A GENERALIZED INTEGRAL II

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1. Introduction. The definition and some of the properties of what may be called a Perron second integral (P^2-integral) were given in a previous paper [4]. This integral starts with a function $f(x)$ defined in an interval (a, c) and goes directly to a second primitive $F(x)$ with the property that the generalized second derivative D^2F is equal to $f(x)$ for almost all x in (a, c). In the present paper the definition is changed slightly and further properties are deduced.

In §**4** it is shown that the P^2-integral provides a solution to a problem discussed by Denjoy in a series of notes published in the Comptes Rendus in 1921 [2]. Denjoy gave rules for the calculation of the second primitive of a function by a process of totalization. He lectured at Harvard in 1938 on this topic, among others, and three volumes of a book based on the lectures were published in 1941 [3]. The title of Volume 3 is *Determination d'une fonction continue par ses nombres dérivés second généralisés extrêmes finis*, and in Chapter VI he proposes the following problem and indicates the solution.

Let $F(x)$ be a continuous function such that $\overline{D}^2F$ and $\underline{D}^2F$ are finite at each point of an interval (a, c). Let $f(x)$ be one of the generalized second derivatives. Denjoy's Problem (U) relative to a set E in (a, c) is the calculation, starting with $f(x)$, of the second variation of $F(x)$,

$$(1.1) \qquad V(F, \alpha, \beta, \gamma) \equiv (\gamma - \beta)\, F(\alpha) + (\alpha - \gamma)\, F(\beta) + (\beta - \alpha)\, F(\gamma),$$

for all α, β, γ in E. After solving the problem in particular cases, Denjoy states at the end of the chapter that the rules for calculating $V(F, \alpha, \beta, \gamma)$ on the whole interval (a, c) will be given in Chap. IX, but the volume containing this chapter has not yet appeared.

The P^2-integral solves Problem (U) on the whole interval (a, c). If $h(x) = D^2F$ where D^2F is defined and finite and $h(x) = 0$ elsewhere, then $V(F, \alpha, \beta, \gamma)$ is equal to $\gamma - \alpha$ times the P^2-integral of $h(x)$ over (α, β, γ).

In §**5** it is shown that a Cesàro-Perron integrable function is also P^2-integrable [1], and that, in a certain sense the integrals agree. An example is given of a function which is P^2-integrable but not Cesàro-Perron integrable.

Section **6** is concerned with the application of the P^2-integral to trigonometric series. Among other things it is proved that if a trigonometric series converges in the interval $(0, 2\pi)$ to a function $f(x)$, then $f(x)$ is necessarily P^2-integrable. Furthermore, with a suitable modification of the definition of the Fourier coefficients for the P^2-integral, the trigonometric series is the Fourier series of $f(x)$. Marcinkiewicz and Zygmund [5] introduced a trigo-

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nometric integral which is probably equivalent, in a certain sense, to the P^2-integral. The problem of the relationship between the two integrals will be considered in another paper.

2. The new definition of the integral. Not all the restrictions placed on the major and minor functions $M(x)$ and $m(x)$ in §4 of [4] are strictly necessary. The point of the definition is to make sure that the difference $M(x) - m(x)$ is a convex function in (a, c) and for this the following definition suffices.

Let $f(x)$ be defined in an interval (a, c). The functions $M(x)$ and $m(x)$ are called major and minor functions, respectively, of $f(x)$ in (a, c) if

(2.1) *$M(x)$ and $m(x)$ are continuous in (a, c);*

(2.2) $M(a) = M(c) = m(a) = m(c) = 0;$

(2.3) $\underline{D}^2M \geqq f(x) \geqq \overline{D}^2m,\ \underline{D}^2M > -\infty,\ \overline{D}^2m < +\infty,$
for all x in (a, c) with the possible exception of a denumerable set E_0;

(2.4) *$M(x)$ and $m(x)$ are smooth for all x in E_0.*

The difference between this definition and the old one is simply that the condition of smoothness is imposed on the functions $M(x)$ and $m(x)$ only when x is in the exceptional set E_0. The proof that $M(x) - m(x)$ is convex follows as in [4]. By conditions (2.3), $\overline{D}^2(M - m) \geqq \underline{D}^2M - \overline{D}^2m \geqq 0$ for all x in (a, c) with the possible exception of a denumerable set E_0. Since $M(x) - m(x)$ is smooth for x in E_0, it follows from a lemma of Zygmund [8, p. 275] that $M(x) - m(x)$ is convex.

