma \text{ a rectifiable curve between } x \text{ and } y\}.$$

We assume always that M is complete in this metric.

We will summarize the assumptions made so far by saying that *M is immersed in $\mathbf{R}^{n+1}$ as a complete connected locally convex hypersurface.*

In this paper we prove the following analogue of the theorems of Sacksteder, Hartman, and Nirenberg [**6**; **3**; **2**] that concern complete hypersurfaces of non-negative sectional curvature in a Euclidean space:

THEOREM. *Let M be an n-dimensional connected topological manifold immersed in $\mathbf{R}^{n+1}$ as a complete locally convex hypersurface. Then either $\xi(M)$ is a hypercylinder (the product of $\mathbf{R}^{n-1}$ with a curve) or else it is the boundary of an open convex subset of $\mathbf{R}^{n+1}$.*

This theorem depends on and generalizes a result of Van Heijenoort [**8**]. We give a somewhat shorter proof of Van Heijenoort's theorem in Proposition 2.

2. Preliminary results. We must first introduce some further terminology. If $x \in M$ and K_x has a hyperplane of support at $\xi(x)$ that meets K_x only at $\xi(x)$, then we say that ξ is *strictly locally convex at* x and that $\xi(M)$ is strictly locally convex at $\xi(x)$. This condition on $\xi(x)$ is also expressed in the literature by saying that $\xi(x)$ is an *exposed point of* K_x (see [**5**]). We remind the reader that a point p on a convex body K is called an *extreme point* of K if p does not lie in the interior of any line segment contained in K.

By a *hyperplane of support* T_x at $x \in M$ we shall mean any hyperplane of support for K_x at $\xi(x)$. $\tau(x)$ will denote the set of hyperplanes of support at x.

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A *line* in M is a subset $l \subset M$ such that $\xi|l$ is a homeomorphism of l onto a line in $\mathbf{R}^{n+1}$. A line segment is defined similarly. By a *flat r-space* in M we shall mean a subset $L \subset M$ such that $\xi|L$ is a homeomorphism of L onto a linear r-manifold in $\mathbf{R}^{n+1}$. A *flat convex set* in M is a connected subset C of M such that $\xi|C$ maps C homeomorphically onto a convex subset of $\mathbf{R}^{n+1}$.

Without loss of generality we will always assume that the sets U_x introduced in the introduction are such that for some $T_x \in \tau(x)$ the orthogonal projection of $\xi(U_x)$ into T_x is a homeomorphism onto an open ball centred at $\xi(x)$ (see Buseman [**1**, Theorem (1.12)]). It follows that if $y \in U_x$ and if π is a plane containing $\xi(x)$, $\xi(y)$, and the direction normal to this preferred hyperplane of support T_x, then x and y lie in the same connected component of $\xi^{-1}(\pi)$.

Let $x \in M$, and $\xi(x) \in L$ where L is a linear submanifold of $\mathbf{R}^{n+1}$. If for some $T_x \in \tau(x)$ we have $L \subset T_x$ we say that *M and L are tangent at x*; if not, we say that *M and L are transverse at x.*

PROPOSITION 1. *Let $\xi : M \to \mathbf{R}^{n+1}$ be an immersion of M as a complete locally convex hypersurface. Let $L \subset \mathbf{R}^{n+1}$ be any linear submanifold of $\mathbf{R}^{n+1}$. Let $N \subset M$ be a connected component of $\xi^{-1}(L)$. Then N is complete, and there are only two possibilities:*

(a) *M and L are transverse everywhere on N and N is an embedded submanifold of M, and $\xi|N : N \to L$ is a locally convex immersion.*

(b) *M and L are tangent everywhere on N and N is an embedded submanifold-with-boundary of M and $\xi(N)$ is a convex subset of L. In this case, if N does not contain a flat $(n-1)$-space, N has a neighbourhood U such that $U - N$ is connected and does not meet $\xi^{-1}(L)$.*

Proof. Certainly $\xi^{-1}(L)$ is closed since ξ is continuous.

