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Monodromy Groups and Self-Invariance

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Abstract. For every polytope $\mathcal{P}$ there is the universal regular polytope of the same rank as $\mathcal{P}$ corresponding to the Coxeter group $\mathcal{C} = [\infty, \ldots, \infty]$. For a given automorphism d of $\mathcal{C}$, using monodromy groups, we construct a combinatorial structure $\mathcal{P}^d$. When $\mathcal{P}^d$ is a polytope isomorphic to $\mathcal{P}$ we say that $\mathcal{P}$ is self-invariant with respect to d, or d-invariant. We develop algebraic tools for investigating these operations on polytopes, and in particular give a criterion on the existence of a d-automorphism of a given order. As an application, we analyze properties of self-dual edge-transitive polyhedra and polyhedra with two flag-orbits. We investigate properties of medials of such polyhedra. Furthermore, we give an example of a self-dual equivelar polyhedron which contains no polarity (duality of order 2). We also extend the concept of Petrie dual to higher dimensions, and we show how it can be dealt with using self-invariance.

1 Introduction

The concept of monodromy groups has been used recently in studying maps and hypermaps (see [2, 18–20, 25]). It was also used by Hartley in his work on quotients of abstract polytopes [8, 9]. Monodromy groups of highly symmetrical structures, such as regular or orientably regular maps and regular polytopes, are permutation groups isomorphic to the automorphism groups of these structures. The true power of monodromy groups is exhibited in dealing with structures having less symmetry, for example, when working with edge-transitive rather than regular or chiral maps [18]. In this paper we make use of this property of monodromy groups when generalizing the concept of duality to d-invariance in Section 3. This generalized concept can be applied to various other operations on polytopes, for example in studying self-Petrie polyhedra, as we shall show. One can similarly deduce how to use it in studying certain operations on maps and hypermaps.

This paper was motivated by some questions arising from the classical theory of polytopes. The concept of duality for polyhedra is well known and has been dealt with recently. In 1988 Grünbaum and Shephard [7] asked whether every self-dual polyhedron is of degree 2, that is, if it possesses a polarity (an involutory duality). In 1989 Jendrol [12] constructed a polyhedron giving a negative answer to this question. A few years later more examples were found (for a brief survey of literature see [1]). All these examples have very small symmetry groups and one naturally expects a positive answer for highly symmetrical polyhedra and polytopes in general. The concept of duality can be extended to abstract polytopes, but so far has been studied

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for regular and chiral polytopes only [10]. It is well known that all regular abstract polytopes possess a polarity, and it was proved that chiral polytopes of odd rank also do [10]. (In the same paper an example was given showing that for even rank this need not be the case.)

The paper is organized as follows. In Section 2 we provide some basic definitions on abstract polytopes and dualities. We also show that every self-dual polytope belongs to one of two classes; that is, modifying the definition used with chiral polytopes, we can say that $\mathcal{P}$ is properly or improperly self-dual.

We develop algebraic tools, which we use in the analysis of dualities and existence of polarities, in Section 3. Operations on abstract polytopes are defined using the group of automorphisms $\mathrm{Aut}(\mathcal{C})$ of a Coxeter group $\mathcal{C}$. In particular, duality turns out to be one such operation. The concept of a duality is generalized to a bijection called a d-automorphism, induced by $d \in \mathrm{Aut}(\mathcal{C})$. The concept of self-duality is generalized to that of self-invariance. Motivated by the problem of existance of polarity, we extend it to the problem of existance of a d-automorphism of a prescribed order. Using monodromy groups, we characterize self-invariance and d-automorphisms, and provide a complete description of actions of d-automorphisms on the flag-orbits (under the action of the automorphism group). In particular, we give necessary and sufficient conditions on the monodromy group of a self-invariant polytope for existence of a d-automorphism of a given degree. This approach is especially useful in dealing with non-regular polytopes having a small number of flag-orbits.

The results obtained in Section 3 are used in Section 4 to classify and analyze self-dual polyhedra with two orbits. In Section 5 we then provide a negative answer to the open problem on the existance of polarities for equivelar polyhedra [21] (Equivelar polyhedra are easily described as polyhedra that can be assigned a Schläfli type.)

The last section deals with medials of polyhedra. We analyze the monodromy groups of medials and characterize all two-orbit medial polyhedra.

2 Polytopes, Automorphisms and Dualities

We briefly introduce several basic definitions and concepts from the theory of abstract polytopes. For more detailed account the reader is referred to [16].

A *flagged poset of rank n* is a partially ordered set $\mathcal{P}$ with order $\leqslant_{\mathcal{P}}$ (or simply $\leqslant$, when it is clear from the context), unique minimal and maximal elements F_{-1} and F_n, respectively, and such that every maximal chain (called a *flag*) of $\mathcal{P}$ contains exactly $n+2$ elements. The order induces the strictly monotone rank function having range $\{-1, \dots, n\}$.

The elements of a flagged poset $\mathcal{P}$ are called *faces*. The faces of rank j, $-1 \leqslant j \leqslant n$, are called *$j$-faces*, and a typical j-face is denoted by F_j. Rank 0, 1, and $n-1$ faces are usually called the *vertices*, *edges* and *facets* of $\mathcal{P}$, respectively. Two faces F and G are said to be *incident* if $F \leqslant G$ or $G \leqslant F$. We denote the set of flags by $\mathcal{F}(\mathcal{P})$ and a group of all bijections on flags by $\mathrm{Sym}(\mathcal{F}(\mathcal{P}))$. Given two faces F and G of a flagged poset $\mathcal{P}$ such that $F \leqslant G$, the *section* G/F of $\mathcal{P}$ is the set of faces $\{H \mid F \leqslant H \leqslant G\}$. If F_0 is a vertex, then the section F_n/F_0 is called the *vertex-figure* of F_0. A flagged poset $\mathcal{P}$ is *connected* if its rank is 0 or 1, or the incidences in the poset without F_{-1} and F_n induce a connected graph, and it is *strongly connected* if every section of $\mathcal{P}$ (including

$\mathcal{P}$ itself) is connected.

An *abstract polytope* of rank n, or an *n-polytope*, is a strongly connected flagged poset of rank n with the *diamond condition*: whenever $F \leqslant G$, with $\operatorname{rank}(F) = j - 1$ and $\operatorname{rank}(G) = j + 1$, there are exactly two faces H of rank j such that $F \leqslant H \leqslant G$. The definition of an abstract polytope encapsulates many combinatorial properties of classical polytopes.

If $\mathcal{P}$ is an n-polytope and Φ a flag of $\mathcal{P}$, the diamond condition tells us that there is exactly one flag that differs from Φ in the i-face. Such a flag is called the *i-adjacent* to Φ and it is denoted Φ^i. Note that $(\Phi^i)^i = \Phi$. By Proposition 2A1 of [16], the strong connectivity of a flagged poset $\mathcal{P}$ is equivalent to the *strong flag connectivity*, that is, for any two flags Φ and Ψ of $\mathcal{P}$ there exists a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$ such that each two successive flags Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i.

In this paper we shall often consider polytopes of rank 3, which we simply call *(abstract) polyhedra*. Note that polyhedra are maps in the sense of [14, 23]. But not every map is a polyhedron. For a map, the diamond condition need not be satisfied, as can be seen in the example of the map given in Figure 1. The map has one vertex, one face, but five edges, and therefore necessarily violates the diamond condition.

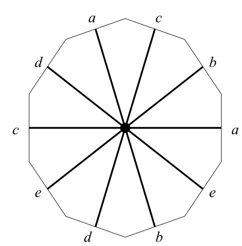

Figure 1: An example of a map which is not a polyhedron.

Let $\mathcal{P}$ and $\mathcal{P}'$ be posets. An *isomorphism* from $\mathcal{P}$ to $\mathcal{P}'$ is a bijection preserving the partial order. An *anti-isomorphism* $\delta\colon \mathcal{P} \to \mathcal{P}'$ is a bijection reversing the order, in which case $\mathcal{P}$ and $\mathcal{P}'$ are said to be *duals* of each other. An anti-isomorphism or isomorphism from $\mathcal{P}$ to $\mathcal{P}$ is called a *duality of* $\mathcal{P}$ (sometimes also *self-duality*) or an *automorphism* of $\mathcal{P}$, respectively, and $\mathcal{P}$ is said to be *self-dual* if there exists a duality of $\mathcal{P}$. The set of all automorphisms and dualities of a polytope $\mathcal{P}$ forms a group, the *extended group* $\mathcal{D}(\mathcal{P})$ *of* $\mathcal{P}$, which contains $\operatorname{Aut}(\mathcal{P})$, the subgroup of all automorphisms of $\mathcal{P}$, as a subgroup of index at most 2. Note that we can consider the extended group of $\mathcal{P}$ as acting on the set of flags $\mathcal{F}(\mathcal{P})$ (as defined in Section 3.) All the actions in this paper will be written as acting on an object on the right. On the other hand, all the functions (excluding automorphisms and dualities) will be acting on the left. The following lemma, which will be used repeatedly, can be easily proved.

Lemma 2.1 *Let $\mathcal{P}$ be an n-polytope, $\phi \in \operatorname{Sym}(\mathcal{F}(\mathcal{P}))$ and Φ a flag of $\mathcal{P}$.*

(i) *If ϕ is an automorphism, then it preserves adjacencies, that is*

$$\Phi^j\phi = (\Phi\phi)^j, \; j = 0, 1, \ldots, n-1.$$

(ii) *If ϕ is a duality, then*

$$\Phi^j\phi = (\Phi\phi)^{n-1-j}, \; j = 0, 1, \ldots, n-1.$$

The *degree of a duality* is its order. The *degree* of a self-dual polytope $\mathcal{P}$ is defined as the smallest integer d such that $\mathcal{P}$ has a duality of degree d. If the degree of $\mathcal{P}$ is finite, it must be a power of 2. This is clear, since if d is the degree of $\mathcal{P}$, r an odd divisor of d, and δ a duality of degree d, then δ^r is a duality of $\mathcal{P}$ of degree at most $d/r \leq d$. A duality of order 2 is called a *polarity*. Regular self-dual polytopes of any rank have degree 2 [16, p. 37], as do all chiral polytopes of odd rank [10].

Let $\mathrm{Orb}(\mathcal{P})$ denote the set of all flag-orbits of $\mathcal{P}$ under the action of $\mathrm{Aut}(\mathcal{P})$. We now consider the action of the dualities on $\mathrm{Orb}(\mathcal{P})$ and the consequences of such action on the degree of $\mathcal{P}$.

Lemma 2.2 *Let $\mathcal{P}$ be a self-dual polytope, δ a duality of $\mathcal{P}$, and $O_1, O_2 \in \mathrm{Orb}(\mathcal{P})$. If δ maps a flag in O_1 to a flag in O_2, then all dualities map flags in O_1 to flags in O_2.*

Proof Let $\Phi, \Psi \in O_1$ and $\Phi\delta \in O_2$. Then for some $\beta \in \mathrm{Aut}(\mathcal{P})$,

$$\Psi\delta = \Phi\beta\delta = \Phi(\delta\delta^{-1})\beta\delta = (\Phi\delta)\delta^{-1}\beta\delta,$$

and since $\delta^{-1}\beta\delta \in \mathrm{Aut}(\mathcal{P})$, $\Psi\delta \in O_2$. If δ' is any other duality of $\mathcal{P}$, $\Phi\delta' = \Phi\delta\delta^{-1}\delta'$, but $\delta^{-1}\delta' \in \mathrm{Aut}(\mathcal{P})$ and $\Phi\delta' \in O_2$. ∎

The following definition was motivated by the corresponding situation for chiral polytopes. (Chiral polytopes, which are formally defined in Section 4, have two distinct flag-orbits.) Polytope $\mathcal{P}$ is said to be *properly self-dual* if its dualities preserve all flag-orbits of $\mathcal{P}$. Otherwise, we say that $\mathcal{P}$ is *improperly self-dual*. The reader can easily verify the following.

Proposition 2.3 *If $\mathcal{P}$ has a duality fixing one flag-orbit, then there exists a duality δ fixing a flag of $\mathcal{P}$ and therefore δ is a polarity. In particular, properly self-dual polytopes are of degree* 2.

3 d-Automorphisms and d-Invariant Polytopes

In this section we give some general results on operations on abstract polytopes of rank n. One can easily extend these results to apply to other structures like maps, hypermaps, and similar combinatorial objects.

