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HOMOGENEOUS BUNDLES AND UNIVERSAL POTENTIALS

BY

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ABSTRACT. This paper studies complex potentials on homogeneous bundles over a compact Lie group. It extends the previous work of V. Guillemin and A. Uribe on potentials isospectral to the zero potential. Then the notion of a universal potential is introduced, that is a potential which acts on sections by a group representation rather than as a scalar. Finally the inverse question of whether the spectral data of a complex potential on all bundles over S^2 determines the potential is answered negatively.

1. **Introduction**. Given a Riemannian manifold M with a Laplacian Δ it is an interesting problem to find potential functions q_1 and q_2 such that the spectra are equal: $Spec(\Delta + q_1) = Spec(\Delta + q_2)$. In the case when $M = S^1$ one way to obtain such isospectral potentials is to use the KdV equation, see [6]. If the potentials are complex valued the situation changes remarkably. Now, in the case that M is a homogeneous space, a whole class of potentials can be found which are isospectral to the zero function, see [3].

The object of this paper is to pursue these questions on bundles. For the manifold, M, we take a compact, semisimple, simply connected Lie group, G, which is simple modulo its center. The bundles are all homogeneous vector bundles, see [7]. A potential is then an endomorphism valued section and we are interested in potentials which are isospectral to the zero potential.

The main result concern universal potentials. Let $\mathbf{C}[G] = \{\Sigma\, \alpha_x x : x \in G$ only finitely many $\alpha_x \neq 0\}$ be the group algebra. If $q : G \to \mathbf{C}[G]$ is given by $q(g) = \Sigma\, \alpha_x(g)x$ then for any homogeneous bundle we obtain a potential. The group acts on each fiber by a representation, say π_τ. Then the potential $q_\tau(g) = \Sigma\, \alpha_x(g)\pi_\tau(x)$. Here π_τ is the irreducible representation with highest weight τ. If the group action is reducible we write it as a sum of irreducible ones and treat each summand separately. Thus the map q into the group algebra gives a potential function on each bundle, and q is called a universal potential. The main result concerns universal potentials of the form

$$q(x) = \sum_{\lambda \in \Lambda, k \in K} \langle \pi_\lambda(x) v_{\lambda k}, u_{\lambda k} \rangle x^k. \tag{1.1}$$

where Λ parameterizes the irreducible representations of G and K is a finite subset of the positive integers.

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THEOREM 1.1. *Let q be as in* (1.1) *with $v_{\lambda k}$ a highest weight vector in E_λ. Then q_τ satisfies $Spec(\Delta + q_\tau) = Spec(\Delta)$ for all τ such that $\lambda > k\tau$ for each $\lambda \in \Lambda$* and *$k \in K$.*

Here the ordering, which is used in $\lambda > k\tau$, is a little stronger than the usual one and is explained in definition 3.3. Throughout this paper the sign of Δ is chosen so that Δ has an increasing sequence of positive eigenvalues. This result is proved in section 3 as theorem 3.4. Section 3 also contains a description of the group algebra and a more detailed account of universal potentials.

Section 2 contains some technical results. There is a description of homogeneous bundles, the Laplacian and the eigensections. The first technical result is lemma 2.2. This is in contrast to theorem 1.1 since it is just the obvious extension of the result in [3]. Then there is the following result.

THEOREM 1.2. *Let q be a class potential, so $q(xgx^{-1}) = q(g)$, then $Spec(\Delta + q) = Spec(\Delta)$ if and only if $Spec(\Delta_T + q|T) = Spec(\Delta_T)$. Here Δ_T is the Laplacian on the bundle over the maximal torus.*

This is proved as theorem 4.1 and we shall comment on it after describing the contents of section 5.

