BULL. AUSTRAL. MATH. SOC. 32F05, 31B35
VOL. 53 (1996) [123–129]

M-HARMONIC FUNCTIONS WITH M-HARMONIC SQUARE

HONG OH KIM

$\mathcal{M}$-harmonic functions with $\mathcal{M}$-harmonic square are proved to be either holomorphic or antiholomorphic in the unit ball of complex n-space under certain additional conditions. For example, if u and u^2 are $\mathcal{M}$-harmonic in the unit ball of $\mathbb{C}^2$ and if u is continuously differentiable up to the boundary then u is either holomorphic or antiholomorphic.

1. INTRODUCTION

It is well known and easy to prove that if u and u^2 are harmonic in an open connected region $\Omega \subset \mathbb{C}$ then at least one of u and $\overline{u}$ is holomorphic in Ω. The analogue of this in the open unit ball B_n of $\mathbb{C}^n$ $(n > 2)$ and with "$\mathcal{M}$-harmonic" in place of harmonic was unexpectedly proved to be false by Ahern and Rudin in [1]. It is not known whether the analogue for $n = 2$ is true or not. In this paper we prove the analogue is true under certain additional conditions for $n \geqslant 2$. For example, if u and u^2 are $\mathcal{M}$-harmonic in the unit ball of $\mathbb{C}^2$ and if u is continuously differentiable up to the boundary then u is either holomorphic or antiholomorphic.

We say that a function u is $\mathcal{M}$-harmonic in B_n if

$$\widetilde{\Delta} u(z) = 0$$

for every $z \in B_n$, where $\widetilde{\Delta}$ is the Moebius-invariant Laplacian:

$$\widetilde{\Delta} u = \left(1 - |z|^2\right) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}. \tag{1}$$

This is related to the ordinary Laplacian $\Delta = \sum \partial^2 / \partial z_j \overline{\partial} z_j$ as $\left(\widetilde{\Delta} u\right)(a) = \Delta(u \circ \phi)(0)$, where ϕ is an automorphism of B_n mapping the origin to a.

It is clear from (1) that all holomorphic or antiholomorphic functions are $\mathcal{M}$-harmonic, as are the pluriharmonic ones. The pluriharmonic functions are those functions that can be represented as a sum of a holomorphic function and an antiholomorphic

Received 23rd March, 1995
Partially supported by KOSEF and TGRC. The author wishes to express his sincere gratitude to Professor P.R. Ahern, discussions with whom made these results possible.

https://doi.org/10.1017/S0004972700016786 Published online by Cambridge University Press

function. It is an interesting fact that the pluriharmonic functions in the ball B_n are those $\mathcal{M}$-harmonic functions that are also ordinary harmonic [3].

We recall that any $\mathcal{M}$-harmonic function u has a spherical harmonic expansion, which converges uniformly on compact subsets of B_n,

$$u(z) = \sum_{p,q\geqslant 0} R_{pq}\left(|z|^2\right) h_{pq}(z) \tag{2}$$

where h_{pq} is a homogeneous polynomial of degree p in z and of degree q in $\overline{z}$, and $R_{pq}(t)$ is a hypergeometric function, normalised so that $R_{p,q}(1) = 1$:

$$R_{pq}(t) = \frac{{}_2F_1(p,q,p+q+n;t)}{{}_2F_1(p,q,p+q+n;1)}.$$

See [2]. Finally, we recall the invariant Poisson kernel $P(z,\zeta)$ is given by

$$P(z,\zeta) = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^n, z \in B_n, \zeta \in S = \partial B_n$$

[3], and the invariant Poisson integral of a function u on S is given by $P[u](z) = \int_S P(z,\zeta)u(z)d\sigma(\zeta)$, where $d\sigma$ is the normalised Lebesgue measure on S with $d\sigma(S) = 1$.

2. The case $n = 2$

It is known in [1] that if $u \in C^2(\overline{B}_2)$ and $\widetilde{\Delta}u = \widetilde{\Delta}u^2 = 0$ then one of u and $\overline{u}$ is holomorphic in B_2. The smoothness condition $u \in C^2(\overline{B}_2)$ is relaxed to $u \in C^1(\overline{B}_2)$ in the following main theorem of this paper.