The action of a group G on a set Z is an operation $\cdot : Z \times G \to Z$, such that $z \cdot 1 = z$ and $(z \cdot g) \cdot h = z \cdot (gh)$, for every $z \in Z$ and $g, h \in G$. We denote it by the triple $(Z, G, \cdot)$, and when the action is clear from the context, we abbreviate it with (Z, G). Let $(Z, G, \cdot)$ and $(Z', G', *)$ be two actions. A pair (p, q) consisting of a surjective mapping $p \colon Z \to Z'$ and a group epimorphism $q \colon G \to G'$ is called an *action epimorphism* if for every $z \in Z$ and every $g \in G$ it follows that $p(z \cdot g) = p(z) * q(g)$. If both p and q are one-to-one, we refer to it as an *action isomorphism*.

In what follows we shall require the following two elementary lemmas dealing with action epimorphisms, as well as the well-known fourth isomorphism theorem for groups together with some straightforward consequences, stated as Theorem 3.3. We denote by $[G{:}H]$ the index of a subgroup H in the group G.

Lemma 3.1 *Let (Z, G) be a transitive action $z \in Z$ and $N = \mathrm{Stab}_G(z)$. Then (Z, G) is isomorphic to the natural action of G on the cosets of N, namely $(G/N, G)$. The action isomorphism is (p, Id), where $\mathrm{Id}\colon G \to G$ is the identity automorphism and $p\colon Z \to G/N$ is defined by $p\colon z \cdot g \mapsto Ng$, for every $g \in G$.*

Lemma 3.2 *Let G be a group, $N \leqslant G$, $q \in \mathrm{Aut}(G)$, and $w \in G$. Then there exists a permutation p on the right cosets in G/N taking N to Nw, such that*

$$(p, q)\colon (G/N, G) \to (G/N, G)$$

is an action isomorphism if and only if $q(N) = w^{-1}Nw$. Furthermore, p is defined by $p\colon Ng \mapsto Nwq(g)$, for every $g \in G$.

Theorem 3.3 *Let G and G' be groups, $f\colon G \to G'$ an epimorphism,*

$$\mathcal{A} = \{K : \ker f \leqslant K \leqslant G\} \quad \text{and} \quad \mathcal{B} = \{K' : K' \leqslant G'\}.$$

The mapping $F\colon \mathcal{A} \to \mathcal{B}$ defined by $F\colon K \mapsto f(K)$ is a bijection. Under this bijection normal subgroups correspond to normal subgroups. If $K \in \mathcal{A}$, $K' \in \mathcal{B}$, and $F(K) = K'$, then for any $w \in G$, $F(w^{-1}Kw) = f(w)^{-1}K'f(w)$, and for any $v \in G'$ and any $z \in f^{-1}(v)$, $F^{-1}(v^{-1}K'v) = z^{-1}Kz$. Furthermore, if $[G{:}K]$ is finite, $[G{:}K] = [G'{:}K']$ and for any two groups $K, H \in \mathcal{A}$, if $K \leqslant H$, then $F(K) \leqslant F(H)$.

For a polytope $\mathcal{P}$ of rank n we define the involutions $s_i \in \mathrm{Sym}(\mathcal{F}(\mathcal{P}))$, $i = 0, \dots, n-1$, where for each $\Phi \in \mathcal{F}(\mathcal{P})$, $s_i\colon \Phi \mapsto \Phi^i$. The involutions s_i generate a subgroup $\mathcal{M}(\mathcal{P})$ of $\mathrm{Sym}(\mathcal{F}(\mathcal{P}))$ which we call the *monodromy group* of $\mathcal{P}$. As already noted by Hartley [8], the monodromy group is a quotient of the Coxeter group $\mathcal{C} = [\infty, \dots, \infty]$ generated by involutions $s_0, \dots, s_{n-1}$ (note that here we use the labels s_i as words in $\mathcal{C}$). The quotient epimorphism $f_{\mathcal{P}}\colon \mathcal{C} \to \mathcal{M}(\mathcal{P})$ maps the generating word s_i to the generator labelled as s_i in $\mathcal{M}(\mathcal{P})$. The epimorphism defines the action of $\mathcal{C}$ on $\mathcal{F}(\mathcal{P})$ by $\Phi \cdot w := \Phi^{f_{\mathcal{P}}(w)}$, and $(\mathrm{Id}, f_{\mathcal{P}})\colon (\mathcal{F}(\mathcal{P}), \mathcal{C}) \to (\mathcal{F}(\mathcal{P}), \mathcal{M}(\mathcal{P}))$ an action epimorphism (where Id denotes the identity mapping). It follows from Lemma 2.1 that

$$(\Phi \cdot w)\phi = \Phi^{f_{\mathcal{P}}(w)}\phi = (\Phi\phi)^{f_{\mathcal{P}}(w)} = (\Phi\phi) \cdot w,$$

for any $\phi \in \mathrm{Aut}(\mathcal{P})$, $\Phi \in \mathcal{F}(\mathcal{P})$, and $w \in \mathcal{C}$.

We now fix a flag $\Phi \in \mathcal{F}(\mathcal{P})$ and henceforth call it a *base flag* of $\mathcal{P}$. The strong flag connectivity of $\mathcal{P}$ implies the transitivity of $\mathcal{M}(\mathcal{P})$ on flags. From this and the equation above we see that an automorphism is completely determined by the image of the base flag. For $w \in \mathcal{C}$, we use α_w to denote the automorphism taking Φ to $\Phi \cdot w$, whenever such an automorphism exists. Let $\mathrm{Stab}_H(\Phi)$ denote the stabilizer of Φ in H, and let $\mathrm{Norm}_H(K)$ denote the normalizer of K in H. In a sequence of theorems in [9] (Theorem 3.4, Lemma 3.6, Theorem 3.6), Hartley essentially proved the following result. Similar claims in different contexts were also proved in [2, 20].

Proposition 3.4 *An automorphism α_w taking a base flag Φ to $\Phi \cdot w$ exists if and only if $w \in \mathrm{Norm}_{\mathcal{C}}(\mathrm{Stab}_{\mathcal{C}}(\Phi))$. Furthermore, since $\alpha_w = \alpha_v$ if and only if $w^{-1}v \in \mathrm{Stab}_{\mathcal{C}}(\Phi)$, the mapping $w \mapsto \alpha_w$ induces the isomorphism*

$$\mathrm{Norm}_{\mathcal{C}}(\mathrm{Stab}_{\mathcal{C}}(\Phi))/\,\mathrm{Stab}_{\mathcal{C}}(\Phi) \cong \mathrm{Aut}(\mathcal{P}).$$

In what follows we shall be dealing with more than one polytope at a time, and hence it is convenient to define $\mathcal{C}_{\mathcal{P},\Phi} := \mathrm{Stab}_{\mathcal{C}}(\Phi)$ (or simply $\mathcal{C}_\Phi$, when $\mathcal{P}$ is clear from the context). Note that since for different polytopes the group $\mathcal{C}$ acts on different sets of flags, the notation introduced above denotes stabilizers of different actions of the same group. Since $f_{\mathcal{P}}$ is an epimorphism, it follows that $\mathcal{C}_\Phi = f_{\mathcal{P}}^{-1}(\mathrm{Stab}_{\mathcal{M}(\mathcal{P})}(\Phi))$. Hartley [8] showed that the polytope $\mathcal{P}$ is isomorphic to the quotient $\mathcal{U}/\mathcal{C}_\Phi$, where $\mathcal{U}$ is the universal regular polytope corresponding to the (universal string) Coxeter group $\mathcal{C} = [\infty, \dots, \infty]$ of the same rank. On the other hand, McMullen and Schulte [15, 16] proved that by taking a subgroup $N \leqslant \mathcal{C}$ which admits certain conditions, the quotient $\mathcal{U}/N$ yields a polytope. The construction of the poset from the action of the monodromy group can be found in [8, 15, 16].

We shall be making use of the following proposition proved by Hartley [8].

Proposition 3.5 *Two polytopes $\mathcal{P}$ and $\mathcal{Q}$, both of which are quotients of $\mathcal{C}$, with base flags Φ and Ψ, respectively, are isomorphic if and only if $\mathcal{C}_{\mathcal{P},\Phi}$ and $\mathcal{C}_{\mathcal{Q},\Psi}$ are conjugate in $\mathcal{C}$.*

Proposition 3.4 naturally extends to the following.

Proposition 3.6 *Let $\mathcal{P}$ be a polytope with finite number of flag-orbits under the action of its automorphism group, and let Φ be its base flag. Let $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$ and $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$. Then $|\,\mathrm{Orb}(\mathcal{P})| = [\mathcal{C}:\mathcal{N}]$. Furthermore, all orbits are of the same cardinality. In particular, if the polytope is finite, the size of each orbit equals $|\,\mathrm{Aut}(\mathcal{P})|$ and*

$$|\,\mathrm{Orb}(\mathcal{P})| = \frac{|\mathcal{F}(\mathcal{P})|}{|\,\mathrm{Aut}(\mathcal{P})|} = [\mathcal{M}(\mathcal{P})\colon \mathrm{Norm}_{\mathcal{M}(\mathcal{P})}(\mathrm{Stab}_{\mathcal{M}(\mathcal{P})}(\Phi))].$$

Proof Let $\Psi_1, \Psi_2 \in \mathcal{F}(\mathcal{P})$. Then $\Psi_1 = \Phi \cdot w$, $\Psi_2 = \Phi \cdot v$, for some $v, w \in \mathcal{C}$. By Proposition 3.4, Ψ_1 and Ψ_2 are in the same orbit if and only if there exists $m \in \mathcal{N}$ such that $\Psi_1\alpha_m = (\Phi \cdot w)\alpha_m = \Phi \cdot mw = \Phi \cdot v = \Psi_2$, or equivalently, $mwv^{-1} \in N$, implying $\mathcal{N}w = \mathcal{N}v$. Conversely, if the latter is true, then $wv^{-1} \in \mathcal{N}$ and $\Psi_2\alpha_{wv^{-1}} = \Psi_1$. We conclude that the orbits are in one-to-one correspondence with cosets in $\mathcal{C}/\mathcal{N}$. Since the action of the automorphism group is regular (sharply transitive) on the orbits, the cardinality of every orbit equals the cardinality of $\mathrm{Aut}(\mathcal{P})$. The last equality follows by Theorem 3.3. ∎

The structure of the automorphism group of the universal string Coxeter group $\mathcal{C}$ of rank n was determined by James [11]. For a given $d \in \mathrm{Aut}(\mathcal{C})$ we now define a polytope operation induced by d. Let $f_{\mathcal{P}^d} := f_{\mathcal{P}} \circ \mathrm{d}$ and $s_i' := f_{\mathcal{P}^d}(s_i)$. Changing from $f_{\mathcal{P}}$ to $f_{\mathcal{P}^d}$, while keeping $\mathcal{M}(\mathcal{P})$ and its action on $\mathcal{F}(\mathcal{P})$, has an effect equivalent to choosing a new set of labelled generators of the same permutation group acting on the same set of flags. This procedure introduces different incidences between flags yielding a new structure, which we denote by $\mathcal{P}^d$. We will not be interested in structures that with such induced incidences are not polytopes, and henceforth assume $\mathcal{P}^d$ is a polytope.

If $\mathcal{P} \cong \mathcal{P}^d$, we say that $\mathcal{P}$ is *self-invariant with respect to d* or *d-invariant*. In Theorem 3.7(iv) we shall prove that this isomorphism induces a bijection δ on $\mathcal{F}(\mathcal{P})$ such that $(\Psi \cdot a)\delta = (\Psi\delta) \cdot d(a)$, for each $a \in \mathcal{C}$ and each $\Psi \in \mathcal{F}(\mathcal{P})$. The bijection δ is called a *d-automorphism* of $\mathcal{P}$. Note that $\mathcal{P}$ and $\mathcal{P}^d$ are defined on the same set of flags $\mathcal{F}(\mathcal{P})$. The change induced by d results in a new action of $\mathcal{C}$ on the same set of flags. In addition to the previously defined action $\Phi \cdot w = \Phi^{f_{\mathcal{P}}(w)}$, we now also have the following action $\Phi * w = \Phi^{f_{\mathcal{P}^d}(w)} = \Phi^{f_{\mathcal{P}}(d(w))}$.