Suppose that there is a point $x \in N$ such that L is tangent to M at x. That is, L is contained in a hyperplane of support at x. If L meets $\xi(U_x)$ at another point y, then clearly L is tangent to M at y as well. Let $O \subset N$ be the subset of all points at which M and L are tangent: we have just shown that O is open in N (for $x \in N \cap U_x \subset O$). O is also closed (even in M), for if $x \in \bar{O}$ then $x \in O$ by [**1**, (1.6)]. Since N is connected it follows that $N = O$, and N is a closed subset of M.

Moreover, N is locally convex in the sense that each point of N has an N-neighbourhood which is flat convex subset of M. To see this, note that if $y, z \in U_x \cap N$ then

$$\xi(y), \xi(z) \in L \cap \xi(U_x) \subset L \subset \partial K_x,$$

whence $[\xi(y), \xi(z)] \subset \partial K_x$. Since $\xi(U_x)$ is projected homeomorphically onto a ball in some hyperplane, $[\xi(y), \xi(z)] \subset L \cap \xi(U_x)$. Since y and z are arbitrary points in $U_x \cap N$ it follows that $U_x \cap N$ is a flat convex subset of M. There are two consequences.

Since N is connected, these flat convex neighbourhoods have the same dimension throughout N. Thus N is an embedded submanifold-with-boundary of M.

On the other hand, if a line segment lies in N then its endpoints lie in N. This, together with the local convexity of N, allows a successful application of an argument of Klee [**4**, Propositions (5.1) and (5.2)] to show that $\xi(N)$ is a convex subset of L.

Now suppose N does not contain a flat $(n - 1)$-space. Then one of the following three situations occurs: (1) N has dimension $\leqq n - 2$ so that $U_x - N$ is always connected for $x \in N$, (2) N is $(n - 1)$-dimensional and bounded so that $U_x - N$ is connected for at least one $x \in N$, (3) N is n-dimensional and bounded so that $U_x' - N$ is connected for all $x \in N$ if $U_x' \subset U_x$ is a suitably chosen neighbourhood of x. In each case a simple argument shows that if $U = \bigcup_{x \in N} U_x$, then $U - N$ is connected.

The only remaining possibility is that M and L are transverse at all points of N. It is clear that in this case N is an embedded submanifold and that $\xi|N : N \rightarrow L$ is a locally convex embedding.

COROLLARY 1. *The convex bodies K_x for M may be chosen in such a way that on the complement of the unions of the flat $(n - 1)$-spaces contained in M, if $U_x \cap U_y \neq \emptyset$ then* $\operatorname{int} K_x \cap \operatorname{int} K_y \neq \emptyset$.

Proof. Let $z \in U_x \cap U_y$. Suppose z is not contained in an n-dimensional convex subset of M. Then clearly $\operatorname{int} K_x \cap \operatorname{int} K_y \neq \emptyset$. If N is an n-dimensional convex set not including a flat $(n - 1)$-space, then if U is as in the proof of Proposition 1, $\xi(U)$ lies entirely on one side of the hyperplane L containing $\xi(N)$. Hence the sets K_x, $x \in \partial N$, all lie on one side of L. For interior points x of N, if K_x does not already lie on that side of L we can achieve this by reflecting K_x in L. The result is now obvious.

PROPOSITION 2 (Van Heijenoort's theorem). *Let M be a connected topological manifold, immersed by ξ in $\mathbf{R}^{n+1}$ as a complete locally convex hypersurface, and suppose M has at least one exposed point x. Then $\xi(M)$ bounds an open convex subset of $\mathbf{R}^{n+1}$.*

Proof. It is easy to see that coordinates $(u^1, \ldots, u^{n+1})$ may be selected on $\mathbf{R}^{n+1}$ to make $T_x = \{u^{n+1} = 0\}$ a support plane at $x = (0, \ldots, 0)$. Since x is exposed we may assume in addition that on $\xi(U_x)$ we have $u^{n+1} = f(u^1, \ , u^n)$ where f is convex and $f(u^1, \ldots, u^n) = 0 \Leftrightarrow u^1 = u^2 = \ldots = u^n = 0$.