The definition of the integral now reads as it did in [4], but is here restated for convenience.

DEFINITION 2.1. *A function $f(x)$ defined in an interval (a, c) is said to be integrable over (a, b, c) where $a < b < c$, if, for every $\epsilon > 0$ there exist a major function $M(x)$ and a minor function $m(x)$ such that $0 \leqq m(x) - M(x) < \epsilon$. The notation for the P^2-interval is $\int_{a,b,c} f(x)\, dx$.*

No further changes are necessary in the statements or proofs of the results of [4].

3. Modification of conditions (2.3). It is well known that in the corresponding situation for the ordinary Perron integral the exceptional denumerable set for the first set of inequalities in (2.3) may be replaced by one of measure zero. A similar modification is possible for the P^2-integral and it is not difficult to prove the following result.

THEOREM 3.1. *A function $f(x)$ defined in an interval (a, c) is integrable over (a, b, c), where $a < b < c$, if, and only if, for every $\epsilon > 0$ there exist functions $T(x)$ and $t(x)$ with the properties*

(3.1) $T(x)$ *and* $t(x)$ *are continuous in* (a, c);

(3.2) $T(a) = T(c) = t(a) = t(c) = 0$;

(3.3) $\underline{D}^2T \geqq f(x) \geqq \overline{D}^2t$ *for all* x *in* (a, c) *except for a set* E *of measure zero*;

(3.4) $\underline{D}^2T > -\infty, \overline{D}^2t < +\infty$ *for all* x *in* (a, c) *with the possible exception of a denumerable set* E_0;

(3.5) $T(x)$ *and* $t(x)$ *are smooth for all* x *in* E_0;

(3.6) $0 \leqq t(b) - T(b) < \epsilon$.

Proof. The statement of the theorem and its proof are modelled on McShane's proof [6] of the corresponding result for the ordinary Perron integral.

If $f(x)$ is integrable, conditions (2.1)-(2.4) and Definition 2.1 show that $T(x) = M(x)$ and $t(x) = m(x)$ have the properties (3.1)-(3.6) with $E = E_0$. Hence it is only necessary to show that (3.1)-(3.6) imply the integrability of $f(x)$.

For every $\delta > 0$ there is a non-negative, non-decreasing, absolutely continuous function $\varphi(x)$ such that

(3.7) $\varphi'(x) = +\infty$ for x in the set E of measure zero, and

(3.8) $$0 \leqq \varphi(c) < \delta/(b - a).$$

([7], §11.8. Lemma 1, with $\varphi(x) = \chi(x) - \chi(a)$.) Let

(3.9) $$\Phi(x) = \int_a^x \varphi(\xi)d\xi - (x - a)/(c - a) \int_a^c \varphi(\xi)\, d\xi.$$

Since $\Phi(x)$ is the integral of a bounded non-decreasing function, it is convex ([7], Ex. 8. p. 372), and

(3.10) $$\underline{D}^2\Phi \geqq \underline{D}\varphi \geqq 0.$$

By (3.9), $\Phi(a) = \Phi(c) = 0$, and, by (3.8),

(3.11) $$0 \leqq -\Phi(b) \leqq (b - a)/(c - a) \int_a^c \varphi(\xi)\, d\xi < \delta.$$

It can be shown that the functions

(3.12) $$M(x) = T(x) + \Phi(x), \quad m(x) = t(x) - \Phi(x)$$

are major and minor functions, respectively, of $f(x)$ in (a, c), and that

(3.13) $$0 \leqq m(b) - M(b) < \epsilon + 2\delta.$$

This will prove that $f(x)$ is integrable over (a, b, c).