The well-known dual operation on a polytope $\mathcal{P}$ can be represented by $d \in \mathrm{Aut}(\mathcal{C})$ defined by the mapping $d\colon s_i \mapsto s_{n-1-i}$, $i = 0, \dots, n-1$. By reversing the sequence of generators, we essentially describe the incidences of the flags in the poset obtained by reversing the poset representing $\mathcal{P}$ to get the poset $\mathcal{P}^d$, which is isomorphic to the dual of $\mathcal{P}$. In this case, d-invariant polytopes are exactly self-dual ones and d-automorphisms are dualities.

As another example, consider a rank 3 polytope (polyhedron) and the automorphism $p \in \mathrm{Aut}(\mathcal{C})$ defined by the mapping (see [13,24]) $p\colon (s_0, s_1, s_2) \mapsto (s_0s_2, s_1, s_2)$. The induced operation is precisely the Petrie dual (or Petrial) on polyhedra (or maps). It can easily be seen that d, as defined above, and p generate a subgroup of $\mathrm{Aut}(\mathcal{C})$ isomorphic to S_3. All automorphisms in that subgroup induce operations. In particular, the automorphism $p \circ d \circ p = d \circ p \circ d$ induces the so-called operation *opposite* (*i.e.,* $\mathrm{opp}(\mathcal{P})$) of a polyhedron $\mathcal{P}$ (or a map). We observe that if $\mathcal{P}$ is d-invariant and p-invariant, then it is self-invariant for any element in the group $\langle p, d\rangle$.

For rank n greater than 3 (see [11]), one can generalize the above defined automorphism to $p \in \mathrm{Aut}(\mathcal{C})$ given on the generators by

$$p\colon (s_0, \dots, s_{n-4}, s_{n-3}, s_{n-2}, s_{n-1}) \mapsto (s_0, \dots, s_{n-4}, s_{n-3}s_{n-1}, s_{n-2}, s_{n-1}),$$

that induces the operation on a poset representing a rank n polytope. We call this operation the *generalized Petrie dual* of a polytope. Note that the operation on the polytope induced by p, when restricted to a co-face of rank 3, is precisely the Petrial operation on the co-face. It is easy to see that for $n \geqslant 4$ the automorphisms p and d generate a subgroup of $\mathrm{Aut}(\mathcal{C})$ isomorphic to the dihedral group $D_4 = \langle d, p \mid d^2, p^2, (dp)^4\rangle$. Note that for $n \geqslant 6$, both operations induced by $d \circ p \circ d$ and $(d \circ p)^2 = (p \circ d)^2$, restricted to a face of rank 3, yield the same operation, namely the opposite operation.

The following theorem, characterising d-automorphisms of a d-invariant polytope, is a powerful tool in analysis of self-invariance.

Theorem 3.7 *Assume $\mathcal{P}$ is a polytope and Φ its base flag. Let $N = \mathcal{C}_\Phi$, $G = f_{\mathcal{P}}(N) =$ $\mathrm{Stab}_{\mathcal{M}}(\Phi)$, $\mathcal{M} = \mathcal{M}(\mathcal{P})$, and $d \in \mathrm{Aut}(\mathcal{C})$. Then the following are equivalent.*

(i) $\mathcal{P} \cong \mathcal{P}^d$.
(ii) $d(N) = w^{-1}Nw$ *for some* $w \in \mathcal{C}$.
(iii) *For some* $v \in \mathcal{M}$ *there exists* $d' \in \operatorname{Aut}(\mathcal{M})$ *such that* $d'(G) = v^{-1}Gv$ *and* $f_{\mathcal{P}} \circ d = d' \circ f_{\mathcal{P}}$.
(iv) *There exists a bijection* $\delta \in \operatorname{Sym}(\mathcal{F})$ *such that for every flag* $\Psi \in \mathcal{F}(\mathcal{P})$ *and every* $a \in \mathcal{C}$

$$(\Psi^{f_{\mathcal{P}}(a)})\delta = (\Psi \cdot a)\delta = (\Psi\delta) * a = (\Psi\delta)^{f_{\mathcal{P}}(d(a))}.$$

(v) *There exists* $w \in \mathcal{C}$, *such that* $\Delta\colon \operatorname{Aut}(\mathcal{P}) \to \operatorname{Aut}(\mathcal{P})$, $\alpha_r \mapsto \alpha_{wd(r)w^{-1}}$, *is a group automorphism.*

Proof Proposition 3.5 implies that $\mathcal{P} \cong \mathcal{P}^d$ if and only if there exists $r \in \mathcal{C}$, such that $N = r^{-1}\mathcal{C}_{\mathcal{P}^d,\Phi}r$. Note that

$$\mathcal{C}_{\mathcal{P}^d,\Phi} = f_{\mathcal{P}^d}^{-1}(G) = (f_{\mathcal{P}} \circ d)^{-1}(G) = d^{-1}(f_{\mathcal{P}}^{-1}(G)) = d^{-1}(N).$$

Therefore $d(N) = d(r^{-1}d^{-1}(N)r) = d(r)^{-1}Nd(r)$, and by taking $w = d(r)$, (ii) is equivalent to (i).

Now we show that (ii) is equivalent to (iii). By Theorem 3.3 we have a bijective correspondence between subgroups of $\mathcal{C}$ containing $\ker f_{\mathcal{P}}$ and subgroups of $\mathcal{M}$. The correspondence between two subgroups $H \leqslant \mathcal{M}$ and $K \leqslant \mathcal{C}$ is denoted by $H \leftrightarrow K$, meaning that $f_{\mathcal{P}}(K) = H$ and $f_{\mathcal{P}}^{-1}(H) = K$. The definiton of the action of $\mathcal{C}$ on $\mathcal{F}(\mathcal{P})$ through $f_{\mathcal{P}}$ and $\mathcal{M}$ implies $G = \operatorname{Stab}_{\mathcal{M}}(\Phi) \leftrightarrow \mathcal{C}_{\mathcal{P},\Phi} = N$.

Assuming (iii), $v^{-1}Gv = d'(G) = d'(f_{\mathcal{P}}(N)) = f_{\mathcal{P}}(d(N)) \leftrightarrow d(N)$. By Theorem 3.3, $v^{-1}Gv \leftrightarrow w^{-1}Nw$ for any $w \in f_{\mathcal{P}}^{-1}(v)$, therefore $d(N) = w^{-1}Nw$, and (ii) follows.

Assuming (ii), it remains to show that d' exists. Since G is a stabilizer of a transitive permutation group, it follows that $\operatorname{Core}_{\mathcal{M}}(G) = \{1\}$ (*i.e.,* the maximal normal subgroup in $\mathcal{M}$ contained in G is trivial). By Theorem 3.3, $\ker f_{\mathcal{P}} = \operatorname{Core}_{\mathcal{C}}(N) = \operatorname{Core}_{\mathcal{C}}(w^{-1}Nw)$. Since $d \in \operatorname{Aut}(\mathcal{C})$ and $d(N) = w^{-1}Nw$, it follows that

$$d(\ker f_{\mathcal{P}}) = \ker f_{\mathcal{P}} \quad \text{and} \quad \ker f_{\mathcal{P}} = \ker(f_{\mathcal{P}} \circ d).$$

Let $q\colon \mathcal{C} \to \mathcal{C}/\ker f_{\mathcal{P}}$ be the natural epimorphism. By the first isomorphism theorem on groups, it follows that there exist unique group isomorphisms $i_1, i_2\colon \mathcal{M} \to \mathcal{C}/\ker f_{\mathcal{P}}$ such that $q = i_1 \circ f_{\mathcal{P}}$ and $q = i_2 \circ (f_{\mathcal{P}} \circ d)$. Defining $d' := i_2^{-1} \circ i_1$, (iii) follows. Note that for $a \in \mathcal{M}$, $d'(a) = f_{\mathcal{P}}(d(b))$ for any $b \in f_{\mathcal{P}}^{-1}(a)$.

Assume (ii) and (iii). Denote by (p, Id) the action isomorphism from Lemma 3.1 between $(\mathcal{F}(\mathcal{P}), \mathcal{M})$ and $(\mathcal{M}/G, \mathcal{M})$. Then by Lemma 3.2 and the fact that $d'(G) = v^{-1}Gv$ there exists a bijection $p'\colon \mathcal{M}/G \to \mathcal{M}/G$ such that (p', d') is an action isomorphism. Define $\delta := p^{-1} \circ p' \circ p$. Then (δ, d') is an action isomorphism from $(\mathcal{F}(\mathcal{P}), \mathcal{M})$ to $(\mathcal{F}(\mathcal{P}), \mathcal{M})$ such that $\Psi^r\delta = (\Psi\delta)^{d'(r)}$ for any $r \in \mathcal{M}$, and (iv) follows.

Statement (iv) says that (δ, Id) is an action isomorphism of the two actions $(\mathcal{F}(\mathcal{M}), \mathcal{C}, \cdot\,)$ and $(\mathcal{F}(\mathcal{M}), \mathcal{C}, *)$, or equivalently, (δ, d) is an action isomorphism of $(\mathcal{F}(\mathcal{M}), \mathcal{C}, \cdot\,)$ and $(\mathcal{F}(\mathcal{M}), \mathcal{C}, \cdot\,)$. Let $\Phi\delta = \Phi^a$ for some $a \in \mathcal{M}$. Note that since (δ, d) is an action isomorphism, $d(N) = \mathcal{C}_{\mathcal{P}^d,\Phi^a}$. But since $f_{\mathcal{P}}(N) = G$ and

$$f_{\mathcal{P}}(\mathcal{C}_{\mathcal{P}^d,\Phi^a}) = \operatorname{Stab}_{\mathcal{M}}(\Phi^a) = a^{-1}Ga,$$

then by Theorem 3.3, $\mathcal{C}_{\mathcal{P}^d,\Phi^a} = b^{-1}Nb$ for any $b \in f_{\mathcal{P}}^{-1}(a)$. Therefore $d(N) = b^{-1}Nb$, implying (ii).

Assuming (v), by Proposition 3.4, it follows that $r \in N$ if and only $wd(r)w^{-1} \in N$, implying that $d(N) = w^{-1}Nw$ and (ii).

Finally, assume (ii) to be true. Then $wd(N)w^{-1} = N$ and $d^{-1}(w)Nd^{-1}(w^{-1}) = d^{-1}(N)$. Denote by $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$. For any $r \in \mathcal{N}$ it follows that

$$\begin{aligned}(wd(r)w^{-1})^{-1}Nwd(r)w^{-1} &= wd(r)^{-1}w^{-1}Nwd(r)w^{-1} = wd(r^{-1})d(N)d(r)w^{-1}\\ &= wd(r^{-1}Nr)w^{-1} = wd(N)w^{-1} = N\end{aligned}$$

and $wd(r)w^{-1} \in \mathcal{N}$. Since also for any $a, b \in \mathcal{N}$,

$$\alpha_a = \alpha_b \Leftrightarrow ab^{-1} \in N \Leftrightarrow d(ab^{-1}) \in w^{-1}Nw \Leftrightarrow wd(a)w^{-1}(wd(b)w^{-1})^{-1} \in N,$$

it follows that the mapping Δ is well defined and one-to-one. From Proposition 3.4, it follows that $\alpha_{wv} = \alpha_w \alpha_v$, which implies that Δ is a homomorphism. For $x \in \mathcal{N}$, there exists $a \in \mathcal{C}$, such that $x = wd(a)w^{-1}$, namely $a = d^{-1}(w^{-1}xw)$. To show that Δ is onto, it remains to prove that $a \in \mathcal{N}$. But this is true, since

$$\begin{aligned}d^{-1}(w^{-1}xw)^{-1}Nd^{-1}(w^{-1}xw) &= d^{-1}(w^{-1}x^{-1})d^{-1}(w)Nd^{-1}(w^{-1})d^{-1}(xw)\\ &= d^{-1}(w^{-1}x^{-1})d^{-1}(N)d^{-1}(xw) = d^{-1}(w^{-1}x^{-1}Nxw)\\ &= d^{-1}(w^{-1}Nw) = d^{-1}(d(N)) = N.\end{aligned}$$

■

With $N = \mathcal{C}_\Phi$ as above, the next corollary follows immediately from the proof of Theorem 3.7.