Theorem 1. *Suppose $\widetilde{\Delta}u = \widetilde{\Delta}u^2 = 0$ in B_2. If u is continuously differentiable up to the boundary of B_2, then one of u and $\overline{u}$ is holomorphic in B_2.*

Proof: Let $T = \overline{z}_2\partial/\partial z_1 - \overline{z}_1\partial/\partial z_2$ be a tangential Cauchy-Riemann operator and $R = z_1\partial/\partial z_1 + z_2\partial/\partial z_2$ a radial differential operator on B_2. Then the hypothesis $\widetilde{\Delta}u = \widetilde{\Delta}u^2 = 0$ implies that

$$T\overline{T}u = \left(r^2-1\right)R\overline{R}u - \overline{R}u, \tag{3}$$

$$Tu\overline{T}u = \left(r^2-1\right)Ru\overline{R}u. \tag{4}$$

Since $u \in C^1(\overline{B}_2)$, (4) implies that $Tu\overline{T}u = 0$ on S. There are two open subsets V and W of S such that

$$\begin{aligned} \overline{T}u &= 0 \text{ on } V,\\ Tu &= 0 \text{ on } W,\\ \overline{V \cup W} &= S. \end{aligned} \tag{5}$$

We fix $\zeta_0 \in V$ and take $\phi \in C_c^\infty(V)$ such that $\phi \equiv 1$ near ζ_0. We set $u_1 = \phi u$ and let $U_1 = P[u_1]$ be the invariant Poisson integral of u_1. Then

$$\begin{aligned} u(z) &= \int_S P(z,\zeta)u(\zeta)d\sigma(\zeta) \\ &= \int_S P(z,\zeta)\phi(\zeta)u(\zeta)d\sigma(\zeta) + \int_S P(z,\zeta)(1-\phi(\zeta))u(\zeta)d\sigma(\zeta) \\ &= U_1(z) + U_2(z). \end{aligned} \tag{6}$$

We can easily check that

$$\begin{aligned} T\overline{T}U_2(z) &\to 0, \\ \overline{R}U_2(z) &\to 0 \end{aligned} \tag{7}$$

as $z \to \zeta_0$. If we note that

$$\begin{aligned} \overline{T}u_1 &= (\overline{T}\phi)u + \phi\overline{T}u \\ &= \overline{T}\phi u \end{aligned}$$

on S, we see that $\overline{T}u_1 \in C^1(S)$ and so $T\overline{T}u_1 \in C(S)$. Since $\widetilde{\Delta}T\overline{T}U_1 = T\overline{T}\widetilde{\Delta}U_1 = 0$, we also have

$$T\overline{T}U_1(z) = P[T\overline{T}u_1](z) \to T\overline{T}u_1(\zeta), \quad z = r\zeta, \tag{8}$$

as $r \to 1$. We write $U_1(z) = \sum R_{pq}\left(|z|^2\right)h_{pq}(z)$ as in (2). An easy computation gives

$$T\overline{T}U_1(z) = -\sum R_{pq}\left(|z|^2\right)q(p+1)h_{pq}(z). \tag{9}$$

Therefore, we have

$$\int \left|T\overline{T}U_1(r\zeta)\right|^2 d\sigma(\zeta) = \sum_{p,q\geqslant 0} R_{pq}\left(|z|^2\right)^2 q^2(p+1)^2 r^{2p+2q} \int |h_{pq}(\zeta)|^2 \, d\sigma(\zeta). \tag{10}$$

If we let $r \to 1^-$ in (10), we get, by (8),

$$\int \left|T\overline{T}u_1(\zeta)\right|^2 d\sigma(\zeta) = \sum_{p,q\geqslant 0} q^2(p+1)^2 \int |h_{pq}(\zeta)|^2 \, d\sigma(\zeta) < \infty \tag{11}$$

On the other hand, we have

$$\overline{R}U_1(z) = \sum_{p,q\geqslant 0} \{R'_{pq}\left(|z|^2\right)|z|^2 + qR_{pq}\left(|z|^2\right)\}h_{pq}(z),$$

and so

$$\int_S \left|\overline{R}U_1(r\zeta)\right|^2 d\sigma(\zeta) = \sum_{p,q\geqslant 0} \{R'_{pq}(r^2)r^2 + qR_{pq}(r^2)\}^2 r^{2p+2q} \int_S |h_{pq}(\zeta)|^2 d\sigma(\zeta)$$

$$\leqslant \sum_{p,q\leqslant 0} q^2(p+1)^2 \int |h_{pq}(\zeta)|^2 d\sigma(\zeta)$$

$$< \infty. \tag{12}$$

The last inequality comes from (11). Therefore,

$$\begin{aligned}(13)\qquad &\int_S \left|\overline{R}U_1(r\zeta) - T\overline{T}u(\zeta)\right|^2 d\sigma(\zeta)\\ &= \sum_{p,q\geqslant 0} \{R'_{pq}(r^2)r^2 + qR_{pq}(r^2)r^{2p+2q} - q(p+1)\}^2 \int_S |h_{pq}(\zeta)|^2 d\sigma(\zeta)\\ &\to 0\end{aligned}$$