Clearly, (2.1) and (2.2) are satisfied and, since

(3.14) $$\underline{D}^2M \geqq \underline{D}^2T + \underline{D}^2\Phi, \quad \overline{D}^2m \leqq \overline{D}^2t - \underline{D}^2\Phi,$$

it follows from (3.4) and (3.10) that $\underline{D}^2M > -\infty$, $\overline{D}^2m < +\infty$ with the possible exception of the denumerable set E_0.

Similarly, by (3.3) and (3.10),

$$\underline{D}^2 M \geqq f(x) \geqq \overline{D}^2 m \tag{3.15}$$

for all x not in E. If x is in E, but not in E_0, then $\underline{D}^2\Phi = +\infty$, $\underline{D}^2 T > -\infty$, $\overline{D}^2 t < +\infty$ and

$$\underline{D}^2 M = +\infty \geqq f(x) \geqq -\infty = \overline{D}^2 m.$$

Thus, (3.15) holds for all x in (a, c) with the possible exception of the denumerable set E_0.

Finally, $m(b) - M(b) = t(b) - T(b) - 2\Phi(b)$ and (3.13) follows from (3.12), (3.11), and (3.6). Hence $f(x)$ is integrable over (a, b, c).

COROLLARY. *If $f(x)$ is integrable over (a, b, c) and the functions $T(x)$ and $t(x)$ satisfy* (3.1)-(3.6), *then*

$$-t(b) \leqq \int_{a,b,c} f(x)\, dx \leqq -T(b).$$

Proof. Since $M(x)$ and $m(x)$, defined by (3.12) are major and minor functions, respectively, of $f(x)$ in (a, c), it follows that

$$-\{t(b) - \Phi(b)\} \leqq \int_{a,b,c} f(x)\, dx \leqq -\{T(b) + \Phi(b)\}.$$

Hence, by (3.11),

$$-t(b) - \delta \leqq \int_{a,b,c} f(x)\, dx \leqq -T(b) + \delta.$$

Since δ is an arbitrary positive number, the inequalities stated in the corollary must hold.

4. Additional properties of the integral.

THEOREM 4.1. *If $f_1(x)$ is integrable over (a, b, c) and $f_2(x) = f_1(x)$ almost everywhere in (a, c) then $f_2(x)$ is integrable over (a, b, c) and*

$$\int_{a,b,c} f_2(x)\, dx = \int_{a,b,c} f_1(x)\, dx.$$

Proof. The functions $T(x)$ and $t(x)$ which satisfy (3.1)-(3.6) for the function $f_1(x)$ clearly satisfy the same conditions for the function $f_2(x)$. Hence $f_2(x)$ is integrable over (a, b, c). From the corollary to Theorem 3.1, it follows that the integrals of $f_2(x)$ and $f_1(x)$ both lie in the interval $(-t(b), -T(b))$ of length less than ϵ. Since ϵ is arbitrary, the two integrals must be equal.

THEOREM 4.2 (GENERALIZATION OF THEOREM 13 OF [1]). *Suppose that $F(x)$ is continuous in (a, c), that D^2F is defined for all x in (a, c) except for a set E of measure zero, and that $\overline{D}^2F$, $\underline{D}^2F$ are finite for all x in (a,c) with the possible exception of a denumerable set E_0, where, however, $F(x)$ is smooth. If $f(x) = D^2F$ where D^2F is defined and $f(x) = 0$ elsewhere, then $f(x)$ is integrable over (a, b, c) and*

$$(c - a) \int_{a,b,c} f(x)\, dx = (c - b)\, F(a) + (a - c)\, F(b) + (b - a)\, F(c) \tag{4.1}$$

Proof. If the functions $T(x)$ and $t(x)$ are defined by

$$T(x) = t(x) = F(x) - \frac{(c - x)\, F(a) + (x - a)\, F(c)}{(c - a)},$$

all the conditions of Theoerem 3.1 are satisfied for the function $f(x)$. Thus $f(x)$ is integrable over (a, b, c) and, by the corollary to Theorem 3.1, $\int_{a,b,c} f(x)dx$ is equal to $-T(b) = -t(b)$. Formula (4.1) follows at once.