Corollary 3.8 *Let $\mathcal{P}$ be a polytope and Φ its base flag. If any (and therefore all) of the equivalent statements from Theorem 3.7 are satisfied, then for any $w \in \mathcal{C}$ with $d(N) = w^{-1}Nw$, there exists a d-automorphism δ, such that*

$$\Phi\delta = \Phi \cdot w = \Phi * d^{-1}(w) \quad \text{and} \quad d'(G) = f_{\mathcal{P}}(w)^{-1}Gf_{\mathcal{P}}(w).$$

Furthermore, this exactly describes all d-automorphisms.

As we have seen in the previous section, the action of dualities on flags induces a permutation on the flag-orbits. A similar action occurs with other d-automorphisms. When dealing with self-invariant polytopes, we shall find the following more general theorem useful.

Theorem 3.9 *Let $d \in \mathrm{Aut}(\mathcal{C})$ and $\mathcal{P}$ be a d-invariant polytope with a base flag Φ. Let δ be a d-automorphism, $N := \mathcal{C}_{\mathcal{P},\Phi}$, $\mathrm{Conj}(N)$ the set of all conjugates of N in $\mathcal{C}$, and $\mathrm{Orb}(\mathcal{P})$ the set of all orbits of flags under the action of $\mathrm{Aut}(\mathcal{P})$. Then there exists a bijection $\Sigma\colon \mathrm{Conj}(N) \to \mathrm{Orb}(\mathcal{P})$ such that $\Sigma(d^i(K)) = \Sigma(K)\delta^i$, for any integer i and any $K \in \mathrm{Conj}(N)$.*

Proof Let $w \in \mathcal{C}$ be such that $\Phi\delta = \Phi \cdot w$. By Theorem 3.7, $d(N) = w^{-1}Nw$. Since $d \in \mathrm{Aut}(\mathcal{C})$, $d^i(K) \in \mathrm{Conj}(N)$ for any i and any $K \in \mathrm{Conj}(N)$.

Let O_Ψ denote the orbit of a flag Ψ under the action of $\mathrm{Aut}(\mathcal{P})$. If Ψ, Ψ' are two flags in the same orbit and $\gamma \in \mathrm{Aut}(\mathcal{P})$ such that $\Psi\gamma = \Psi'$, then for every $r \in \mathcal{C}$ it follows that $(\Psi \cdot r)\gamma = (\Psi\gamma) \cdot r = \Psi' \cdot r$. Therefore, the action $(\mathcal{F}(\mathcal{P}), \mathcal{C}, \cdot\,)$ induces an action on $\mathrm{Orb}(\mathcal{P})$ defined by $O_\Psi \cdot r = O_{\Psi \cdot r}$.

Define $\Sigma\colon a^{-1}Na \mapsto O_\Phi \cdot a$. Then for $a, b \in \mathcal{C}$,

$$\begin{aligned} b^{-1}Nb = a^{-1}Na &\Leftrightarrow (ab^{-1})^{-1}Nab^{-1} = N \\ &\Leftrightarrow ab^{-1} \in \mathrm{Norm}_{\mathcal{C}}(N) \Leftrightarrow \alpha_{ab^{-1}} \in \mathrm{Aut}(\mathcal{P}). \end{aligned}$$

The latter is equivalent to saying that $\Phi \cdot a$ is in the same orbit as $\Phi \cdot b$, which in turn is equivlant to $\mathcal{O}_\Phi \cdot a = \mathcal{O}_\Phi \cdot b$ (note that we used Proposition 3.4). Therefore, Σ is well defined and one-to-one. Obviously, it is onto. It suffices to prove $\Sigma(d(K)) = \Sigma(K)\delta$ for every $K \in \mathrm{Conj}(N)$, as $\Sigma(d^i(K)) = \Sigma(d(d^{i-1}(K))) = \Sigma(d^{i-1}(K))\delta$ for $i > 1$.

Since $d(N) = w^{-1}Nw$, then

$$d(a^{-1}Na) = d(a^{-1})d(N)d(a) = d(a^{-1})w^{-1}Nwd(a) = (wd(a))^{-1}Nwd(a),$$

and therefore $\Sigma(d(a^{-1}Na)) = O_\Phi \cdot wd(a)$. On the other hand,

$$(\Phi \cdot a)\delta = \Phi\delta \cdot d(a) = \Phi \cdot wd(a)$$

(by Theorem 3.7 and Corollary 3.8). The claim of the theorem is implied by the following simple observation: $O_{\Phi\delta} = O_\Phi\delta$, since $\alpha_v \in \mathrm{Aut}(\mathcal{P})$ exists if and only if $\alpha_{d(v)} \in \mathrm{Aut}(\mathcal{P})$ exists, so that $(\Phi\alpha_v)\delta = (\Phi \cdot v)\delta = (\Phi\delta) \cdot d(v) = (\Phi\delta)\alpha_{d(v)}$. ■

The above theorem implies that the orbits in $\mathrm{Orb}(\mathcal{P})$ can be labelled by conjugates in $\mathrm{Conj}(N)$, $N := \mathcal{C}_{\mathcal{P},\Phi}$, in such a way that the action of an automorphism $d \in \mathrm{Aut}(\mathcal{C})$ on conjugates coincides with the action of any d-automorphism δ on the orbits.

According to Theorem 3.7, a d-automorphism is determined by an image of the base flag, similarly as it is the case with automorphisms. Denote by δ_w the d-automorphism taking a base flag $\Phi \mapsto \Phi \cdot w$, $w \in \mathcal{C}$. Note that for $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$, $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$, and $a \in \mathcal{C}$, it follows that $a^{-1}Na = w^{-1}Nw$ if and only if $\mathcal{N}a = \mathcal{N}w$. Theorem 3.7 and Corollary 3.8 now imply the following.

Corollary 3.10 *Let $\mathcal{P}$ be a polytope and Φ a base flag. If δ_w, $w \in \mathcal{C}$, is a d-automorphism, then all d-automorphisms are of the from δ_v, for $v \in \mathrm{Norm}_{\mathcal{C}}(\mathrm{Stab}_{\mathcal{C}}(\Phi))w$.*

Let $d \in \mathrm{Aut}(\mathcal{C})$ be the automorphism defining the dual operation. From Theorem 3.9 we conclude that a self-dual polytope $\mathcal{P}$ is properly self-dual if d fixes all conjugates of $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$, and improperly self-dual otherwise.

Let $d \in \mathrm{Aut}(\mathcal{C})$ be of a finite order $m > 1$ and δ a d-automorphism. Since for any $v \in \mathcal{C}$, $(\Phi \cdot v)\delta^2 = ((\Phi\delta) \cdot d(v))\delta = (\Phi\delta^2) \cdot d^2(v)$, by induction $(\Phi \cdot v)\delta^k = (\Phi\delta^k) \cdot d^k(v)$ for any integer k. Hence δ^k is a d^k-automorphism for any integer k; in particular δ^m is an automorphism. A d-automorphism δ such that $\delta^m = 1$ is said to be of degree m.

Note that when d defines the dual operation, the d-automorphism of degree 2 is called a polarity.

The following theorem describes the existence of d-automorphisms of degree m in terms of existence of certain elements in a monodromy group.

Theorem 3.11 *Let $d \in \mathrm{Aut}(\mathcal{C})$ be of finite order m, $\mathcal{P}$ a d-invariant polytope with a base flag Φ, δ_w a d-automorphism for some $w \in \mathcal{C}$, $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$, $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$, and $\mathcal{M} = \mathcal{M}(\mathcal{P})$. Then $\mathcal{P}$ has a d-automorphism of degree m if and only if there exists $a \in \mathcal{N}w$ such that $\prod_{i=0}^{m-1} d^i(a) \in N$. This is true if and only if there exists $v \in \mathrm{Norm}_{\mathcal{M}}(\mathrm{Stab}_{\mathcal{M}}(\Phi)) f_{\mathcal{P}}(w)$ such that $\prod_{i=0}^{m-1} d'^i(v) \in \mathrm{Stab}_{\mathcal{M}}(\Phi)$ for d' defined in Theorem 3.7.*

Proof According to Corollary 3.10, any d-automorphism can be denoted by δ_a for some $a \in \mathcal{N}w$. By Theorem 3.7, $\Phi\delta_a = \Phi \cdot a$, $\Phi\delta_a^2 = (\Phi \cdot a)\delta_a = \Phi \cdot ad(a)$, and by induction $(\Phi)\delta_a^j = \Phi \cdot \prod_{i=0}^{j-1} d^i(a)$. Since $d^m = 1$, and therefore $\delta_a^m \in \mathrm{Aut}(\mathcal{P})$, to prove $\delta_a^m = 1$ it suffices to show that $\Phi\delta_a^m = \Phi$, *i.e.*, $p := \prod_{i=0}^{m-1} d^i(a) \in N$. Therefore δ_a is of degree m if and only if $p \in N$. By Theorems 3.7 and 3.3, this is true if and only if

$$f_{\mathcal{P}}(p) = \prod_{i=0}^{m-1} f_{\mathcal{P}}(d^i(a)) = \prod_{i=0}^{m-1} d'^i(f_{\mathcal{P}}(a)) \in \mathrm{Stab}_{\mathcal{M}}(\Phi).$$

From $f_{\mathcal{P}} \circ d = d' \circ f_{\mathcal{P}}$, it can be easily verified by induction that $f_{\mathcal{P}} \circ d^i = d'^i \circ f_{\mathcal{P}}$, for $i \geqslant 0$. Note that by Theorem 3.3, $a \in \mathcal{N}w$ if and only if

$$f_{\mathcal{P}}(a) \in \mathrm{Norm}_{\mathcal{M}}(\mathrm{Stab}_{\mathcal{M}}(\Phi)) f_{\mathcal{P}}(w). \qquad \blacksquare$$

4 Two-Orbit and Edge-Transitive Polyhedra

The most symmetrical polytopes are those that have exactly one orbit of flags under the action of the automorphism group. Such polytopes are said to be *regular*. A polytope is said to be *chiral* if it has exactly two distinct flag-orbits with adjacent flags in distinct orbits. Both regular and chiral polytopes are equivelar in the following sense. A polytope $\mathcal{P}$ of rank n is said to be *equivelar* if, for each $i = 1, \ldots, n-1$ and each flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$, the section F_{i+1}/F_{i-2} is a p_i-gon. Note that a polytope $\mathcal{P}$ of rank 3 is equivelar if all of its 2-faces are p-gons and all of its vertex-figures are q-gons, that is, $\mathcal{P}$ is of Schläfli type $\{p, q\}$. A polytope $\mathcal{P}$ is i*-face-transitive* if $\mathrm{Aut}(\mathcal{P})$ is transitive on the set of its i-faces. If a polytope of rank n is i-face-transitive for all $i = 0, \ldots, n-1$, then it is called *fully-transitive*. Clearly, fully-transitive polytopes of rank 3 are equivelar.

Moving away from regular polytopes, we now consider polytopes with precisely two flag-orbits and refer to them as *two-orbit polytopes*. The following lemma gives an important property which cannot be extended to polytopes with more than two orbits.

Lemma 4.1 *Let $\mathcal{P}$ be a two-orbit polytope and Φ a flag of $\mathcal{P}$. If for $i \in \{0, \ldots, n-1\}$, Φ and Φ^i are in the same flag-orbit, then any flag Ψ and its i-adjacent flag Ψ^i are in the same orbit.*

Proof Let O_1 and O_2 be the two orbits. Assume that $\Psi \in \mathcal{F}(\mathcal{P})$, such that Ψ and Ψ^i are in different orbits. Since $\mathcal{P}$ is a two-orbit polytope, without loss of generality we may assume $\Phi, \Phi^i, \Psi \in O_1$, and $\Psi^i \in O_2$. There exist $\alpha, \rho_i \in \mathrm{Aut}(\mathcal{P})$ such that $\Phi\alpha = \Psi$ and $\Phi\rho_i = \Phi^i$. Hence, $\Psi\alpha^{-1}\rho_i\alpha = \Phi\rho_i\alpha = \Phi^i\alpha = (\Phi\alpha)^i = \Psi^i$, implying that $\Psi^i \in O_1$, which is a contradiction. ■

We note that if $\mathcal{P}$ is an improperly self-dual two-orbit polytope, then $\mathcal{D}(\mathcal{P})$ is regular (sharply transitive) on flags. Furthermore, if Φ is a fixed flag and δ a duality, then there is at least one $i \in \{0, \dots, n-1\}$, such that Φ^i is in the same orbit as $\Phi\delta$. Then at least one of the dualities δ_{s_i} exists.