In Section 5 the example of the circle is given. The main result concerns the bundle of degree k and potentials of the form

$$q_k(x) = \sum c_{jr} e^{i(r+kj)x}. \tag{1.2}$$

THEOREM 1.3. *The operators $\Delta + q_k$ and Δ are isospectral providing $c_{jr} = 0$ for $r \leq -kj$.*

The point of this is that, given a universal potential, it is sometimes possible to find infinitely many bundles for which the universal potential is isospectral. In view of theorem 1.2 one would like to find a potential invariant under the map $x \to -x$. Then this would give rise to a class potential on $SU(2)$. This is not possible with our methods since if $c_{jr} \neq 0$ then, as explained at the end of section 5, $r > |kj|$, and invariance under $x \to -x$ requires $c_{-j-r} = c_{jr}$ giving the additional condition $-r > |kj|$. Thus no isospectral potentials arise this way.

As an application of this work we consider an inverse spectral problem. The question is "On the sphere S^2 does the spectrum of $\Delta + q$ determine q?" The answer is that it does not determine q. However, recently in [1] there is the result that if q is real valued and the spectrum is extended to include the spectrum on sections of the line bundles over the torus $\mathbb{R}^n/L$ then this spectrum greatly restricts the possible potentials q. There is also the suggestion that it might be possible to recover the potential q from the spectrum. Our concern is with complex valued potentials rather than real ones; for work on real valued potentials we refer the reader elsewhere. The work in [1] does raise the question of the inverse spectral problem and whether the extended (or Floquet) spectrum determines the potential. Working with complex potentials on S^2 we answer the question negatively in the complex case by exhibiting a complex potential which is isospectral to the zero potential on all complex homogeneous line bundles over S^2. This is done in section 6 of this paper.

I should like to thank Professor Victor Guillemin for a very useful discussion on this work. It was Professor Guillemin who brought to my attention the inverse spectral problem discussed in this paper and it was in discussion with him that we realized the other results in this paper essentially settled that question. The author would also like to thank C. T. Stretch for his helpful comments concerning the group algebra.

2. **Potentials on Homogeneous Bundles**. Let G be a compact, simply connected and semisimple Lie group which is simple modulo its center. We follow [7] in defining the homogeneous bundle $\boldsymbol{E}_\tau$, associated to the irreducible representation $\pi_\tau : G \to Aut\ E_\tau$, by requiring the following diagram to commute:

$$
\begin{tikzcd}
G \times G \times E_\tau \arrow[r] \arrow[d] & \boldsymbol{E}_\tau \arrow[d] \\
G \times G \arrow[r] & G.
\end{tikzcd}
\tag{2.1}
$$

The lower horizontal map is $(x, y) \to y^{-1}x$ thus we are factoring out by the diagonal action:

$$g(x, y) = (gx, gy). \tag{2.2}$$

Sections of $\boldsymbol{E}_\tau$ are realized as maps $S : G \times G \to E_\tau$ which are invariant under this diagonal action:

$$S(gx, gy) = \pi_\tau(g)S(x, y). \tag{2.3}$$

The Laplacian Δ on sections is realized as half the usual Laplacian on these invariant functions.

For any representation $\pi_\lambda : G \to Aut\ E_\lambda$ we let $\langle , \rangle$ denote a G-invariant innerproduct. Define the matrix coefficient sections $S_{\lambda v u}$ by

$$S_{\lambda v u}(x, y) = \sum_i \langle \pi_\lambda(y^{-1}x)v, u_{i\lambda} \rangle \pi_\tau(x)u_{i\tau}, \tag{2.4}$$

Where $u = \Sigma\, u_{i\lambda} \otimes u_{i\tau}$ is an element of E_μ, $u_{i\lambda}$ are in E_λ, $u_{i\tau}$ in E_τ, and E_μ is an irreducible summand of $E_\lambda \otimes E_\tau$. These maps are clearly invariant and so define sections of E_τ. By a routine extension of the main result in [2] we have the following:

LEMMA 2.1. *The matrix coefficient sections are the eigensections of Δ and the eigenvalues are* $(1/2)(C(\lambda) + C(\mu))$.

That is

$$\Delta S_{\lambda v u} = \frac{1}{2}(C(\lambda) + C(\mu))S_{\lambda v u} \tag{2.5}$$

where $C(\lambda) = \|\lambda + \rho\|^2 - \|\rho\|^2$ is the eigenvalue of the Casimir operator on G. Notice that if λ^* indexes the dual representation to π_λ then $C(\lambda^*) = C(\lambda)$.