The right-hand side of (4.1) is the second variation of $F(x)$ over (a, b, c) and Denjoy [3] denotes it by $V(F, a, b, c)$. Thus the formula of the theorem may be written

$$(4.2) \qquad (c - a) \int_{a,b,c} f(x)\, dx = V(F, a, b, c).$$

COROLLARY. *Suppose that $F_1(x)$ and $F_2(x)$ are two functions satisfying the hypotheses of Theorem 4.2, and that $D^2F_1 = D^2F_2$ almost everywhere in (a, c). Then $F_1(x)$ and $F_2(x)$ differ only by a linear function in (a, c).*

Proof. For $i = 1, 2$, let $f_i(x) = D^2F_i$ where D^2F_i is defined and let $f_i(x) = 0$ elsewhere. By (4.2)

$$(4.3) \qquad (c - a) \int_{a,b,c} f_i(x)\, dx = V(F_i, a, b, c).$$

Since $f_1(x) = f_2(x)$ almost everywhere, it follows that the integrals in (4.3) are equal. Hence $V(F_1, a, b, c) = V(F_2, a, b, c)$ and this means that

$$F_2(x) - F_1(x) = \frac{c - x}{c - a} \{F_2(a) - F_1(a)\} + \frac{x - a}{c - a} \{F_2(c) - F_1(c)\}.$$

The expression on the right is a linear function.

If $F(x)$ is a function satisfying the conditions of Problem (U) as stated in the introduction, and if $f(x)$ is one of the generalized second derivatives of $F(x)$, Denjoy shows in Chap. V [3] that $f(x) = D^2F$ almost everywhere in (a, c). Thus, it follows from Theorems 4.2 and 4.1 that $f(x)$ is integrable over (a, b, c). By Theorem 8 of [4], $f(x)$ is integrable over (α, β, γ) where $a \leqq \alpha < \beta < \gamma \leqq c$, and, from (4.2),

$$(4.4) \qquad (\gamma - \alpha) \int_{\alpha,\beta,\gamma} f(x)dx = V(F, \alpha, \beta, \gamma).$$

Thus the P^2-interval provides a solution to Problem (U) over the entire interval (a, c).

5. The generality of the integral. It will be shown in this section that a function $f(x)$ which is CP-integrable over (a, c) is P^2-integrable over (a, b, c), where $a < b < c$. The definitions which Burkill [1] gives for C-continuity and C-derivates are equivalent to the following:

A function $g(x)$ is C-continuous at $x = x_0$ if

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$$(5.1)\qquad \lim_{h\to 0+} \frac{1}{h}\int_0^h \{g(x_0+\xi) - g(x_0)\}\, d\xi$$
$$= \lim_{h\to 0+} \frac{1}{h}\int_0^h \{g(x_0) - g(x_0-\xi)\}\, d\xi = 0.$$

The larger of the upper limits, as $h \to 0+$, of

$$(5.2)\qquad \frac{2}{h^2}\int_0^h \{g(x_0+\xi) - g(x_0)\}\, d\xi$$

and

$$(5.3)\qquad \frac{2}{h^2}\int_0^h \{g(x_0) - g(x_0-\xi)\}\, d\xi$$

is the upper C-derivate of $g(x)$ at $x = x_0$, and is denoted by $\overline{CD}g$. The smaller of the lower limits, as $h \to 0+$, is the lower C-derivate, denoted by $\underline{CD}g$.