In the remainder of the paper we consider only rank 3 polytopes, that is, polyhedra, but most of what we state will also apply to maps.

Let $\mathcal{P}$ be a two-orbit polyhedron. Lemma 4.1 enables us to divide polyhedra with two distinct orbits into the following (disjoint) classes.

- $\mathcal{P}$ is in class 2 if no adjacent flags of $\mathcal{P}$ are in the same orbit. In this case $\mathcal{P}$ is chiral.
- $\mathcal{P}$ is in class 2_i with $i \in \{0, 1, 2\}$ if every flag of $\mathcal{P}$ and its i-adjacent are in the same orbit, but every flag and its j-adjacent are in distinct orbits, for $j \neq i$. In these classes polytopes are fully-transitive.
- $\mathcal{P}$ is in class $2_{i,j}$ with $i, j \in \{0, 1, 2\}$ and $i \neq j$ if every flag of $\mathcal{P}$ and its i- and j-adjacent are in the same orbit, but every flag and its k-adjacent, for $k \neq i, j$, are in distinct orbits.

It follows from this classification and Lemmas 2.1 and 4.1 that the duals of polyhedra in classes 2_i, $2_{i,j}$ belong to the classes 2_{2-i}, $2_{2-i,2-j}$, respectively. Hence polyhedra in the classes 2_0, 2_2, $2_{0,1}$, and $2_{1,2}$ cannot be self-dual.

Lemma 4.1 also implies that two-orbit polyhedra in all classes but $2_{0,2}$ are edge-transitive, *i.e.,* 1-face-transitive). In [6, 22] edge-transitive maps were classified into 14 types according to possession of certain automorphisms. Edge-transitive maps can have 1, 2, or 4 flag-orbits under the action of the automorphism group. In Table 1 we list the classes of two-orbit polyhedra and the corresponding types according to Graver and Watkins [6] (G–W type), when the polyhedra in the classes are edge-transitive. The ranks of the faces, on which polyhedra in a given class are transitive, are listed in the second row of Table 1.

Class	2	2_0	2_1	2_2	$2_{0,2}$	$2_{0,1}$	$2_{1,2}$
Transitivity ranks	0, 1, 2	0, 1, 2	0, 1, 2	0, 1, 2	0, 2	0, 1	1, 2
Dual classes	2	2_2	2_1	2_0	$2_{0,2}$	$2_{1,2}$	$2_{0,1}$
G–W type	2^Pex	2^*ex	2^P	2ex	–	2^*	2

Table 1: Classes of two-orbit polyhedra.

Recall the notation introduced in Section 3, where $\Phi\alpha_w = \Phi \cdot w$ for $w \in \mathcal{C}$ and Φ a base flag of a polytope $\mathcal{P}$. Here $w \in \mathcal{C}$ is a word in the generators s_0, s_1, and s_2.

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For convenience, in Tables 2 and 3, we shall abbreviate the notation replacing s_i in subscripts of α and δ by i.

By Lemma 4.1, given a class of a two-orbit polyhedron $\mathcal{P}$, the local arrangement of flag-orbits in the neighbourhood of any flag is completely determined and extends to the whole map in a unique way. Up to duality, all possible arrangements corresponding to classes of two-orbit polyhedra are given in Figure 2.

For each polyhedron in class C, this arrangement of flag orbits in the neighborhood of a base flag Φ can in turn be used to select generators for the automorphism group of the polyhedron. Our selection of generators is listed in the second column of Table 2, and we refer to them as the *distinguished generators with respect to* Φ for the automorphism group of the polyhedron. The distinguished generators satisfy certain relations which are common to all polyhedra in the class. These are listed in the third column of Table 2.

Consider now the set S_C of elements $w \in \mathcal{C}$ such that α_w is a distinguished generator for the automorphism group of some polyhedron in class C. The *universal group* for the class C is the group generated by S_C and the presentation given in the third column of Table 2. The automorphism groups of two-orbit polyhedra in each class are quotients of the corresponding universal groups.

To improve readability of the presentations of the universal groups given in the third column of the table, we label subsequent generators of the second column by a, b, c, d, where unnecessary generators are simply omitted. Note that polyhedra in classes 2, 2_i, $i = 0, 1, 2$, and $2_{0,2}$ have Schläfli type $\{p, q\}$. Polyhedra in class $2_{0,1}$ have two face orbits with p_1- and p_2-gons while its vertex-figures are all q-gons. Dually, polyhedra in class $2_{1,2}$ have p-gonal faces, while its vertex-figures are q_1- and q_2-gons.

As noted in the introduction, regular and chiral self-dual polyhedra possess polarities, that is, are of degree 2. Chiral polyhedra can be properly or improperly self-dual (see [10, §3]). We now discuss the degree of the remaining self-dual two-orbit polyhedra, that is, the polyhedra in the classes 2_1 and $2_{0,2}$. As seen in Table 1, they are at least vertex and face-transitive and hence are equivelar. As we shall see, such polyhedra are of degree 2 or 4. Using Theorem 3.11 we have conducted a brief survey of known self-dual edge-transitive polyhedra and found out that, although most of

Class	Generators a, b, c, d	Presentation of the universal group
2	α_{01}, α_{12}	$\langle a, b \mid a^p, b^q, (ab)^2 \rangle$
2_0	α_0, α_{12}	$\langle a, b \mid a^2, b^q, (bab^{-1}a)^{p/2} \rangle$
2_1	$\alpha_1, \alpha_{010}, \alpha_{02}$	$\langle a, b, c \mid a^2, b^2, c^2, (ab)^{p/2}, (acbc)^{q/2} \rangle$
2_2	α_2, α_{10}	$\langle a, b \mid a^2, b^p, (bab^{-1}a)^{q/2} \rangle$
$2_{0,2}$	$\alpha_0, \alpha_2, \alpha_{101}, \alpha_{121}$	$\langle a, b, c, d \mid a^2, b^2, c^2, d^2, (ac)^2, (ac)^{p/2}, (bd)^{q/2}, (cd)^2 \rangle$
$2_{0,1}$	$\alpha_0, \alpha_1, \alpha_{212}$	$\langle a, b, c \mid a^2, b^2, c^2, (ab)^{p_1}, (bc)^{q/2}, (ac)^{p_2} \rangle$
$2_{1,2}$	$\alpha_1, \alpha_2, \alpha_{010}$	$\langle a, b, c \mid a^2, b^2, c^2, (ab)^{q_1}, (ac)^{p/2}, (bc)^{q_2} \rangle$

Table 2: Generators and relations of the universal groups for two-orbit polyhedra.

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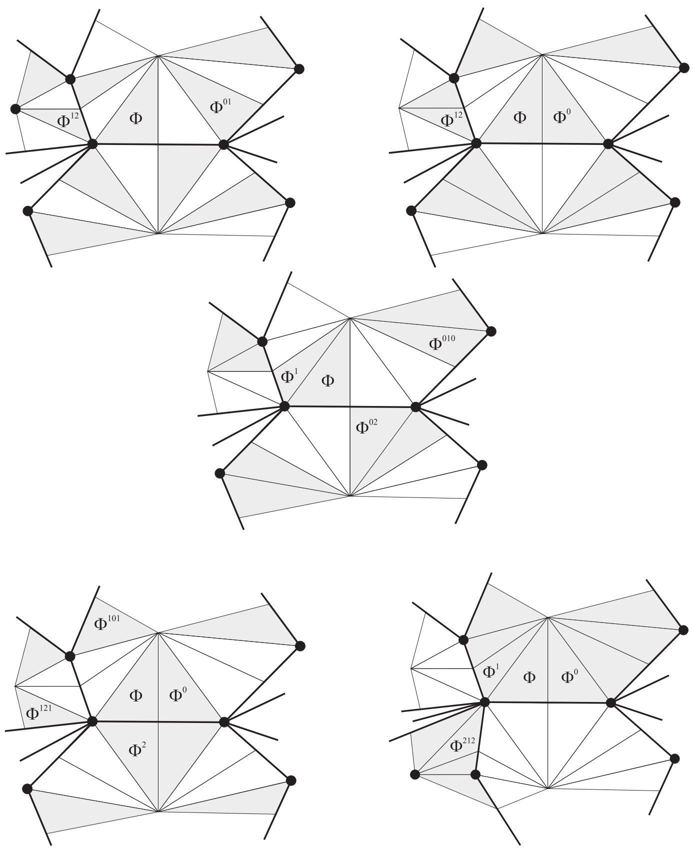

Figure 2: Edges of polyhedra in classes 2, 2_0, 2_1, $2_{0,2}$ and $2_{0,1}$, respectively.

them are of degree 2, some, such as the polyhedra described in Section 5, contain no polarities, and therefore must be of degree 4.

Theorem 4.2 *Let $\mathcal{P}$ be a two-orbit polyhedron in class 2_1, let Φ be a base flag of $\mathcal{P}$, and let $\rho_1 := \alpha_{s_1}$, $\rho := \alpha_{s_0 s_1 s_0}$, $\tau := \alpha_{s_0 s_2}$ be the distinguished generators of* $\mathrm{Aut}(\mathcal{P})$ *with respect to Φ.*

(i) *$\mathcal{P}$ is properly self-dual if and only if there exists a group automorphism*

$$\tilde{\delta}\colon \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$$

of order 2 such that

$$\tilde{\delta}(\rho_1) = \rho_1, \quad \tilde{\delta}(\rho) = \tau\rho\tau, \quad \tilde{\delta}(\tau) = \tau.$$

(ii) *$\mathcal{P}$ is improperly self-dual if and only if there exists a group automorphism*

$$\widehat{\delta}\colon \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$$

of order 4 *such that*

$$\widehat{\delta}(\rho_1) = \rho, \quad \widehat{\delta}(\rho) = \tau\rho_1\tau, \quad \widehat{\delta}(\tau) = \tau.$$

Proof The orbits O_Φ and $O_\Phi \cdot s_0$ are distinct. By Theorem 3.9, $\mathcal{P}$ is properly self-dual, if $d(N) = N$ and improperly self-dual if $d(N) = s_0 N s_0$, where $d \in \mathrm{Aut}(\mathcal{C})$ is the automorphism that induces the dual operation. The proof follows from Theorem 3.7(v). For part (i) take $\tilde{\delta}\colon \alpha_w \mapsto \alpha_{d(w)}$ and for (ii) take $\widehat{\delta}\colon \alpha_w \mapsto \alpha_{s_0 d(w) s_0}$. ∎

We observe that if $\mathcal{P}$ is an improperly self-dual two-orbit polyhedron in class 2_1, then its left and right Petrie polygons must have the same length. To see this, we note that the right Petrie motion is given by, for example, $\Pi_R = \rho\tau$, and therefore the left Petrie motion by $\Pi_L = \rho_1\tau$. Since $\mathcal{P}$ is improperly self-dual, we can apply the automorphism $\widehat{\delta} \in \mathrm{Aut}(\mathrm{Aut}(\mathcal{P}))$ from Theorem 4.2(ii) on Π_L to obtain $\widehat{\delta}(\Pi_L) = \widehat{\delta}(\rho_1\tau) = \rho\tau = \Pi_R$. Therefore the orders of Π_L and Π_R are equal. However, note that if $\mathcal{P}$ is properly self-dual of type 2_1, then the left and right Petrie polygons need not have the same length.