A potential is a section of the endomorphism bundle, $End(\boldsymbol{E}_\tau)$, that is the homogeneous bundle with the endomorphisms $End(E_\tau)$ as fibers. These can be realized as invariant functions

$$q: G \times G \to End(E_\tau) \tag{2.6}$$

with

$$q(gx, gy) = \pi_\tau(g) q(x, y) \pi_\tau(g^{-1}) \tag{2.7}$$

as the invariance property. Pick a basis, $\{\alpha_k\}$, for the space $End(E_\tau)$ then as a sum of matrix coefficients we have

$$q(x, y) = \sum_{\nu, k} \langle \pi_\nu(y^{-1}x) w_{\nu k}, u_{\nu k} \rangle \pi_\tau(y) \alpha_k \pi_\tau(y^{-1}), \tag{2.8}$$

with $w_{\nu k}$ and $u_{\nu k}$ elements in E_ν. A simple calculation shows

$$qS_{\lambda\nu u} = \sum_{\lambda, \nu, i, k} \langle \pi_{\nu\otimes\lambda}(y^{-1}x) w_{\nu k} \otimes v, u_{\nu k} \otimes u_{i\lambda} \rangle \pi_\tau(y) \alpha_k \pi_\tau(y^{-1}x) u_{i\tau}. \tag{2.9}$$

Thus from [3] we can deduce our main lemma.

LEMMA 2.2. *The potential q has the property that $Spec\,\Delta = Spec(\Delta + q)$ if $w_{\nu k}$ is a highest weight vector in E_ν.*

The point of this lemma is that if the matrix coefficients in q, the terms $\langle \pi_\nu(y^{-1}x) w_{\nu k}, u_{\nu k} \rangle$ in equation (2.8), are of a special type then so is the potential q. These complex valued functions are known to be isospectral to the zero function and the endomorphism potential q is also isospectral to the zero endomorphism.

3. **Universal potentials and the group algebra**. In the case of a discrete group G the group algebra is defined by

$$\boldsymbol{C}[G] = \{\Sigma\, \alpha_x x : x \in G, \alpha_x \in C \text{ only finitely many } \alpha_x \neq 0\} \tag{3.1}$$

Thus the elements of the group algebra can be viewed as maps, $x \to \alpha_x$, which are zero almost everywhere.

In the case when G is a compact Lie group there are two other candidates for the group algebra: the L^2-functions on G and the distributions on G. Of these the second, that is the L^2-functions on G, is most usually taken as the group algebra. However, in our case the definition in (3.1) with G a compact Lie group is the most convenient. There is little difference for our purposes between these definitions and so as a matter of convenience we make the following definition.

DEFINITION 3.1. *The group algebra, $\boldsymbol{C}[G]$, for a compact Lie group is the set $\{\Sigma\, \alpha_x x : x \in G\}$ where only finitely many α_x are nonzero.*

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If $\pi_\lambda : G \to Aut\, E_\lambda$ is a representation of G there is a representation of $C[G]$ given by

$$(3.2) \qquad \pi_\lambda(\Sigma\, \alpha_x x) = \Sigma\, \alpha_x \pi_\lambda(x).$$

Notice that this is endomorphism valued, $\pi_\lambda : C[G] \to End\, E_\lambda$. Using this extension of π_λ we can define a universal potential.

DEFINITION 3.2. *A universal potential is a map $q : G \to C[G]$. For the homogeneous bundle associated to $\pi_\tau : G \to Aut\, E_\tau$ the endomorphism valued potential is $q_\tau = \pi_\tau \circ q$.*

The main theorem of this section concerns universal potentials of the form $q(x) = \Sigma_k f_k(x) x^k$. Using the Peter-Weyl theorem, to write f_k as a sum of matrix coefficients, gives

$$(3.3) \qquad q(x) = \sum_{\lambda \in \Lambda, k \in K} \langle \pi_\lambda(x) v_{\lambda k}, u_{\lambda k} \rangle x^k,$$

where Λ and K are the indexing sets for λ and k. Notice that K is finite by our definition of the group algebra.