A function $f(x)$ is CP-integrable over (a, c) if, and only if, for every $\epsilon > 0$ there exist major and minor functions $M(x)$ and $m(x)$, respectively, such that

(5.4) *$M(x)$ and $m(x)$ are C-continuous in (a, c);*

(5.5) $M(a) = m(a) = 0$;

(5.6) *$\underline{CD}M \geqq f(x) \geqq \overline{CD}m$ for all x in (a, c) except for a set E of measure zero;*

(5.7) *$\underline{CD}M > -\infty$, $\overline{CD}m < +\infty$ for all x in (a, c) with the possible exception of a denumerable set E_0;*

(5.8) $0 \leqq M(c) - m(c) < \epsilon$.

The main result of this section is

THEOREM 5.1. *If $f(x)$ is CP-integrable over (a, c), then it is P^2-integrable over (a, b, c), where $a < b < c$.*

Proof. Let the functions $T(x)$ and $t(x)$ be defined by

$$(5.9)\qquad T(x) = \int_a^x M(\xi)\, d\xi - \frac{x-a}{c-a}\int_a^c M(\xi)\, d\xi,$$

$$(5.10)\qquad t(x) = \int_a^x m(\xi)\, d\xi - \frac{x-a}{c-a}\int_a^c m(\xi)\, d\xi.$$

Since $M(x)$ and $m(x)$ are C-continous, the integrals in (5.9) and (5.10) exist as Perron integrals. Clearly, $T(x)$ and $t(x)$ satisfy conditions (3.1) and (3.2) of Theorem 3.1. Since

$$(5.11)\qquad \Delta_h^2\, T \equiv T(x+h) - 2T(x) + T(x-h)$$
$$= \int_0^h \{M(x+\xi) - M(x)\}\, d\xi + \int_0^h \{M(x) - M(x-\xi)\}\, d\xi,$$

it follows from (5.1) that $\frac{\Delta_h^2 T}{h} \to 0$ as $h \to 0$. Hence $T(x)$, and similarly $t(x)$, is smooth for all x in (a, c).

By (5.11),

$$\frac{\Delta_h^2 T}{h^2} = \frac{1}{h^2}\left[\int_0^h \{M(x+\xi) - M(x)\}\, d\xi + \int_0^h \{M(x) - M(x-\xi)\}\, d\xi\right]$$

and therefore, if x is not in E, it follows from (5.2), (5.3) that

$$\underline{D}^2 T \geqq \tfrac{1}{2}\{\underline{CD}M + \underline{CD}M\} \geqq f(x). \tag{5.12}$$

Similarly, if x is not in E,

$$\overline{D}^2 t \leqq \tfrac{1}{2}\{\overline{CD}m + \overline{CD}m\} \leqq f(x) \tag{5.13}$$

and conditions (3.3) are satisfied. It also follows from the first parts of (5.12) and (5.13) that (3.4) is satisfied.

Finally, the function $T(x) - t(x)$ is equal to the integral of the non-decreasing function $M(x) - m(x)$ minus a linear function, and is therefore convex in the interval (a, c). Since $T(a) - T(c) = t(a) - t(c) = 0$, it follows from (5.9), (5.10), and (5.8) that

$$0 \leqq t(b) - T(b) < \frac{b-a}{c-a}\int_a^c \{M(\xi) - m(\xi)\}\, d\xi < (b-a)\,\epsilon. \tag{5.14}$$

Thus, the final condition (3.6) is satisfied and $f(x)$ is P^2-integrable over (a, b, c).

COROLLARY 1. *If $f(x)$ is CP-integrable over (a, c) and $G(\xi) = (CP)\int_a^\xi f(x)dx$, $a \leqq \xi \leqq c$, then*

$$\int_{a,b,c} f(x)dx = \frac{b-a}{c-a}\int_a^c G(\xi)\, d\xi - \int_a^b G(\xi)\, d\xi. \tag{5.15}$$

The integrals on the right-hand side of (5.15) *are Perron integrals.*

Proof. Let major and minor functions $M(x)$, $m(x)$, $T(x)$, and $t(x)$ be defined as in the proof of Theorem 5.1. Let

$$F(x) = \int_a^x G(\xi)\, d\xi - \frac{x-a}{c-a}\int_a^c G(\xi)\, d\xi. \tag{5.16}$$

The integrals are Perron integrals since $G(x)$ is C-continuous.