Theorem 4.3 *Let $\mathcal{P}$ be a two-orbit polyhedron in class $2_{0,2}$, Φ be a base flag and let $\rho_0 := \alpha_{s_0}$, $\rho_2 := \alpha_{s_2}$, $r_0 := \alpha_{s_1 s_0 s_1}$, and $r_2 := \alpha_{s_1 s_2 s_1}$ be the distinguished generators of* $\mathrm{Aut}(\mathcal{P})$ *with respect to Φ. Then $\mathcal{P}$ is of degree* 2*.*

(i) *$\mathcal{P}$ is properly self-dual if and only if there exists a group automorphism*

$$\tilde{\delta}\colon \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$$

of order 2 *such that*

$$\tilde{\delta}(\rho_0) = \rho_2, \quad \tilde{\delta}(r_0) = r_2.$$

(ii) *$\mathcal{P}$ is improperly self-dual if and only if there exists a group automorphism*

$$\widehat{\delta}\colon \mathrm{Aut}(\mathcal{P}) \to \mathrm{Aut}(\mathcal{P})$$

of order 2 *such that*

$$\widehat{\delta}(\rho_0) = r_2, \quad \widehat{\delta}(\rho_2) = r_0.$$

Proof The orbits O_Φ and $O_\Phi \cdot s_1$ are distinct. By Theorem 3.9, $\mathcal{P}$ is properly self-dual, if $d(N) = N$ and improperly self-dual if $d(N) = s_1 N s_1$, where $d \in \mathrm{Aut}(\mathcal{C})$ is the automorphism that induces the dual operation. The proof follows from Theorem 3.7(v). For part (i) take $\tilde{\delta}\colon \alpha_w \mapsto \alpha_{d(w)}$ and for (ii) take $\widehat{\delta}\colon \alpha_w \mapsto \alpha_{s_1 d(w) s_1}$. ∎

We conclude the section by considering self-dual edge-transitive polyhedra with four orbits and show that they must be of degree 2. According to [22], the orbits are O_Φ, $O_\Phi \cdot s_0$, $O_\Phi \cdot s_2$, $O_\Phi \cdot s_0 s_2$. By Theorem 3.9, the corresponding conjugates are N, $s_0 N s_0$, $s_2 N s_2$, $s_0 s_2 N s_2 s_0$. Note that $d \in \mathrm{Aut}(\mathcal{C})$, inducing the dual operation, has order 2, preserves s_1, and interchanges s_0 and s_2. This implies that $d(N) = N$ or $d(N) = s_0 s_2 N s_2 s_0$, otherwise we have a contradiction. In the first case $d(s_0 s_2 N s_2 s_0) =$

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$s_0s_2Ns_2s_0$, since $d(s_0s_2) = s_0s_2$, but d interchanges s_0Ns_0 and s_2Ns_2. In the second case,

$$d(s_0Ns_0) = d(s_0)d(N)d(s_0) = d(s_0)s_0s_2Ns_2s_0d(s_0) = s_2s_0s_2Ns_2s_0s_2 = s_0Ns_0$$

and similarly $d(s_2Ns_2) = s_2Ns_2$. By Theorem 3.9, any duality must fix two of the orbits and interchange the other two. Therefore, by Proposition 2.3, self-dual edge-transitive maps with 4 orbits are all of degree 2.

5 Examples of Two-Orbit Polyhedra of Degree 4

The smallest, in terms of number of flags, (non-degenerate) edge-transitive self-dual map with two flag-orbits and no dualities of degree two, is the one-vertex map in Figure 1. As mentioned in Section 2, the map is not a polyhedron; however, it is an example of what is commonly referred to as a *polyhedral map*, that is, a map having the property that the intersection of any two distinct faces is either empty, a vertex, or an edge [3]. The map is embedded into a non-orientable surface of genus 5. It is non-degenerate in the sense that the degrees of vertices, faces, and lengths of its Petrie-polygons are at least 3 (see [18]). The map is of type 2^P (according to Graver and Watkins) and Schläfli type $\{10, 10\}$, its Petrie polygons are of length 5, and all of its dualities of degree 4.

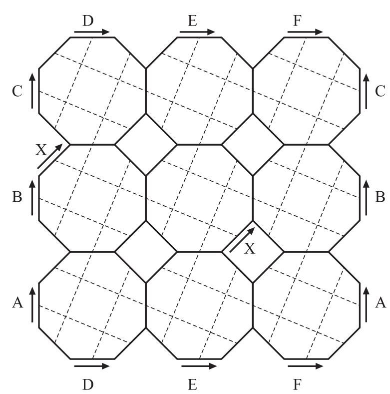

Figure 3: The smallest self-dual polyhedron in class 2_1 possesing no polarity.

By Theorems 4.2 and 4.3, a self-dual two-orbit polyhedron of degree 4 must be improperly self-dual in the class 2_1. The smallest example of such a polyhedron, again in terms of number of flags, is C_3 given in Figure 3. It is an orientable map on a surface of genus 10 with 9 vertices, 9 faces and 36 edges. The nine faces of the polyhedron are the nine octagonal faces of the truncated toroidal map $\{4, 4\}_{(3,0)}$, where the edges along the 4-gons are identified when connected by the dotted lines (as, for example, edges labelled by X in Figure 3). The map is the smallest member

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of the infinite family of polyhedra C_a obtained from the truncated toroidal maps $\{4,4\}_{(a,0)}$, where the edges of the 4-gonal holes are identified according to the same rule (where the dotted lines of C_a have the same slopes as the dotted lines of C_3.) The extended Schläfli symbols of the polyhedra C_a are $\{8,8|4\}$, where the 2-holes are the 4-gons of the toroidal map (for the notation see, for example, [21]). The maps in the family are all in the class 2_1.

In terms of the distinguished generators defined in Theorem 4.2, the automorphism group of the map C_a is

$$\mathrm{Aut}(C_a) = \langle \rho_1, \rho, \tau \mid \rho_1^2, \rho^2, \tau^2, (\rho_1\rho)^4, (\rho_1\rho\tau)^4, (\tau\rho_1\rho\rho_1\rho)^a, (\tau\rho_1\rho\rho_1)^2 \rangle, \quad a \geqslant 3.$$

Using Theorem 4.2, one can easily check that C_3 is improperly self-dual. Furthermore, C_a is self-dual only if $a = 3$. By using the software package Magma [4], one can verify that there is no element $u \in \mathcal{M}(C_3)$, $u \notin \mathrm{Norm}_{\mathcal{M}(C_3)}(\mathrm{Stab}_{\mathcal{M}(C_3)}(\Phi))$, such that $ud(u)$ is in a stabilizer of any chosen base flag Φ. Therefore, by Theorem 3.11, C_3 does not contain a polarity.

The underlaying graph (1-skeleton) of the polyhedron C_3 has pairs of parallel edges (that is, edges sharing the same vertices). Furthermore, every two adjacent faces have precisely two edges in common, and hence C_3 is not a polyhedral map. However, this does not violate the definition of an abstract polyhedron.

In Figure 4 we give an example of a self-dual abstract polyhedron D_1 of degree four that does not have parallel edges and which is a cover of C_3. The faces of D_1 are octagons, which belong to two copies of truncated toroidal maps, each with 18 octagonal faces (see Figure 4). The edges along the 4-gonal 2-holes of $\mathcal{P}$ are identified again as indicated by the dotted lines, but changing from one copy of a toroidal map to the other (for example, see the identifications by X and Y in Figure 4). The polyhedron also belongs to an infinite family of polyhedra D_k, all in class 2_1, consisting of two copies of truncated toroidal maps each with $\frac{1}{2}(4k+2)^2$ faces and extended Schläfli symbol $\{8,8|4\}$. In terms of the distinguished generators defined in Theorem 4.2, the automorphism group of D_k is

$$\begin{aligned}\mathrm{Aut}(D_k) = \Big\langle \rho_1, \rho, \tau \;\Big|\; &\rho_1^2, \rho^2, \tau^2, (\rho_1\rho)^4, (\rho_1\rho\tau)^4, (\tau\rho_1\rho\rho_1\rho)^{4k+2}, (\tau\rho_1\rho\rho_1)^4, \\ &(\rho_1\tau\rho\tau\rho_1\rho)^2, (\rho_1\rho\tau\rho_1\tau\rho)^2, (\tau\rho_1\tau\rho)^4, \\ &(\rho\rho_1\tau\rho\rho_1)^{2k+1}(\rho_1\rho\rho_1\rho\tau)^{2k+1}\Big\rangle, \quad k \geqslant 1.\end{aligned}$$

We have not been able to produce an example of a self-dual equivelar polyhedral map without a polarity. Existence of such a map is still an open problem.

6 Medials of Polyhedra and Self-Duality

Starting from the standard combinatorial and topological definition of the medial of a map, we reinterpret it as an operation on the poset of an abstract polytope. We show that the medial of a polyhedron is a polyhedron. It will be clear from the definition that the medial of a polyhedron and the medial of its dual are isomorphic. In the

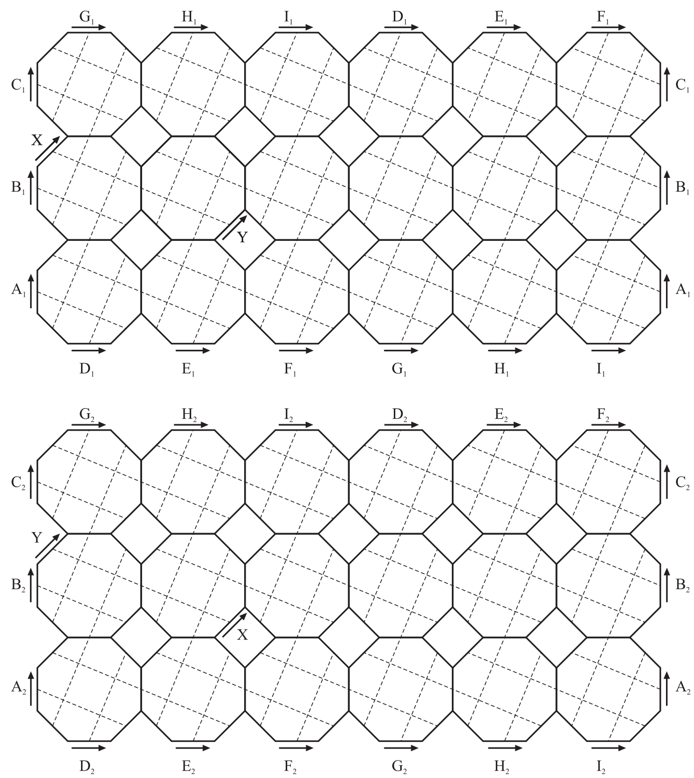

Figure 4: A polyhedron in class 2_1 without parallel edges possesing no polarity.

classical theory, the medial of a regular convex polyhedron with Schläfli type $\{p, q\}$ is a quasi-regular polyhedron of type $\left\{ \begin{smallmatrix} p \\ q \end{smallmatrix} \right\}$ (see [5]), when the polyhedron is not self-dual, and it is regular otherwise. In the remaider of the section, using monodromy groups, we investigate the connection between the self-duality of a polyhedron and symmetries of its medial.

For a polyhedron $\mathcal{P}$, we denote by $\mathcal{F}_i(\mathcal{P})$ the set of all i-faces, for $i = 0, 1, 2$. Furthermore, for a given flag $\Phi \in \mathcal{F}(\mathcal{P})$ we denote by $(\Phi)_i$ its i-face. It is convenient to omit the improper faces (that is, the unique minimal and maximal faces) of the poset, when specifying a flag. Hence, without loss of generality, we denote a flag Φ as $\{(\Phi)_0, (\Phi)_1, (\Phi_2)\}$.

The *medial polytope* $\mathrm{M}_{\mathcal{P}}$ of $\mathcal{P}$ is a poset with i-faces $\mathcal{F}_i(\mathrm{M}_{\mathcal{P}})$ defined as follows:

$$\mathcal{F}_0(\mathrm{M}_{\mathcal{P}}) = \mathcal{F}_1(\mathcal{P}),$$
$$\mathcal{F}_1(\mathrm{M}_{\mathcal{P}}) = \{\{(\Phi)_0, (\Phi)_2\} \mid \Phi \in \mathcal{F}(\mathcal{P})\},$$
$$\mathcal{F}_2(\mathrm{M}_{\mathcal{P}}) = \mathcal{F}_0(\mathcal{P}) \cup \mathcal{F}_2(\mathcal{P}).$$

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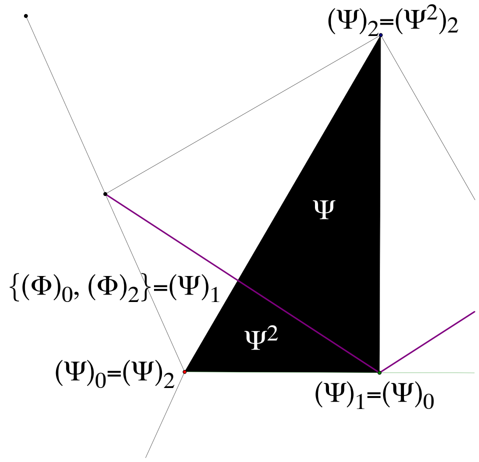

Figure 5: A flag of $\mathcal{P}$ "divided" into two flags of $\mathrm{M}_{\mathcal{P}}$

The partial order on the faces $G_i \in \mathcal{F}_i(\mathrm{M}_{\mathcal{P}})$ is given by

$$G_0 \leqslant_{\mathrm{M}_{\mathcal{P}}} G_1 \Leftrightarrow \{G_0\} \cup G_1 \in \mathcal{F}(\mathcal{P}),$$
$$G_1 \leqslant_{\mathrm{M}_{\mathcal{P}}} G_2 \Leftrightarrow G_2 \in G_1.$$

Any flag Φ of $\mathcal{P}$ induces in a natural way (see Figure 5) exactly two (adjacent) flags of $\mathrm{M}_{\mathcal{P}}$ related to Φ, namely,

$$\Psi := \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_0\} \quad \text{and} \quad \Psi^2 = \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_2\}.$$

Conversely, if Ψ is a flag of $\mathrm{M}_{\mathcal{P}}$, then $\{(\Psi)_0\} \cup (\Psi)_1$ is a flag of $\mathcal{P}$. Denoting it by Φ, we have that $(\Psi)_0 = (\Phi)_1$, $(\Psi)_1 = \{(\Phi)_0, (\Phi)_2\}$, and $(\Psi)_2 = (\Phi)_0$ or $(\Phi)_2$.