as $r \to 1^-$. We can choose a sequence $r_j \nearrow 1$ so that $\overline{R}U_1(r\zeta) \to T\overline{T}u_1(\zeta)[\sigma]$ almost everywhere on S. We can easily see that $\overline{R}U_1(r\zeta) \to \overline{R}u_1(\zeta)$ near ζ_0 and so we have $T\overline{T}u_1(\zeta) = \overline{R}u(\zeta)$ near ζ_0 by continuity. We have proved that $\overline{R}u = 0$ on V and so on $\overline{V}$ by continuity. Similarly, we can show $Ru = 0$ on $\overline{W}$. Since $Ru\overline{R}u = 0$ on S, we have, by orthogonality of the $u_{p,q}$'s,

$$\begin{aligned}0 &= \int_S \overline{R}u(\zeta)\overline{\overline{R}u}(\zeta)d\sigma(\zeta)\\ &= \sum pq(p+1)(q+1)\int |u_{pq}(\zeta)|^2 d\sigma(\zeta),\end{aligned}$$

where $u = \sum u_{pq}$ is the homogeneous expansion of u on S in $L^2(\sigma)$. Therefore, $pq = 0$ unless $u_{pq} \equiv 0$. This means that u is pluriharmonic in B_2. If we write $u = f + \overline{g}$ where f, g are holomorphic in B_2, then

$$\overline{T}Tu = \overline{T}Tf = -\left(z_2\frac{\partial f}{\partial z_2} + z_1\frac{\partial f}{\partial z_1}\right)$$

is holomorphic in B_2 and vanishes on W. Suppose $W \neq \phi$. Then $\overline{T}Tu \equiv 0$ on S and so $Ru \equiv 0$ on B_2. Therefore u is antiholomorphic in B_2 by a Theorem of Forelli [**3**]. Similarly, we can show that u is holomorphic in B_2 if $V \neq \phi$. This completes the proof. ∎

THEOREM 2. *Suppose $\widetilde{\Delta}u = \widetilde{\Delta}u^2 = 0$ in B_2. If u is holomorphic in one of two variables then one of u and $\overline{u}$ is holomorphic in B_2.*

PROOF: Suppose $u(z_1, z_2)$ is holomorphic in z_2 in B_2. $\widetilde{\Delta} u = \widetilde{\Delta} u^2 = 0$ implies that

$$\frac{\partial u^2}{\partial z_1 \partial z_2} = |z_1|^2 \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \overline{z}_1 z_2 \frac{\partial^2 u}{\partial \overline{z}_1 \partial z_2}, \tag{14}$$

$$\frac{\partial u}{\partial \overline{z}_1}\left(\frac{\partial u}{\partial z_1} - |z_1|^2 \frac{\partial u}{\partial z_1} - \overline{z}_1 z_2 \frac{\partial u}{\partial z_2}\right) = 0. \tag{15}$$

We have either $\partial u / \partial \overline{z}_1 \equiv 0$ or

$$\frac{\partial u}{\partial z_1} \equiv |z_1|^2 \frac{\partial u}{\partial z_1} + \overline{z}_1 z_2 \frac{\partial u}{\partial z_2}. \tag{16}$$

If $\partial u / \partial \overline{z}_1 \equiv 0$, then u is holomorphic in both variables, so it is holomorphic in B_2. Suppose (16) is true. If we take $\partial / \partial \overline{z}_1$ on both sides of (16) then we have $Ru = 0$ by (14). Therefore u is antiholomorphic in B_2 by a theorem of Forelli [**3**]. This completes the proof. ∎

3. $\mathcal{M}$-HARMONIC FUNCTIONS WITH PLURIHARMONIC SQUARE

Finally we prove that any $\mathcal{M}$-harmonic function with pluriharmonic square is either holomorphic or antiholomorphic.

THEOREM 3. *Suppose $\widetilde{\Delta} u = \widetilde{\Delta} u^2 = 0$ in B_n. If $\Delta u^2 = 0$, in addition, then one of u and $\overline{u}$ is holomorphic in B_n. In other words, if $\mathcal{M}$-harmonic function u has a pluriharmonic square, then either u or $\overline{u}$ is holomorphic.*

PROOF: Since u^2 is pluriharmonic, it can be written as $u^2 = f + \overline{g}$, where f and g are holomorphic in B_n. Hence $u = (f + \overline{g})^{1/2}$, a branch, where u does not vanish. Since

$$\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} = -\frac{1}{4}(f + \overline{g})^{-3/2} \frac{\partial f}{\partial z_j} \overline{\frac{\partial g}{\partial z_k}},$$

we have

$$\begin{aligned} 0 &= \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial^2 u}{\partial z_j \overline{\partial} z_k} \\ &= -\frac{1}{4}(f + \overline{g})^{-3/2} \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial f}{\partial z_j} \overline{\frac{\partial g}{\partial z_k}}. \end{aligned}$$