For $a \in (0, \infty)$ let

$$R_a = \{(u^1, \ldots, u^{n+1}) | u^{n+1} = a\}.$$

For $a \in (0, \infty]$ let

$$J_a = \{(u^1, \ldots, u^{n+1}) | u^{n+1} < a\}$$

and let P_a be the connected component of $\xi^{-1}(J_a)$ that contains x. Note that

$J_\infty = \mathbf{R}^{n+1}$ and that $P_\infty = M$. For any $a \in (0, \infty]$ let K_a denote the closed convex hull of $\xi(P_a)$. For $a \in (0, \infty]$ let $A(a)$ denote the condition:

$$A(a)\colon \xi|P_a \text{ is a homeomorphism of } P_a \text{ onto } \partial K_a \cap J_a.$$

It is clear that $A(a)$ holds for sufficiently small a. We will show that $A(a)$ is true for all $a \in (0, \infty)$ and then that this implies $A(\infty)$, which is the conclusion of the theorem.

Let $S \subset \mathbf{R}_+$ be the set $\{b \in \mathbf{R}_+ | A(b) \text{ is true}\}$. Since $A(b)$ clearly implies $A(a)$ for all $a \leqq b$ it is clear that S is connected.

S is closed. For suppose $A(a)$ is true for all $a < b$. Let $y \in P_b$. Then $y \in P_a$ for some $a < b$. Let H_y be any closed half-space containing K_a such that $\xi(y) \in \partial H_y$. Let $z \in P_b$. Then $z \in P_{a'}$ for some $a' \in (a, b)$. $A(a')$ implies that because ∂H_y is a supporting hyperplane for a neighbourhood of $\xi(y) \in K_{a'}$, it is a supporting hyperplane for $K_{a'}$. Thus $z \in K_{a'} \subset H_y$. Thus $\xi(P_b) \subset H_y$ and $\xi(y) \in \partial K_b$. Therefore $\xi(P_b) \subset \partial K_b$. It is now easy to see that ξ is a homeomorphism of P_b onto $\partial K_b \cap J_b$.

To show that S is also open we need to know that ∂P_b is connected. This is proved in the lemma below. Assuming it for now, let $y \in \partial P_b$ and suppose M and R_b are tangent at y. Let N be the connected component of $\xi^{-1}(R_b)$ that contains y. By Proposition 1, N is convex. Clearly $\partial P_b \subset \partial N$. It is clear that N cannot contain an $(n - 1)$-flat, and so by Proposition 1 there is a neighbourhood U of N such that $U - N$ is connected and

$$(U - N) \cap \xi^{-1}(R_b) = \emptyset.$$

Thus $U - N \subset P_b$. Hence $\partial P_b = \partial N$ and $M = P_b \cup N$ since M is connected. It follows that for $c > b$, $P_c = M$ so that $A(c)$ is trivially true.

Now suppose M and R_b are transverse throughout ∂P_b. Then ∂P_b is a connected component of $\xi^{-1}(R_b)$. On the other hand, $\xi(\partial P_b) = \partial'(K_b \cap R_b)$ where ∂' denotes the boundary taken in R_b. For every $y \in \partial P_b$,

$$\xi(U_y \cap \partial P_b) = \xi(U_y) \cap R_b$$

is an $(n - 1)$-dimensional submanifold of R_b. Hence $K_b \cap R_b$ must have an interior in R_b. Let z lie in this interior. Let C be the solid cylinder

$$C = \{(u^1, \ldots, u^{n+1}) | (u^1, \ldots, u^n, b) \in K_b \cap R_b\}.$$

Then any line l_y from z to $\xi(y)$, $y \in \partial P_b$, passes through int C. For at least one point $y \in \partial P_b$, l_y also passes through int K_y. Since K_b cannot contain any flat $(n - 1)$-space it follows from Corollary 1 that l_y passes through int K_y for all $y \in \partial P_b$. But then y has a neighbourhood V_y such that cylindrical projection from the line l through z parallel to the u^{n+1}-axis maps $\xi(V_y)$ homeomorphically to a neighbourhood W_y of $\xi(y)$ in ∂C. Since $\bar{P}_b$ is compact we can choose an open neighbourhood V of ∂P_b so that $\xi(V)$ is protected from l homeomorphically onto $W = \partial C \cap \{b - \epsilon < u^{n+1} < b + \epsilon\}$. That implies in particular that each ray $h(t)$, $0 \leqq t < \infty$, perpendicular to l at