Proposition 6.1 *The poset* $\mathrm{M}_{\mathcal{P}}$ *is a polyhedron.*

Proof Verifying the diamond condition is straightforward. To see that $\mathrm{M}_{\mathcal{P}}$ is strongly connected, we have to show that each section of $\mathrm{M}_{\mathcal{P}}$ is connected. Clearly, $\mathrm{M}_{\mathcal{P}}$ is connected, since $\mathcal{P}$ is. Since 1-sections are diamonds, they are connected. The 2-sections of $\mathrm{M}_{\mathcal{P}}$ are its facets and its vertex-figures. Since the facets of $\mathrm{M}_{\mathcal{P}}$ correspond to the facets and vertex-figures of $\mathcal{P}$, and since $\mathcal{P}$ is strongly connected, the facets of $\mathrm{M}_{\mathcal{P}}$ are connected. Furthermore, the vertices of $\mathrm{M}_{\mathcal{P}}$ are the edges of $\mathcal{P}$. Given an edge F_1 of $\mathcal{P}$, the set $\{H \in \mathcal{P} \mid H \leqslant F_1 \text{ or } F_1 \leqslant H\}$ corresponds to a 4-gon (vertex-figure) in $\mathrm{M}_{\mathcal{P}}$ (see Figure 6). Hence, the vertex-figures of $\mathrm{M}_{\mathcal{P}}$ are connected. ■

Let s_0, s_1, s_2 be the generators of the C-group $\mathcal{C}$ of rank 3 as in Section 3. Recall that the semi-direct product $N \rtimes_\theta Q$ of two groups N and Q by the homomorphism

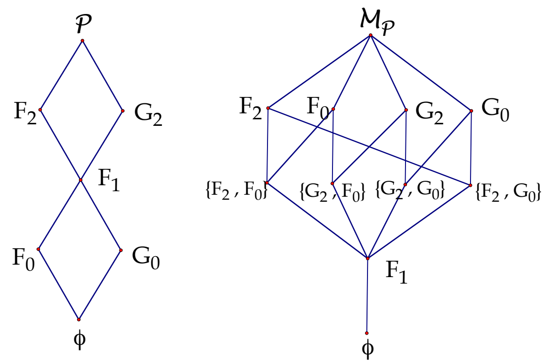

Figure 6: Faces adjacent to an edge of $\mathcal{P}$ and the corresponding 4-gonal vertex-figure in $M_{\mathcal{P}}$.

$\theta\colon Q \to \mathrm{Aut}(N)$ is defined as follows. The group elements are the elements of the Cartesian product $N \times Q$ and the operation is defined by

$$(n, q)(n', q') = (n\theta(q)(n'), qq').$$

When the homomorphism θ is clear from the context, we write $N \rtimes Q$. Let d be the automorphism of $\mathcal{C}$ that induces the dual operation. Take $Q = \langle d \rangle$, $N = \mathcal{C}$ and $\theta\colon \langle d \rangle \to \mathrm{Aut}(\mathcal{C})$, the inclusion homomorphism. Let $\mathcal{D} = \mathcal{C} \rtimes \langle d \rangle$, $r_0 = (s_1, \mathrm{id})$, $r_1 = (s_0, \mathrm{id})$, and $r_2 = (1, d)$, where 1 and id denote the identity elements in the corresponding groups. Clearly, $\mathcal{D}$ is generated by r_0, r_1, and r_2 and in terms of a finite presentation, it can be given by $\mathcal{D} = \langle r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_0r_2)^2, (r_1r_2)^4 \rangle$. Therefore, the mapping $\Omega\colon \mathcal{C} \to \mathcal{D}$, taking s_i to r_i, $i = 0, 1, 2$, is a group epimorphism. We define the actions of r_0, r_1, r_2 on the flags of $M_{\mathcal{P}}$ as follows:

$$\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_i\} \diamond r_0 = \{(\Phi)_1 \cdot s_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_i\}, \quad i = 0, 2;$$
$$\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_0\} \diamond r_1 = \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2 \cdot s_2\}, (\Phi)_0\};$$
$$\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_2\} \diamond r_1 = \{(\Phi)_1, \{(\Phi)_0 \cdot s_0, (\Phi)_2\}, (\Phi)_2\};$$
$$\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_0\} \diamond r_2 = \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_2\};$$
$$\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_2\} \diamond r_2 = \{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_0\}.$$

We show that the definition extends to an action of $\mathcal{D}$ on the set of flags of $M_{\mathcal{P}}$. Let $\phi\colon \mathcal{F}(M_{\mathcal{P}}) \to \mathcal{F}(\mathcal{P}) \times \langle d \rangle$ be defined by $\{(\Phi)_1, \{(\Phi)_0, (\Phi)_2\}, (\Phi)_i\} \mapsto (\Phi, d^{(2-i)/2})$. Clearly, ϕ is a bijection. Define the action of the group $\mathcal{D} = \mathcal{C} \rtimes \langle d \rangle$ on the set $\mathcal{F}(\mathcal{P}) \times \langle d \rangle$ by $(\Phi, q) \bullet (w, p) = (\Phi \cdot q(w), q \circ p)$, for any $\Phi \in \mathcal{F}(\mathcal{P})$, $q, p \in \langle d \rangle$, and $w \in \mathcal{C}$. To see that the action is well defined, notice that $(1, \mathrm{id}) \in \mathcal{D}$ is the identity

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element and $(\Phi, q) \bullet (1, \mathrm{id}) = (\Phi \cdot \mathrm{id}(1), q \circ \mathrm{id}) = (\Phi, q)$. Furthermore,

$$
\begin{aligned}
((\Phi, q) \bullet (w, f)) \bullet (v, h) &= (\Phi \cdot q(w), q \circ f) \bullet (v, h) = (\Phi \cdot q(w)(q \circ f)(v), q \circ f \circ h),\\
(\Phi, q) \bullet \big((w, f)(v, h)\big) &= (\Phi, q) \bullet (wf(v), f \circ h) = (\Phi \cdot q(wf(v)), q \circ f \circ h)\\
&= (\Phi \cdot q(w)(q \circ f)(v)), q \circ f \circ h).
\end{aligned}
$$

Using the bijection $\phi\colon \mathcal{F}(\mathrm{M}_{\mathcal{P}}) \to \mathcal{F}(\mathcal{P}) \times \langle d \rangle$ defined above, we transfer the action of $\mathcal{D}$ from the set $\mathcal{F}(\mathcal{P}) \times \langle d \rangle$ to the set $\mathcal{F}(\mathrm{M}_{\mathcal{P}})$, where $\phi^{-1}((\Phi, q) \bullet (w, f)) = \phi^{-1}((\Phi, q)) \diamond (w, f)$, making $(\phi^{-1}, \mathrm{Id})\colon (\mathcal{F}(\mathcal{P}) \times \langle d \rangle, \mathcal{D}) \to (\mathcal{F}(\mathrm{M}_{\mathcal{P}}), \mathcal{D})$ an action isomorphism. It can be easily verified that this is exactly the required extension of the action $\diamond$.

Given a base flag Φ of $\mathcal{P}$, we choose (Φ, id) as a base flag of $\mathrm{M}_{\mathcal{P}}$. Let $(w, f) \in \mathcal{D}$ be in the stabilizer of (Φ, id) under the action $(\mathcal{F}(\mathcal{P}) \times \langle d \rangle, \mathcal{D})$. Then $(\Phi, \mathrm{id}) \bullet (w, f) = (\Phi \cdot \mathrm{id}(w), f) = (\Phi \cdot w, f)$, implying that $f = \mathrm{id}$ and $w \in \mathrm{Stab}_{\mathcal{C}}(\Phi) =: N$. Hence, $\mathrm{Stab}_{\mathcal{D}}((\Phi, \mathrm{id})) = (N, \mathrm{id})$ is the embedding of N in $\mathcal{D}$. Likewise $(\mathcal{C}, \mathrm{id})$ is the embedding of $\mathcal{C}$ in $\mathcal{D}$. Therefore, the monodromy group $\mathcal{M}(\mathrm{M}_{\mathcal{P}})$ of the medial $\mathrm{M}_{\mathcal{P}}$ is isomorphic to the group $\mathcal{D}/\operatorname{Core}_{\mathcal{D}}(N)$. Clearly, $\operatorname{Core}_{\mathcal{D}}(N) \leqslant (\operatorname{Core}_{\mathcal{C}}(N), \mathrm{id})$, with equality if and only if $(1, d)(\operatorname{Core}_{\mathcal{C}}(N), \mathrm{id})(1, d) = (\operatorname{Core}_{\mathcal{C}}(N), \mathrm{id})$. That is, $d(\operatorname{Core}_{\mathcal{C}}(N)) = \operatorname{Core}_{\mathcal{C}}(N)$. By Theorem 3.7, this always happens when the polyhedron is self-dual, but it might also happen in some other cases. Defining $d' \in \mathrm{Aut}(\mathcal{M}(\mathcal{P}))$ as in Theorem 3.7(iii), we now have the following proposition.

Proposition 6.2 *Let $\mathcal{P}$ be a self-dual polyhedron. The monodromy group $\mathcal{M}(\mathrm{M}_{\mathcal{P}})$ is isomorphic to $\mathcal{M}(\mathcal{P}) \rtimes_\theta \langle d' \rangle$, where $\theta\colon \langle d' \rangle \to \mathrm{Aut}(\mathcal{M}(\mathcal{P}))$ is the inclusion homomorphism.*

Proof As above, denote by $N := \mathrm{Stab}_{\mathcal{C}}(\Phi)$, where Φ is a base flag of $\mathcal{P}$. Both $d \in \mathrm{Aut}(\mathcal{C})$ and $d' \in \mathrm{Aut}(\mathcal{M}(\mathcal{P}))$ are of order 2. Denoting by λ the isomorphism $\lambda\colon \langle d \rangle \to \langle d' \rangle$, the mapping $(f_{\mathcal{P}}, \lambda)\colon \mathcal{D} \to \mathcal{M}(\mathcal{P}) \rtimes_\theta \langle d' \rangle$, $(f_{\mathcal{P}}, \lambda)(w, h) = (f_{\mathcal{P}}(w), \lambda(h))$ is surjective. By Theorem 3.7, for any $h \in \langle d \rangle$ it follows that $f_{\mathcal{P}} \circ h = \lambda(h) \circ f_{\mathcal{P}}$. Hence, for every $w, v \in \mathcal{C}$ and $h, k \in \langle d \rangle$,

$$
\begin{aligned}
\big((f_{\mathcal{P}},\lambda)((w, h))\big)\big((f_{\mathcal{P}}, \lambda)((v, k))\big) &= \big(f_{\mathcal{P}}(w), \lambda(h)\big)\big(f_{\mathcal{P}}(v), \lambda(k)\big)\\
&= \big(f_{\mathcal{P}}(w)\lambda(h)(f_{\mathcal{P}}(v)), \lambda(h) \circ \lambda(k)\big) = \big(f_{\mathcal{P}}(w)(f_{\mathcal{P}} \circ h)(v), \lambda(h \circ k)\big)\\
&= (f_{\mathcal{P}}(wh(v)), \lambda(h \circ k)) = (f_{\mathcal{P}}, \lambda)(wh(v), h \circ k) = (f_{\mathcal{P}}, \lambda)((w, h)(v, k)).
\end{aligned}
$$

Then $(f_{\mathcal{P}}, \lambda)$ is an epimorphism with the kernel $(\operatorname{Core}_{\mathcal{C}}(N), \mathrm{id})$, which is, as discussed above, exactly $\operatorname{Core}_{\mathcal{D}}(N)$. Therefore, by the first isomorphism theorem for groups, it follows that

$$
\mathcal{M}(\mathrm{M}_{\mathcal{P}}) \cong \mathcal{D}/\operatorname{Core}_{\mathcal{D}}(N) \cong \mathcal{D}/\ker(f_{\mathcal{P}}, \lambda) \cong (f_{\mathcal{P}}, \lambda)(\mathcal{D}) = \mathcal{M}(\mathcal{P}) \rtimes_\theta \langle d' \rangle. \qquad \blacksquare
$$

The proposition implies that the monodromy group of the medial of a finite self-dual polyhedron is twice as big as the monodromy group of the polyhedron.