The function $T(x) - F(x)$ is equal to the integral of the non-decreasing function $M(x) - G(x)$ minus a linear function, and is therefore convex in (a, c). Since $T(a) = T(c) = F(a) = F(c) = 0$, it follows that $T(b) - F(b) \leqq 0$. Similarly, it can be shown that $0 \leqq t(b) - F(b)$. Hence

$$-\,t(b) \leqq -\,F(b) \leqq -\,T(b). \tag{5.17}$$

But $T(x)$ and $t(x)$ are major and minor functions, respectively, in the P^2 sense, and by the Corollary to Theorem 3.1,

$$-t(b) \leqq \int_{a,b,c} f(x)\,dx \leqq -T(b). \tag{5.18}$$

It follows from (5.16), (5.17), and (5.18) that the right and left-hand sides of (5.15) lie in the same interval of length $t(b) - T(b) < (b-a)\,\epsilon$. Since ϵ is arbitrary the two sides must be equal.

COROLLARY 2. *If $f(x)$ is CP-integrable, if $G(\xi) = (CP)\int_a^\xi f(x)\,dx$, and if $F(x) = -\int_{a,x,c} f(x)\,dx$ then $F'(x)$ exists for all x in (a, c) and*

$$F'(x) = G(x) - \frac{1}{c-a}\int_a^c G(\xi)\,d\xi.$$

Proof. By (5.16),

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_0^h G(x+\xi)\,d\xi - \frac{1}{c-a}\int_a^c G(\xi)\,d\xi$$

and the first term on the right tends to $G(x)$, since $G(x)$ is C-continuous.

The following is an example of a function which is P^2-integrable, but not CP-integrable. Let

$$F(x) = \begin{cases} x\cos(1/x), & x \neq 0, \\ 0, & x = 0; \end{cases} \qquad f(x) = \begin{cases} \dfrac{-\cos(1/x)}{x^3}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $F(x)$ is continuous and smooth, and $D^2F = f(x)$ for all values of x including $x = 0$. Hence $f(x)$ is P^2-integrable over an interval (a, c) which includes the origin, and

$$F(x) = -\int_{a,x,c} f(x)\,dx + \frac{c-x}{c-a}F(a) + \frac{x-a}{c-a}F(c).$$

If $f(x)$ were CP-integrable, the function $F(x)$ would, by Corollary 2, have a derivative at $x = 0$. But, $(F(h) - F(0))/h = \cos(1/h)$ does not tend to a limit as $h \to 0$. It follows that $f(x)$ is not CP-integrable.

6. Trigonometric series. Before the main result of this section can be stated, some preliminary definitions and theorems are needed. Throughout the section, $f(x)$ and $g(x)$ denote periodic functions with period 2π.

THEOREM 6.1. *If $g(x)$ is CP-integrable over $(-2\pi, 2\pi)$ it is P^2-integrable over $(-2\pi, 0, 2\pi)$ and*

$$\int_{-2\pi,0,2\pi} g(x)\,dx = \pi\int_{-\pi}^{\pi} g(x)\,dx. \tag{6.1}$$

The integral on the right is a CP-integral.

Proof. Let $G(\xi) = \int_{-2\pi}^{\xi} g(x)\,dx$. By Theorem 5.1 and the first corollary,

$$\int_{-2\pi,0,2\pi} g(x)\,dx = \tfrac{1}{2}\int_{-2\pi}^{2\pi} G(\xi)\,d\xi - \int_{-2\pi}^{0} G(\xi)\,d\xi \tag{6.2}$$
$$= \tfrac{1}{2}\int_{-\pi}^{\pi} \{G(\xi+\pi) - G(\xi-\pi)\}\,d\xi.$$

But

$$G(\xi+\pi) - G(\xi-\pi) = \int_{\xi-\pi}^{\xi+\pi} g(x)\,dx = \int_{-\pi}^{\pi} g(x)\,dx,$$

and (6.1) follows at once from (6.2).