Therefore

$$\sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial f}{\partial z_j} \overline{\frac{\partial g}{\partial z_k}} = 0,$$

where u does not vanish and so u does not vanish everywhere by continuity. We apply the result of Ahern and Rudin [1, Theorem I]. If $n = 2$, then one of f and g is a constant function, and so one of u and $\overline{u}$ is holomorphic. Assume $n \geqslant 3$, and suppose that neither f nor g is a constant function. Then there exist

(i) an interger m, $2 \leqslant m \leqslant n-1$,
(ii) a utitary transformation $U : \mathbb{C}^n \to \mathbb{C}^n$,
(iii) entire functions $\phi : \mathbb{C}^{m-1} \to \mathbb{C}$ and $\psi : \mathbb{C}^{n-m} \to \mathbb{C}$, such that

$$f(Uz) = \phi\left(\frac{z_2}{1-z_1}, \cdots, \frac{z_m}{1-z_1}\right), \quad g(Uz) = \psi\left(\frac{z_{m+1}}{1-z_1}, \cdots, \frac{z_n}{1-z_1}\right).$$

Therefore we may assume

$$u^2(z) = \phi\left(\frac{z_2}{1-z_1}, \cdots, \frac{z_m}{1-z_1}\right) + \overline{\psi}\left(\frac{z_{m+1}}{1-z_1}, \cdots, \frac{z_n}{1-z_1}\right).$$

We claim that u^2 vanishes somewhere on B_n. First we can choose $\zeta^{(1)} \in \mathbb{C}^{m-1}$ and $\zeta^{(2)} \in \mathbb{C}^{n-m}$ so that $\phi(\zeta^{(1)}) + \overline{\psi}(\zeta^{(2)}) = 0$. We take a value x_1 with $0 < x_1 < 1$ so that $\left|\zeta^{(1)}\right| + \left|\zeta^{(2)}\right| < R = \sqrt{1-x_1^2}/\sqrt{2}(1-x_1)$ and set

$$z^{(1)} = \frac{\sqrt{1-x_1^2}}{\sqrt{2}}\,\xi^{(1)}, \xi^{(1)} \in \mathbb{C}^{m-1}, \ \left|\xi^{(1)}\right| < 1,$$
$$z^{(2)} = \frac{\sqrt{1-x_1^2}}{\sqrt{2}}\,\xi^{(2)}, \xi^{(2)} \in \mathbb{C}^{n-m}, \ \left|\xi^{(2)}\right| < 1.$$

Then $\left(x_1, z^{(1)}, z^{(2)}\right) \in B_n$. We can choose $\xi^{(1)}$ and $\xi^{(2)}$ so that

$$\frac{z^{(1)}}{1-x_1} = \frac{\sqrt{1-x_1^2}}{\sqrt{2}(1-x_1)}\xi^{(1)} = \zeta^{(1)}$$
$$\frac{z^{(2)}}{1-x_1} = \frac{\sqrt{1-x_1^2}}{\sqrt{2}(1-x_1)}\xi^{(2)} = \zeta^{(2)}.$$

Therefore

$$u^2\left(x_1, z^{(1)}, z^{(2)}\right) = \phi\left(\zeta^{(1)}\right) + \overline{\psi}\left(\zeta^{(2)}\right) = 0.$$

For fixed x_1 and $z^{(2)}$, $u^2\left(x_1, z^{(1)}, z^{(2)}\right)$ is holomorphic in $z^{(1)}$ and hence u^2 takes all values of a neighbourhood of 0. Therefore it cannot have a continuous square root function u, which is a contradicion. This shows that either f or g must be a constant function. That is, one of u or $\overline{u}$ is holomorphic. This completes the proof. ∎

https://doi.org/10.1017/S0004972700016786 Published online by Cambridge University Press

References

[1] P.R. Ahern and Walter Rudin, '$\mathcal{M}$-harmonic products', *Indag. Math.* **2** (1991), 141–147.

[2] G.B. Folland, 'Spherical harmonic expansion of the Poisson-Szago kernel for the ball', *Proc. Amer. Math. Soc.* **47** 2 (1975), 401 – 408.

[3] Walter Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$ (Springer-Verlag, Berlin, Heidelberg, New York, 1980).

Department of Mathematics
Korea Advanced Institute of Science and Technology
373-1 Kusong-Dong Yusong-Gu
Taejon 3-5-701
Kores