In what follows, we will show that any two-orbit medial must be induced by either a regular or a two orbit polyhedron.

Theorem 6.3 *Let $\mathcal{P}$ be a two-orbit polyhedron. Then $\mathrm{M}_{\mathcal{P}}$ has two or four orbits. Furthermore, $\mathrm{M}_{\mathcal{P}}$ is a two-orbit polyhedron if and only if $\mathcal{P}$ is self-dual.*

Proof Let $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$ and $\mathcal{N} = \mathrm{Norm}_{\mathcal{C}}(N)$. Clearly, $(\mathcal{N}, \mathrm{id}) \leqslant \mathrm{Norm}_{\mathcal{D}}((N, \mathrm{id}))$. If $\mathcal{P}$ is a two-orbit polyhedron, then $[\mathcal{C}:\mathcal{N}] = 2$ by Proposition 3.6. Since $[\mathcal{D}:(\mathcal{C}, \mathrm{id})] = 2$, it follows that $[\mathcal{D}:(\mathcal{N}, id)] = 4$. The index $[\mathcal{D}: \mathrm{Norm}_{\mathcal{D}}((N, \mathrm{id}))]$ is either 4 or 2, since no element $(w, \mathrm{id}) \in \mathcal{D}$, $w \notin \mathcal{N}$ normalizes (N, id). The index is 2 if and only if there exists an element of the form (v, d), $v \in \mathcal{C}$, normalizing (N, id). But for any $n \in N$

$$(v, d)^{-1}(n, \mathrm{id})(v, d) = (d(v^{-1}), d)(n, \mathrm{id})(v, d) = (d(v^{-1})d(n)d(v), \mathrm{id}),$$

implying that $(v, d) \in \mathrm{Norm}_{\mathcal{D}}((N, \mathrm{id}))$ if and only if $d(v^{-1})d(N)d(v) = N$ or $d(N) = d(v)Nd(v)^{-1}$. But, by Theorem 3.7, the latter is equivalent to $\mathcal{P}$ being self-dual. ∎

We observe that the medial of a polyhedron can be regular. For example, the medial of the regular toroidal map $\{4, 4\}_{(b,0)}$ with b^2 faces and b^2 vertices is the regular toroidal map $\{4, 4\}_{(b,b)}$ with $2b^2$ and the same number of vertices. Its medial in turn, is the regular toroidal map $\{4, 4\}_{(2b,0)}$ with $4b^2$ faces (and the same number of vertices).

Similarly, arguments used in the proof of Theorem 6.3 can be used to prove the following.

Theorem 6.4 *Let $\mathcal{P}$ be a regular polyhedron and $\mathrm{M}_{\mathcal{P}}$ its medial. Then $\mathrm{M}_{\mathcal{P}}$ is either in class 2_{01} or regular. Furthermore, $\mathrm{M}_{\mathcal{P}}$ is regular if and only if $\mathcal{P}$ is self-dual.*

Each automorphism and each duality of a polyhedron induces in a natural way an automorphism of its medial. The correspondence is given in Table 3. To see how the automorphisms of $\mathrm{M}_{\mathcal{P}}$ induced by $\mathcal{P}$ with a base flag Φ are found, one can take a geometric approach. For example, Figure 7 shows how the automorphisms ρ_1 and ρ_2 of $\mathcal{P}$ correspond to the automorphisms α_{s_0} and $\alpha_{s_2s_1s_2}$ of $\mathrm{M}_{\mathcal{P}}$, respectively. Figure 8 shows how the dualities δ and δ_{s_0} of $\mathcal{P}$ induce the automorphisms α_{s_2} and $\alpha_{s_1s_2}$ of $\mathrm{M}_{\mathcal{P}}$, respectively (Recall that δ_{s_i} and δ are the unique dualities such that $\Phi\delta_{s_i} = \Phi^i$ and $\Phi\delta = \Phi$, respectively.)

Automorphisms and dualities of $\mathcal{P}$	ρ_1	ρ_0	δ	$\rho_0\rho_1$	δ_1
Induced automorphisms of $\mathrm{M}_{\mathcal{P}}$	α_0	α_1	α_2	α_{01}	α_{02}
Automorphisms and dualities of $\mathcal{P}$	δ_0	$\rho_1\rho_0\rho_1$	$\rho_0\rho_1\rho_0$	$\delta_0\rho_2$	ρ_2
Induced automorphisms of $\mathrm{M}_{\mathcal{P}}$	α_{12}	α_{010}	α_{101}	α_{121}	α_{212}

Table 3: Automorphisms of $\mathrm{M}_{\mathcal{P}}$ induced by automorphisms and dualities of $\mathcal{P}$.

Alternatively, using a purely algebraic approach, the correspondences are found as follows. For convenience, denote by $\mathcal{N}_{\mathcal{C}} = \mathrm{Norm}_{\mathcal{C}}(N)$ and $\mathcal{N}_{\mathcal{D}} = \mathrm{Norm}_{\mathcal{D}}((N, \mathrm{id}))$,

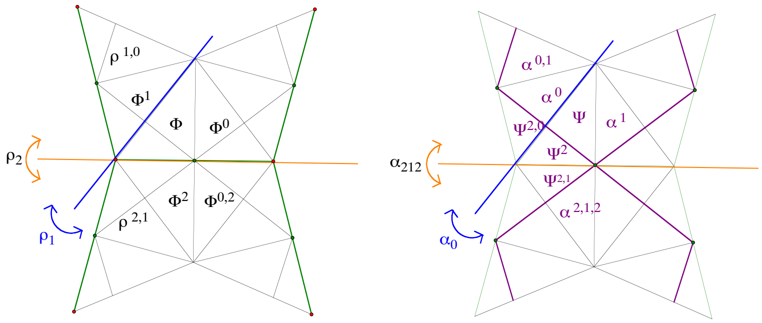

Figure 7: Automorphisms of $\mathcal{P}$ and the corresponding automorphisms of $M_{\mathcal{P}}$.

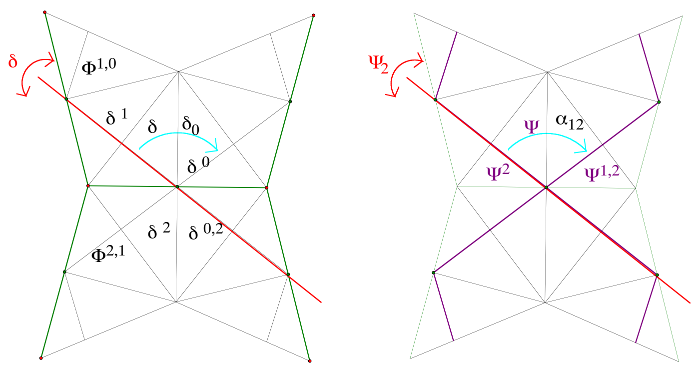

Figure 8: Dualitites of $\mathcal{P}$ and the corresponding automorphisms of $M_{\mathcal{P}}$.

where $N = \mathrm{Stab}_{\mathcal{C}}(\Phi)$ and $(N, \mathrm{id}) = \mathrm{Stab}_{\mathcal{D}}(\Phi)$. Then $r_0(N, \mathrm{id})r_0 = (s_1Ns_1, \mathrm{id})$, implying that $r_0 \in \mathcal{N}_{\mathcal{D}}$ if and only if $s_1 \in \mathcal{N}_{\mathcal{C}}$. Conjugating (N, id) by r_2, we see that $r_2 \in \mathcal{N}_{\mathcal{D}}$ if and only if $d(N) = N$. When $\mathcal{P}$ is self-dual, we define

$$\mathcal{I}(\mathcal{P}) = \{u \in \mathcal{C} \mid d(N) = u^{-1}Nu\}.$$

Therefore $r_2 \in \mathcal{N}_{\mathcal{D}}$ if and only if $\mathcal{I}(\mathcal{P}) \subseteq \mathcal{N}_{\mathcal{C}}$. Similarly, the conjugation of (N, id) by $r_2r_1r_0$ yields $(d(s_0s_1Ns_1s_0), \mathrm{id})$. Hence, $r_2r_1r_0 \in \mathcal{N}_{\mathcal{D}}$ if and only if $d(s_0s_1Ns_1s_0) = s_2s_1d(N)s_1s_2 = N$. But this is equivalent to the claim that $w \in \mathcal{I}(\mathcal{P})$ implies $ws_1s_2 \in \mathcal{N}_{\mathcal{C}}$. Using this approach, on certain other elements of $\mathcal{D}$, we obtain the following set of correspondences

$$r_0 \leftrightarrow s_1, \quad r_1 \leftrightarrow s_0, \quad r_2 \leftrightarrow w, \quad r_0r_1 \leftrightarrow s_0s_1,$$
$$r_0r_2 \leftrightarrow ws_1, \quad r_1r_2 \leftrightarrow ws_0, \quad r_0r_1r_0 \leftrightarrow s_1s_0s_1, \quad r_1r_0r_1 \leftrightarrow s_0s_1s_0,$$
$$r_1r_2r_1 \leftrightarrow ws_2s_0, \quad r_2r_1r_2 \leftrightarrow s_0,$$

where w is any element of $\mathcal{I}(\mathcal{P})$ and $a \leftrightarrow b$ stands for "$a \in \mathcal{N}_\mathcal{D}$ if and only if $b \in \mathcal{N}_\mathcal{C}$".

By Proposition 3.4, these correspondences give us the correspondences between automorphisms in $\mathrm{Aut}(\mathrm{M}_\mathcal{P})$ and elements of the extended group of $\mathcal{P}$. For example, the first correspondence tells us that $\alpha_{s_0} \in \mathrm{Aut}(\mathrm{M}_\mathcal{P})$ if and only if $\rho_1 \in \mathrm{Aut}(\mathcal{P})$, where $\Phi\rho_1 = \Phi \cdot s_1$. Denoting by 1 the class of regular polyhedra and using Theorems 6.3 and 6.4, we obtain Table 4 (where P and I in the second row stand for proper and improper self-duality respectively and "–" denotes the case when a polyhedron is not self-dual). By Theorem 6.3, when $\mathcal{P}$ is a two-orbit polyhedron which is not self-dual, $\mathrm{M}_\mathcal{P}$ must have 4 flag-orbits.

Class of $\mathcal{P}$	1		2		2_1		$2_{0,2}$	
Type of duality	P	–	P	I	P	I	P	I
Class of $\mathrm{M}_\mathcal{P}$	1	$2_{0,1}$	2_2	2	$2_{2,0}$	2_0	$2_{2,1}$	2_1

Table 4: Characterization of regular and two-orbit medial polyhedra.

In particular, the polyhedron in Figure 3, which is in class 2_1 and improperly self-dual, induces a medial in class 2_0. Note that there is an error in [22, Lemma 2.2], where it is claimed that the medial of a polyhedron in class 2_1 (*i.e.,* type 2^P) cannot be edge-transitive. But polyhedra in class 2_0 (*i.e.,* type 2^*ex) are edge-transitive.

In conclusion, we observe that every automorphism of the medial $\mathrm{M}_\mathcal{P}$ of a polyhedron $\mathcal{P}$ is induced by an automorphism or a duality of $\mathcal{P}$.

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