DEFINITION 6.1. *If $f(x)\cos kx$ and $f(x)\sin kx$ are P^2-integrable, the Fourier coefficients of $f(x)$ are defined by*

$$\begin{aligned} a_k &= \frac{1}{\pi^2}\int_{-2\pi,0,2\pi} f(x)\cos kx\,dx, \\ b_k &= \frac{1}{\pi^2}\int_{-2\pi,0,2\pi} f(x)\sin kx\,dx. \end{aligned} \tag{6.3}$$

The definition is justified by Theorem 6.1. If $g(x)$ is replaced by $f(x)\cos kx$ in (6.1), then

$$\frac{1}{\pi^2}\int_{-2\pi,0,2\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx.$$

The expression on the right is the usual one for the Fourier coefficients a_k.

THEOREM 6.2. *Suppose that the trigonometric series*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx), \tag{6.4}$$

where $a_n \to 0$, $b_n \to 0$, has upper and lower Riemann sums $\overline{R}(x)$ and $\underline{R}(x)$, respectively, which are finite for all x in $(-2\pi, 2\pi)$ with the possible exception of a denumerable set. Let $f(x)$ denote either one of the functions $\overline{R}(x)$, $\underline{R}(x)$. Then $f(x)\cos kx$ and $f(x)\sin kx$ are integrable over $(-2\pi, 0, 2\pi)$ and a_k, b_k are given by (6.3).

Proof. It is well-known [8, pp. 270-271] that when $a_n \to 0$, $b_n \to 0$ the series

$$\frac{1}{4}a_0x^2 - \sum_{n=1}^{\infty}\frac{a_n\cos nx + b_n\sin nx}{n^2} \tag{6.5}$$

converges absolutely and uniformly to a continuous and smooth function $F(x)$ In addition, $\overline{D}^2F = \overline{R}(x)$ and $\underline{D}^2F = \underline{R}(x)$. Hence, by the same argument that proved (4.4), $f(x)$ is P^2-integrable over $(-2\pi, 0, 2\pi)$ and

$$V(F, -2\pi, 0, 2\pi) = 4\pi\int_{-2\pi,0,2\pi} f(x)\,dx.$$

The left-hand side reduces to $4\pi^3 a_0$, which proves the first of the formulas (6.3) for $k = 0$.

If the series (6.4) is multiplied by $\cos kx$ and the products $\cos nx\cos kx$, $\sin nx\cos kx$ replaced by sums of cosines and sines, it follows from the Rajchman theory of formal multiplication of trigonometric series [8, Sec.11.42,(d)] that the upper and lower Riemann sums of the new series are $\overline{R}(x)\cos kx$ and $\underline{R}(x)\cos kx$, respectively, if $\cos kx > 0$ and $\underline{R}(x)\cos kx$ and $\overline{R}(x)\cos kx$, respectively if $\cos kx < 0$. The constant term of the new series is $\frac{1}{2}a_k$ and the first of formulas (6.3) is established for $k \geqq 1$ by the same argument as that for $k = 0$. The proof of the second of (6.3) is similar.

COROLLARY. *If the trigonometric series* (6.4) *converges to zero for almost all* x *in* $(-2\pi, 2\pi)$ *and if* $\overline{R}(x)$ *and* $\underline{R}(x)$ *are finite for all* x *in* $(-2\pi, 2\pi)$ *with the possible exception of a denumerable set, then* $a_k = b_k = 0$ *for every* k.

Proof. If (6.4) converges to zero, it is also summable (R) to zero [8, Sec. 11.2, (i)] and hence $\overline{R}(x)$ and $\underline{R}(x)$ are zero for almost all x in $(-2\pi, 2\pi)$. The conditions of Theorem 6.2 are satisfied, and the result follows from (6.3).

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