We also need a strong partial ordering on the highest weights of representations.

DEFINITION 3.3. *The highest weight λ is larger than τ, $\lambda > \tau$, if $\lambda - \alpha$ is dominant for all weights α such that $\tau - \alpha$ is positive or zero.*

This is stronger than the usual ordering which only requires $\lambda - \alpha$ to be positive, which incidentally follows from $\lambda - \tau$ positive.

THEOREM 3.4. *Let q be as in (3.3) with $v_{\lambda k}$ a highest weight vector in E_λ. Then q_τ satisfies $Spec(\Delta + q_\tau) = Spec(\Delta)$ for all τ such that $\lambda > k\tau$ for each $\lambda \in \Lambda$ and $k \in K$.*

PROOF. Let $\{u_j\}$ be a basis for E_τ and δ_{ij} be the endomorphism $\delta_{ij}(v_i) = v_j$ and $\delta_{ij}(v_r) = 0$ for $r \neq i$. Then q_τ given by

$$(3.4) \qquad q_\tau(x) = \sum_{\lambda, i, j, k} \langle \pi_\lambda(x) v_{\lambda k}, u_{\lambda k} \rangle \langle \pi_\tau(x^k) v_i, v_j \rangle \delta_{ij}.$$

Now $\langle \pi_\tau(x^k) v_i, v_j \rangle$ is a function of x and so can be written as a sum of matrix coefficients:

$$(3.5) \qquad \langle \pi_\tau(x^k) v_i, v_j \rangle = \sum_\mu \langle \pi_\mu(x) w_{\mu k}, w'_{\mu k} \rangle$$

where $k\tau - \mu$ is positive or zero. Thus

$$(3.6) \qquad q_\tau(x) = \sum_{\lambda, \mu, i, j, k} \langle \pi_{\lambda \otimes \mu}(x) v_{\lambda k} \otimes w_{\mu k}, u_{\lambda k} \otimes w'_{\mu, k} \rangle \delta_{ij}.$$

The condition $\lambda > k_\tau$ implies $\pi_{\lambda \otimes \mu} = \pi_\lambda \otimes \pi_\mu = \Sigma m_\mu(\alpha)\pi_{\lambda - \alpha}$, a well known result, see [5] for example. Continuing our calculation gives

$$q_\tau(x) = \sum_{\lambda, \alpha, i, j} \langle \pi_{\lambda - \alpha}(x)(v_{\lambda\alpha}, u_{\lambda\alpha}\rangle \delta_{ij} \tag{3.7}$$

where $v_{\lambda\alpha}$ is the component in $E_{\lambda - \alpha}$ of the vector $v_{\lambda k} \otimes w_{\mu k}$, and similarly for u. Since $v_{\lambda k}$ is a highest weight vector and $\lambda - k\tau$ we see that $v_{\lambda\alpha}$ are highest weight vectors. The theorem now follows from lemma 2.2.

In this proof we have taken advantage of an abuse of notation. To make sense of δ_{ij} one has to work on the trivial bundle $G \times G \times E_\tau$ over $G \times G$. Our calculation is done on the $G \times \{1\}$ subset of $G \times G$, extended to the whole of $G \times G$ by invariance and then we pass to G by factoring out the diagonal.

4. **Reduction to the maximal torus**. In this section we present a theorem with a slightly different thurst to the other results in this paper.