$$x' \in l \cap \{b - \epsilon < u^{n+1} < b + \epsilon\}$$

meets $\xi(V)$ precisely once, say at $h(1)$. The set of points

$$O = \operatorname{int} K_b \cup \{h(t), 0 \leqq t < 1, h \text{ as above}\}$$

then forms an open set whose boundary is locally convex in such a way that the convex bodies K_y associated with points on the boundary all intersect O. Then by a theorem of Tietze (see [**7**, p. 53]), O is convex. Put $\bar{O} = K_{b+\epsilon}$ and we see easily that $A(b + \epsilon)$ is true. Thus S is open.

It follows that $A(a)$ is true for all $a \in (0, \infty)$. It remains to prove $A(\infty)$. If $\xi(y) \in K_\infty \cap \bar{J}_b$, $\xi(y) \in K_a$ for some $a > b$ and so $\xi(y) \in K_a \cap \bar{J}_b = K_b$. Thus $K_\infty \cap \bar{J}_b = K_b$ for all $b \in (0, \infty)$. ξ maps M onto ∂K_∞, for if $q \in \partial K_\infty$ then $q \in \partial K_a$ for some a, and thus q is in the image of ξ. It is similarly clear that ξ is 1-1 and a local homeomorphism. Thus $A(\infty)$ is true.

LEMMA. *If $A(b)$ holds, then ∂P_b is connected.*

Proof. Suppose to the contrary, that B_1 and B_2 are non-empty disjoint closed subsets of M covering ∂P_b. Since M is metric, hence normal, there are open disjoint sets U_1, U_2 containing B_1 and B_2 respectively. Choose a sequence $\{a_i\}$ of real numbers, such that $a_i < b$, and $\lim a_i = b$. Since ∂P_{a_i} is connected, we may choose $y_i \in \partial P_{a_i} - (U_1 \cup U_2)$. Since $\overline{\xi(P_b)}$ is compact, we may assume that $\{\xi(y_i)\}$ is a Cauchy sequence. Let $\xi(y_i) \to p$; clearly $p \in \partial(\xi P_b)$. If we show that $\{y_i\}$ is Cauchy in M, with limit $y \in \partial P_b - (U_1 \cap U_2)$, we will have a contradiction.

Let T be a support hyperplane to K_b at p such that the orthogonal projection π is a homeomorphism of a compact neighbourhood W of $p \in \partial K_b$ to a closed ball $V \subset T$, centred at p. Then by Busemann [**1**, 1.6], we may assume that for some $\theta \in (0, \pi/2)$ the angle between the support hyperplanes T' at $p' \in W$, and T at p is always less than θ. It follows that if y, z are points of M such that $\xi(y), \xi(z) \in W$, then

$$||\xi(y) - \xi(z)|| \leqq \sec\theta ||\pi\xi(y) - \pi\xi(z)|| \leqq \sec\theta ||\xi(y) - \xi(z)||.$$

Hence if α is a path in W, $\text{length}(\alpha) \leqq \sec\theta \,\text{length}(\pi \circ \alpha)$. Now let $B = \pi(K_b \cap R_b)$. Clearly $p \in B$ and $\pi\xi(y_i) \notin B$. If the affine space L generated by B is $(n-1)$-dimensional and p lies in the interior of B with respect to L we take a subsequence $\pi\xi(y_i)$ that lies entirely on one side of L in T. In any case, it is now possible to find paths α_{ij} in $T - B$ that join $\pi\xi(y_i)$ and $\pi\xi(y_j)$ and such that $\text{length}(\alpha_{ij}) \to 0$ as $i, j \to \infty$. But then the paths $\xi^{-1}\pi^{-1}(\alpha_{ij})$ are paths in P_b and

$$d(y_i, y_j) \leqq \text{length}(\xi^{-1}\pi^{-1}(\alpha_{ij})) \leqq \sec\theta \,\text{length}(\alpha_{ij}).$$

Thus $\{y_i\}$ is a Cauchy sequence in M with $(\lim y_i) \in \partial P_b - (U_1 \cap U_2)$ and we have a contradiction.