THEOREM 4.1. *Let q be a class potential, so $q(xgx^{-1}) = q(g)$, then $Spec(\Delta + q) = Spec(\Delta)$ if and only if $Spec(\Delta_T + q|T) = Spec(\Delta_T)$.*

PROOF. Let $\tilde{\Delta}$, $\tilde{\Delta}_T$, $\tilde{\rho}$, and $\tilde{j}$ denote the Laplacian on $G \times G$, on $T \times T$, half the sum of the positive roots of $G \times G$ and the denominator function of $G \times G$. Then by a well known result of Harish-Chandra, quoted in [3] for example,

$$\tilde{j}\,(\tilde{\Delta} f)|T = (\tilde{\Delta}_T - \|\tilde{\rho}\|^2)\,\tilde{j}\,(f|T). \tag{4.1}$$

To obtain the equation (4.1) we have used Harish-Chandra's result in the context of a group $G \times G$ with maximal torus $T \times T$. By carrying out the passage between the group and maximal torus for these product groups we can then obtain the reduction from G to T for sections of homogeneous bundles.

Hence on sections of E_τ

$$j(\Delta s)|T = (\Delta_T - \|\rho\|^2) j(s|T), \tag{4.2}$$

since $\Delta = (1/2)\tilde{\Delta}$ on invariant functions and $\|\rho\|^2 = (1/2)\|\tilde{\rho}\|^2$. Thus if $C + \|\rho\|^2$ is an eigenvalue of Δ_T then C is an eigenvalue of Δ. Hence if $C + \|\rho\|^2$ is an eigenvalue of $\Delta_T + q|T$ then C is an eigenvalue of $\Delta + q$. All that remains is to check the multiplicities of these eigenvalues. This is done as in [2]. Let A_t be the operator $\Delta + tq$ acting on sections of E_τ. Let D be a small disc containing the eigenvalue C, which is common to A_t for all $0 \leq t \leq 1$, and no other eigenvalues. Let $\Gamma = \partial D$ then the operator

$$P_t = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - A_t} dz \tag{4.3}$$

is well defined and depends continuously on t. The trace of P_t is the multiplicity of the eigenvalue C. Thus $tr\,P_t$ is constant and the multiplicity of $C \in Spec(\Delta + q)$ is the same as the multiplicity of $C \in Spec\,\Delta$.

5. **The example of the circle**. In this section $G = S^1$ which is identified with the unit complex numbers $\{e^{ix}: x \in \mathbb{R}\}$. Since this group is abelian all its irreducible representations are either trivial or two (real) dimensional. These two dimensional ones are indexed by the integers $\boldsymbol{Z}$:

$$\pi_k(e^{ix})v = e^{ikx}v, \tag{5.1}$$

where we have identified $E_k = \boldsymbol{C}$, $v \in \boldsymbol{C}$ and the right hand side is just multiplication. The associated bundle is formed by taking the trivial bundle $G \times G \times E_k$ and dividing out the diagonal action. A basis of eigensection of the Laplacian are given by

$$f_m(x, y) = e^{i(mx + (k-m)y)} \tag{5.2}$$

with

$$\Delta f_m(x, y) = \frac{1}{2}(m^2 + (m - k)^2)f_m(x, y). \tag{5.3}$$

In this $(x, y) \in G \times G$ and since $f_m: G \times G \to \boldsymbol{C} = E_k$ has the correct invariance property we have a section $\boldsymbol{E}_k$ over G.

The group algebra $\boldsymbol{C}[G]$ we take in the form

$$\boldsymbol{C}[G] = \{\Sigma\, a_\alpha e^{i\alpha}: \text{only finitely many } a_\alpha \neq 0\}. \tag{5.4}$$

In this case a universal potential has the form

$$q(x) = \Sigma\, a_j(x)e^{ijx} \tag{5.5}$$

with

$$q_k(x) = \pi_k(q(x)) = \Sigma\, a_j(x)e^{ikjx}. \tag{5.6}$$

Here $a_j(x)$ is a complex valued function of x and q_k is an endomorphism valued function. As at the end of the previous section we are taking advantage of an abuse of notation: it would be more correct to replace x by $x - y$ in these formula and work on $G \times G$ rather than on G. Taking a Fourier series expansion for $a_j(x)$ yields

$$a_j(x) = \Sigma\, c_{jr}e^{irx} \tag{5.7}$$

hence

$$q_k(x) = \Sigma\, c_{jr}e^{i(r+kj)x}. \tag{5.8}$$

Thus

$$(\Delta + q_k)f_m = \frac{1}{2}(m^2 + (m - k)^2)f_m + \Sigma\, c_{jr}f_{m+r+kj}. \tag{5.9}$$

As in the previous work [3] we now have an isospectral result.

THEOREM 5.1. *The operators $\Delta + q_k$ and Δ are isospectral providing $c_{jr} = 0$ for $r \leq -kj$.*

A more interesting way to view this is given in the next corollary.

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COROLLARY 5.2. *The operators $\Delta + q$ and Δ, regarding q as a universal potential, are isospectral on all bundles E_k such that $k > -r/j$ for all r and j with $c_{rj} \neq 0$.*

It is interesting to compare these results with our earlier results for a compact simply connected semisimple Lie group G. The circle S^1 is the maximal torus of $SU(2)$ and we are tempted to transfer these results to $SU(2)$ by conjugation. However, the maximal torus of $SU(2)$ has a Weyl group action: $W = Z_2 = \{\pm 1\}$. The isospectral potentials here are clearly not invariant under $x \to -x$. The irreducible representations of $SU(2)$ decompose on the maximal torus into many of our π_k. In particular both k and $-k$ are always present. Thus the inequality in corollary 4.2 would have to become $k > -r/j$ and $-k > -r/j$ thus $|jk| < r$. This provides some further elaboration on the inequality, $\lambda > k\tau$, in theorem 3.4.

6. **Line bundles on S^2.** For each integer k there is a line bundle E_k on the sphere S^2. This is obtained from the following diagram

$$\begin{array}{ccc} S^3 \times C & \longrightarrow & E_k \\ \downarrow & & \downarrow \\ S^3 & \longrightarrow & S^2 . \end{array} \tag{6.1}$$

Here S^3 is identified with $SU(2)$. The circle S^1 is identified as both the maximal torus in $SU(2)$ and as the complex numbers of unit modulus. This S^1 acts on S^3 by right multiplication and for each integer k by

$$\pi_k(e^{i\theta})(z) = e^{ik\theta}z \tag{6.2}$$

on C. The action on $S^3 \times C$ is then given by $e^{i\theta}(x, z) = (xe^{-i\theta}, e^{ik\theta}z)$ and E_k is the quotient of the trivial bundle $S^3 \times C$ by this action.

We are interested in scalar potentials, that is functions q on S^2 which act by scalar multiplication on sections of E_k. Then an eigenvalue, λ, and corresponding eigen-section, S satisfy

$$(\Delta + q(x))s(x) = \Delta s(x) + q(x)s(x) = \lambda s(x). \tag{6.3}$$

This equation can be lifted to S^3. Let $\tilde{q}: S^3 \to C$ so $\tilde{q}(xe^{-i\theta}) = \tilde{q}(x)$, $\tilde{S}: S^3 \to C$ so that $\tilde{S}(xe^{-i\theta}) = e^{ik\theta}\tilde{S}(x)$ and $\tilde{\Delta}$ be the usual Laplacian on functions on S^3. Then equation (6.3) is equivalent to

$$(\tilde{\Delta} + \tilde{q})\tilde{s} = \lambda \tilde{s}. \tag{6.4}$$

If q is a Fegan potential on S^2 then it is clear from the proof of lemma 2.2, using the identity map $E_k \to E_k$ as the bundle endomorphism, α_k, that q is isospectral to the zero potential. Thus we have the result.

THEOREM 6.1. *Let q be a Fegan potential on S^2 then $\Delta + q$ is isospectral to Δ on sections of the bundle E_k for all k.*

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This result provides a negative answer to the question: "Does the spectrum of $\Delta + q$ on $\boldsymbol{E}_k$ for all k determine q?" One notes, of course, that the complex case is very different from the real case. Here we are concerned with potentials which are isospectral to the zero potential. In the real case this is generally rigid, that is if $\Delta + q$ is isospectral to Δ and q is real valued then in general $q \equiv 0$. So we are in a situation which does not occur with real valued potentials.

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