In the next proposition we prove the main theorem for the case of a surface.

PROPOSITION 3. *Let M be a two-dimensional connected topological manifold immersed in $\mathbf{R}^3$ as a complete locally convex surface. Then either $\xi(M)$ is a cylinder or else it is the boundary of an open convex subset of* $\mathbf{R}^3$.

Proof. If, for some $x \in M$, $\xi(x)$ is an extreme point of K_x then by [**5**, Theorem 2.1], there are exposed points of K_x arbitrarily near $\xi(x)$. In particular there is a point $y \in U_x$ at which ξ is strictly locally convex. But then Van Heijenoort's theorem (Proposition 2) shows that $\xi(M)$ is the boundary of an open convex subset of $\mathbf{R}^3$.

Thus we may assume for the remainder of this proof that every $x \in M$ is contained in the interior of a line segment. Let $T_x \in \tau(x)$ and let $N(T_x)$ be the connected component of $\xi^{-1}(T_x)$ containing x. If $N(T_x)$ is one-dimensional it must be a line for otherwise it would have an endpoint which would not lie in the interior of a line segment. Suppose $N(T_x)$ is two-dimensional. If $\xi(N(T_x)) = T_x$ it is easy to see that $N(T_x) = M$ and that M is immersed as a plane. If $N(T_x)$ is a proper two-dimensional subset it must be a slab between two lines whose images are parallel, for otherwise its boundary would contain a point whose image is extreme. Thus if we exclude the case where $\xi(M)$ is a plane, every point x lies on a unique line l_x. It remains only to show that the images of these lines are parallel.

First we show that the map $x \to \xi(l_x)$ that associates to x the unique line in $\xi(M)$ through $\xi(x)$ is continuous. Let $\{x_i\} \to x$. It follows from the compactness of the projective space of lines through $\xi(x)$ that there must be a subsequence $\{x_i'\}$ of $\{x_i\}$ such that the sequence $\{\xi(l_{x_i'})\}$ converges to a line m through $\xi(x)$. It follows from the completeness of M that m is the image of a line l through x. Then $l = l_x$ since l_x is unique. For the same reason $\{\xi(l_{x_i})\}$ cannot have cluster points other than m. Thus the sequence $\{\xi(l_{x_i})\}$ approaches $\xi(l_x)$.

Now let $x \in M$ and let $y \in U_x$. If x and y belong to a flat convex subset of M, then either for some $T_x \in \tau(x)$, $N(T_x)$ must be a slab with $y \in N(T_x)$ or else y lies on l_x. In either case $\xi(l_x)$ and $\xi(l_y)$ are parallel. If x and y do not belong to a flat convex set, no support plane at x is parallel to any support plane at y; moreover, then $\xi(U_x)$ is not flat and thus for any $z \in U_x$ and $T_z \in \tau(z)$, $\xi(U_x)$ is contained in precisely one of the closed half-spaces bounded by T_z. We will show that $\xi(l_x)$ is parallel to T_y and $\xi(l_y)$ to T_x. Suppose to the contrary that $\xi(l_y)$ meets T_x in $\xi(z)$. Let L be the plane through $\xi(x)$, containing the normal direction to T_x and also containing $\xi(y)$. Because of the conditions on U_x, x and y belong to the same connected component σ of $\xi^{-1}(L) \cap U_x$. σ is a curve which we parametrize in such a way that $\sigma(0) = x$ and $\sigma(1) = y$. We now restrict σ to $0 \leqq t \leqq 1$ and we call the restricted curve γ. Then no two lines of support for this (convex) curve σ at points of γ make an angle greater than or equal to $\pi/2$. Also $l_{\gamma(t)}$ is transverse to L for $0 \leqq t \leqq 1$, for otherwise x and y would belong to the flat convex subset $U_x \cap l_{\gamma(t)}$ of M. Let L' be the

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plane through $\xi(z)$ parallel to L. Let σ' be the curve in $\xi^{-1}(L')$ such that $\sigma'(t) \in l_{\sigma(t)}$. Define γ' similarly. Orient $\mathbf{R}^3$ in such a way that $\xi(U_x)$ lies above T_x. Now since $N(T_x)$ is either l_x, or a slab containing l_x but not y, we may define $t_1 \in [0, 1)$ to be the greatest t such that $l_{\gamma(t)} \subset N(T_x)$. No point of $l_{\gamma(t_1)}$ has a flat neighbourhood; clearly then, all points of $l_{\gamma(t_1)}$ have neighbourhoods whose images lie on the same side of T_x, namely, above T_x. Thus for some $\epsilon > 0$, $\xi\gamma'(t_1 + \epsilon)$ lies above T_x. But the height of $\xi\gamma'(t)$ above $T_x \cap L'$ is zero when $t = 0$ and $t = 1$. Hence for some $t_0 \in (0, 1)$, this height attains a positive maximum. Then necessarily γ' has a line of support at $\gamma'(t_0)$ which is parallel to $T_x \cap L'$, and $\xi(\gamma')$ lies below this line of support. Since the lines of support of σ' at points of γ' are necessarily parallel to the lines of support of σ at corresponding points of γ this means that a line of support to σ at $\gamma(0) = x$ and a line of support to σ at $\gamma(t_0)$ make an angle of π. We already indicated that this cannot happen, so we must conclude that $\xi(l_y)$ does not meet T_x.

This argument may be repeated with the roles of x and y interchanged but using the same set U_x and plane L. Thus also $\xi(l_x)$ does not meet T_y.

Since T_x and T_y are not parallel it is clear that $\xi(l_x)$ and $\xi(l_y)$ are parallel to the line of intersection of T_x and T_y and hence to each other, if $x, y \in U_x$. Since M is connected it now follows that $\xi(l_x)$ and $\xi(l_y)$ are parallel for all $x, y \in M$.

3. Proof of the theorem. Suppose that through $x \in M$ there is a line l. Let $T_x \in \tau(x)$ be the preferred hyperplane of support with the property that the orthogonal projection onto T_x maps $\xi(U_x)$ homeomorphically onto an open ball. Let m be the line normal to T_x. Choose $y \in U_x$. We will show that through y there passes a line l' such that $\xi(l)$ and $\xi(l')$ are parallel. If l already passes through y there is nothing to prove. If not, $\xi(x)$, $\xi(y)$, $\xi(l)$, and m are contained in a unique 3-space π which is transverse to M at x and thus also at y. By our assumptions about the sets U_x, x and y lie in the same connected component N of $\xi^{-1}(\pi)$. N with the immersion

$$\xi|N : N \to \pi$$

satisfies the conditions of Proposition 3, and hence there is a line l' through y such that $\xi(l')$ and $\xi(l)$ are parallel. Since y is an arbitrary point in U_x it follows that every point in U_x lies on a line parallel to l. Since M is connected it follows that every point in M lies on a line parallel to l.

Now suppose r is the largest integer such that x is contained in a flat r-space L_x. It follows from the preceding discussion that through every $y \in M$ there is a parallel r-space L_y and that r is the largest dimension possible at y.

Let H be the $(n + 1 - r)$-space through x orthogonal to L_x. It is now clear that $P = \xi^{-1}(H)$ is connected. It is also clear that P contains a point at which $\xi|P : P \to H$ is strictly locally convex, for otherwise every point of P would lie on the interior of a line segment and hence on a line (see the first two sentences in the proof of Proposition 3).

If $\dim P \geqq 2$, we can apply Van Heijenoort's theorem (Proposition 2) to show that $\xi(P) = \partial K$ where K is an open convex set in H. It then follows immediately that

$$\xi(M) = \partial(K \times \mathbf{R}^r)$$

so that $\xi(M)$ is the boundary of an open convex subset of $\mathbf{R}^{n+1}$.

If $\dim P = 1$, P is a curve and $\xi(M)$ is a hypercylinder.

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